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MEDIUM-FREQUENCY CYCLES AND THE REMARKABLE NEAR TREND-STATIONARITY OF OUTPUT

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Medium-frequency cycles and the remarkable near trend-stationarity of output.

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Abstract: *This paper builds a dynamic stochastic general equilibrium (DSGE) model of endogenous growth that generates large medium-frequency cycles while robustly matching the near trend-stationary path of observed output. This requires a model in which standard business cycle shocks lead to highly persistent movements around trend, without significantly altering the trend itself. The robustness of the trend also requires that we eliminate the scale effects and knife edge assumptions that plague most growth models. In our model, when products go out of patent protection, the rush of entry into their production destroys incentives for process improvements. Consequently, old production processes are enshrined in industries producing non-protected products, and shocks that affect invention rates change the proportion of industries with advanced technologies. In an estimated version of our model, a financial-type shock to the stock of ideas emerges as the key driver of the medium frequency cycle.*

Keywords: *medium frequency cycles, patent protection, scale effects*

JEL Classification: *E32, E37, L16, O31, O33, O34*

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1. Introduction

Viewed from a distance, a log-plot of the last one hundred years of US GDP looks very near linear. However, closer inspection reveals large medium frequency fluctuations around this linear trend. Generating this combination of remarkably near trend-stationary long run growth, and large cycles around the trend, is a challenge for traditional models of endogenous growth. The near linear trend requires scale effects to be removed not just in the long run, but in the shorter run as well. Models that remove these scale effects via knife-edge assumptions will usually fail this test, as temporary business cycle shocks will knock the model away from perfectly removing the scale effect, leading to a permanent break in the trend of the GDP. Equally, models that remove scale effects via new product creation will tend to produce such trend breaks in GDP if the stock of new products can only respond slowly following a shock. On the other hand, if the stock of products can adjust instantly following a shock, then, (in standard models) there would be no movement in productivity at all, let alone the large, persistent medium frequency cycles that Comin and Gertler (2006) document in the data, and that may be seen in our Figure 1 below. In this paper, we present a mechanism capable of reconciling this apparently contradictory low and medium frequency behaviour of output, while also matching the cyclicality of mark-ups: the key determinant of research and invention decisions.

Our story is as follows. The returns to inventing a new product are higher in a boom due to the higher demand. As a result, during periods of expansion, the rate of creation of new products increases, in line with the evidence of Broda and Weinstein (2010). Due to a first mover advantage, patent protection, or reverse-engineering difficulties, the inventors of these new products will be able to extract rents from them, increasing the costs manufacturing firms face if they wish to produce the new product. These higher costs lead to lower competition in new industries, increasing mark-ups and thus increasing firms' incentives to perform the R&D necessary to catch-up with and surpass the frontier, for basically Schumpeterian reasons. Consequently, the higher proportion of industries that are relatively new in a boom will lead to higher aggregate productivity, lower dispersion of both productivity levels and growth rates, as well as higher mark-ups. Since the length of time for which inventors can extract rents will be determined by the effective duration of patent-protection, this effect will naturally work at medium frequencies. However, since we allow both for the creation of new industries (producing new products) and for a varying number of firms within each industry, even in the short-run the demand faced by any given firm will be roughly constant, meaning that our model will not produce large deviations from linear growth.

Evidence for the pro-cyclicality of TFP has been presented by Bils (1998) and Campbell (1998) amongst others, with Comin and Gertler (2006) showing that the evidence is particularly clear at medium-frequencies. The counter-cyclicality of productivity dispersion has been shown by Kehrig (2011), with evidence on the counter-cyclicality of the dispersion of productivity growth rates provided by e.g. Eisfeldt and Rampini (2006) and Bachmann and Bayer (2009). Evidence for the pro-cyclicality of aggregate mark-ups has been presented by Boulhol (2007) and Nekarda and Ramey (2010). Nekarda and Ramey also show that mark-ups lead output at business-cycle frequencies, we will present further evidence in section 2 below that this relationship continues to hold at medium-frequencies. Boulhol (2007) also shows that although aggregate mark-ups are pro-cyclical, the mark-ups in any particular industry tend to be counter-cyclical. This apparent contradiction will be readily explained in our model since the increase in competition in any particular industry will lead to a decline in mark-ups in that industry (much as in the models of Bilbiie, Ghironi, and Melitz (2012) and Jaimovich (2007)), despite the fact that aggregate mark-ups have increased due to the greater proportion of industries with relatively high mark-ups. Formal evidence on the small size of the unit root in output (i.e. its near trend stationarity) was presented by Cochrane (1988), and we will present further evidence in the next section that GDP returns to trend at long lags.

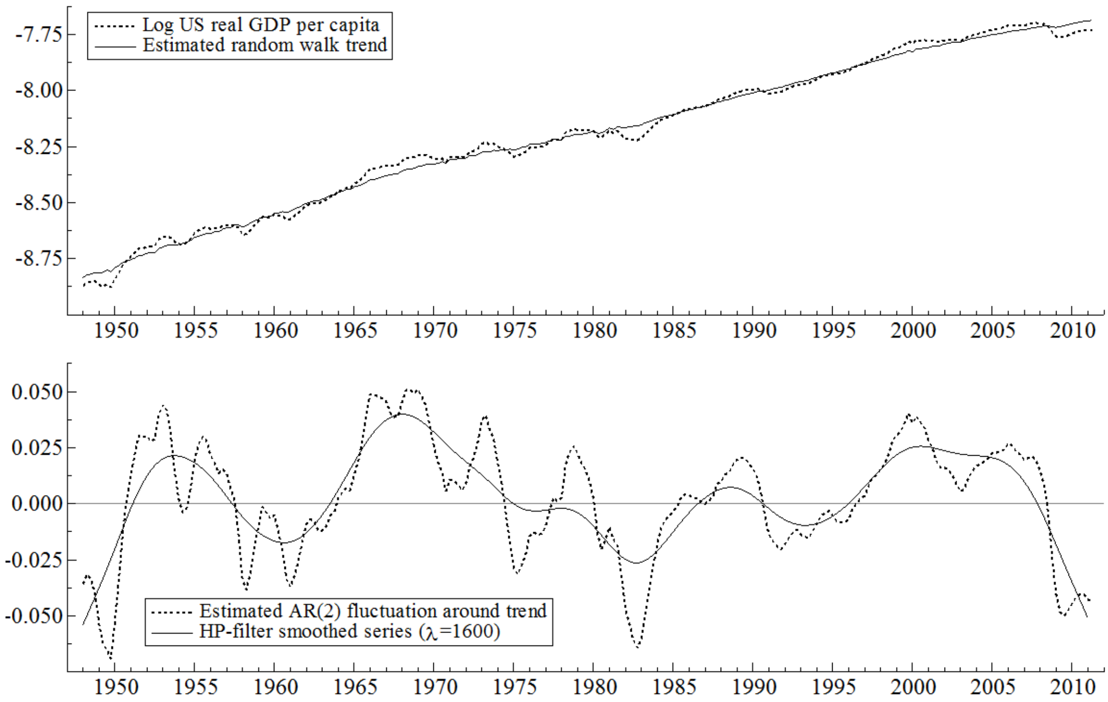

Figure 1: The results of modelling quarterly log real US GDP per capita as a sum of a random walk, an AR(2) process and an idiosyncratic shock.

The solid line in the second graph is a crude representation of the medium-frequency cycle.

Direct evidence for the importance of our mechanism comes from a number of sources. Balasubramanian and Sivadasan (2011) find that firms holding patents have 17% higher TFP levels on average, and additionally find that firms that go from not holding a patent to holding one experience a 7.4% increase in a fixed effects measure of productivity, suggesting that industries producing patent-protected products are indeed significantly more productive. Serrano (2007) finds that although aggregate patenting is only weakly correlated with aggregate TFP, a measure of the number of patents whose ownership is transferred is strongly related to productivity. He argues that there is a great deal of noise in measures of total patent activity, since so many patents are never seriously commercialised. Patent transfers are usually observed though when their purchaser intends to begin exactly such a commercialisation. Thus, patent transfers provide a proxy for the commencement of production of new patented-products, one that is found to be highly pro-cyclical. Finally, we will present new evidence that longer patent protection significantly increases the share of GDP variance attributable to cycles of medium frequency.

Previous papers have introduced endogenous productivity improvement into business cycle models (e.g. Comin and Gertler (2006), Comin (2009), Comin, Gertler, and Santacreu (2009), Phillips and Wrase (2006), Nuño (2008; 2009; 2011)), or looked at cycles in growth models (e.g. Bental and

Peled (1996), Matsuyama (1999), Wälde (2005), Francois and Lloyd-Ellis (2008; 2009), Comin and Mulani (2009)). However, all of these papers have problems with scale effects, either in the long-run, or in the short-run, and thus all of them would predict counter-factually large unit roots in output in the presence of standard DSGE shocks. Furthermore, it is not obvious how these scale effects could be removed without destroying the papers' mechanisms for generating aggregate TFP movements. For example, the papers of Wälde (2005) and Phillips and Wrase (2006) rely on there being a small finite number of sectors. Removing the scale effect would mean allowing this number to grow over time with population, meaning the variance of productivity would rapidly go to zero. Indeed, this happens endogenously in the model of Horii (2011). Many models of endogenous mark-up determination (e.g. Bilbiie, Ghironi, and Melitz (2012) or Jaimovich (2007)) have a similar problem, with the presence of a small finite number of industries being crucial for explaining the observed variance of mark-ups. Indeed, Bilbiie, Ghironi, and Melitz (2011) write that "reconciling an endogenous time-varying markup with stylized growth facts (that imply constant markups and profit shares in the long run) is a challenge to growth theory". By disentangling the margins of firm entry and product creation, we will be able to answer this challenge.

The paper of most relevance to our work is Comin and Gertler (2006), as they made the important contribution of bringing the significance of medium-frequency cycles to the attention of the profession. Additionally, their theoretical model, like ours, stresses the effects of mark-up variations on productivity growth. Unfortunately, however, it counter-factually predicts that increases in mark-ups lead to falls in output, contrary to the empirical evidence of Nekarda and Ramey (2010). Furthermore, its only major sources of productivity persistence are the persistence of the driving mark-up shock, and the counter-factual trend break in productivity following such a shock. We conclude then, that the literature still lacks a model of productivity capable of explaining both its short run and its long run behaviour.

In section 3, we present a model capable of doing this. In order to remove both the long run and the short run scale effect, as discussed above it will feature a varying number of industries, each of which will contain a varying number of firms. We do not wish to make any exogenous assumptions on the differences between industries producing patented products versus those producing unpatented ones, so in order to match the medium-frequency behaviour of productivity and mark-ups it is important that our model allow endogenous variation in these quantities across industries. Were we to assume free transfer of technologies across industries there would be too little difference in productivity between patent-protected and un-patent-protected industries, and hence we would not be able to generate medium-frequency cycles. Equally, were we to assume technology transfer across industries was impossible then it would be legitimate to inquire whether the difference between these industry types was implausibly large, as perhaps firms in non-protected industries would find it optimal to perform technology transfer even if they did not find it optimal to perform any research. Consequently, in modelling the endogenous productivity in each industry we will allow firms both to perform research, and to perform a costly process of catch-up to the frontier we shall term appropriation. To make clear the strength of the amplification and persistence mechanism presented here, we initially omit capital from the model, and we focus on the impulse responses to non-persistent shocks when we discuss our model's qualitative behaviour in section 3.5. Finally, in section 4, we add a few standard additional features to the model (habits, capital with adjustment costs, variable capacity utilisation, sticky wages, Taylor rule monetary policy) and we show that this model matches the data well at low, medium and high frequencies, with financial-type shocks to the stock of ideas playing the key role in driving medium-frequency fluctuations.

2. Empirics

2.1. The near trend stationarity of output

We begin by presenting evidence that GDP returns to trend at long lags. Since statistical tests on regressions with large numbers of lags tend to suffer from a lack of power, we have to find a sparsely parameterised way of capturing this long-run behaviour. It seems implausible that a high-frequency spike in GDP should lead to another spike in GDP many periods later. Instead, if GDP responds at all to its own past fluctuations at long lags, it will only respond to the low frequency (i.e. smoothed) fluctuations. We would like to smooth the data then at a range of frequencies, and regress output on the lags of these smoothed series. It will also help the interpretability of results if each lag of the data affects at most one of these smoothed series, which suggests taking moving averages. We choose then to regress log US quarterly GDP per-capita on a linear trend, the first lag of its one period moving average (i.e. its first lag), the second lag of its two period moving average, the fourth lag of its four period moving average, and so on up to the 32^{nd} lag of its 32 period moving average. I.e. we run the regression:

$$\begin{aligned} y_t = \mu + \delta t + \phi_1 y_{t-1} + \phi_2 \frac{1}{2}(y_{t-2} + y_{t-3}) + \phi_3 \frac{1}{4}(y_{t-4} + y_{t-5} + y_{t-6} + y_{t-7}) \\ + \cdots + \phi_6 \frac{1}{32}(y_{t-32} + \cdots + y_{t-63}) + \varepsilon_t. \end{aligned} \tag{2.1}$$

The full results of this regression are given in Table 1. The key facts to note here though are that ϕ_2, ϕ_3, …, ϕ_6 are all negative, and that ϕ_6 is comfortably significant at 5%, suggesting that GDP is indeed returning towards trend at long lags. ϕ_6 corresponds to a period of eight to sixteen years, which includes the principal band of medium-frequency cycles, as is shown in Figure 3.

Variable	Coefficient	Std. Error	t-value	t-prob.	Part R^2
μ	-1.20281	0.3603	-3.34	**0.0010**	0.0574
δ	0.000572088	0.0001751	3.27	**0.0013**	0.0551
ϕ_1	1.21142	0.06323	19.2	**0.0000**	0.6673
ϕ_2	-0.251229	0.08649	-2.90	**0.0041**	0.0441
ϕ_3	-0.0272064	0.05389	-0.505	*0.6143*	0.0014
ϕ_4	-0.00266296	0.03332	-0.0799	*0.9364*	0.0000
ϕ_5	-0.0139299	0.02365	-0.589	*0.5566*	0.0019
ϕ_6	-0.0531785	0.02489	-2.14	**0.0339**	0.0243

Table 1: Results of the regression (2.1).
Run on log US quarterly real GDP (from NIPA) over X12 seasonally adjusted civilian non-institutional population (CNP16OV from FRED). 1948:1-2011:2.

We would like to know whether the magnitude of ϕ_6 is sufficient to pull GDP completely back to trend, or equivalently, whether log-GDP has a unit root. We can test for this if we transform (2.1) into Augmented Dickey-Fuller (ADF) form (Said and Dickey 1984), giving:

$$\Delta y_t = \mu + \delta t + \left[\sum_{i=1}^{6} \phi_i - 1\right] y_{t-1} - \phi_2 \frac{1}{2}(2\Delta y_{t-1} + \Delta y_{t-2}) - \cdots - \phi_6 \frac{1}{32}(\dots). \qquad (2.2)$$

Since this is an equivalent model, no parameter estimates or standard errors change. However, we can now use the t-value on the y_{t-1} coefficient (-3.36) to perform an ADF test. Our Monte-Carlo experiments[2] indicate that there is only an 11.1% chance we would observe a result as extreme as this if the true data generating process were a random walk.[3] We do not wish to claim because of this that GDP is unambiguously trend-stationary. However, it does suggest that the size of the unit root in US GDP is (at most) very small, reinforcing the findings of Cochrane (1988).

2.2. Mark-ups

Nekarda and Ramey (2010) found that mark-ups were pro-cyclical both when the data was filtered with a standard ($\lambda = 1600$) HP-filter, and when it was filtered by taking first differences. However, Comin and Gertler (2006) report that mark-ups are counter-cyclical when the data is filtered via a band pass filter that keeps cycles of periods from one to fifty years.[4] Given that Comin and Gertler find that the medium-frequency variance of output is concentrated on cycles taking around ten years, the natural question is whether the counter-cyclicality of mark-ups they observe is a consequence of behaviour around these frequencies, or whether it is driven by counter-cyclicality at lower frequencies. Nekarda and Ramey (2010) also found that at business cycle frequencies, mark-ups were strongly correlated with future output, and negatively correlated with past output. Again, we would like to know if this still holds at plausible medium frequencies. The plot in Figure 2 below answers both of these questions.

Each vertical slice of this plot shows the cross-correlation[5] of quarterly log output and log mark-ups[6] when both are filtered by a high pass filter[7] with a cut-off given by the x-axis's value. (Shaded areas indicate positive correlations, with the darker area being significantly different from zero at 5%. The cross-hatched area is negative but insignificantly different from zero at 5%.) We see immediately

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[2] With 2^{20} replications, where in each case the regression (2.2) was run on the second half of a sample from a unit variance random walk, started at zero and twice the length of our data sample. This is broadly the methodology used by Cheung and Lai (1995) in their study of the finite sample properties of the ADF test with varying lag-order.

[3] Standard asymptotic critical values suggest a p-value close to 5%, but given the large number of lags and fairly small sample, it is unsurprising these are inaccurate.

[4] Using annual data, they also find that mark-ups are counter-cyclical at business cycle frequencies, though less so than at medium ones; however, their measure of the mark-up relies on many more questionable assumptions about utility and production functions than the Nekarda and Ramey one does. Additionally, Nekarda and Ramey find that the use of annual data always biases observed correlations towards counter-cyclicality.

[5] Fractional lags are evaluated via linear interpolation.

[6] Mark-ups are measured by the inverse labour share (following Nekarda and Ramey (2010)). Data is from NIPA, 1947:Q1-2011Q2.

[7] Implemented by setting the lower cut-off of a Christiano and Fitzgerald (2003) band-pass filter to two quarters.

that Nekarda and Ramey's finding that mark-ups are positively correlated with future output and negatively correlated with past output holds particularly strongly at medium frequencies. Additionally, tracing along the lead=0 line we see that mark-ups are pro-cyclical when the data is filtered by a high-pass filter with a cut-off less than 16.5 years, suggesting that the Comin and Gertler's medium-frequency counter-cyclicality result was indeed driven by behaviour below the main frequencies of medium-frequency cycles. Indeed, from the spectral decomposition[8] of output growth shown in Figure 3, we see that mark-ups are significantly pro-cyclical when filtered at any frequency corresponding to a peak in the spectral decomposition, including the medium-frequency peak at twelve years. This establishes that the relevant medium-frequency cycles feature pro-cyclical movements in mark-ups.

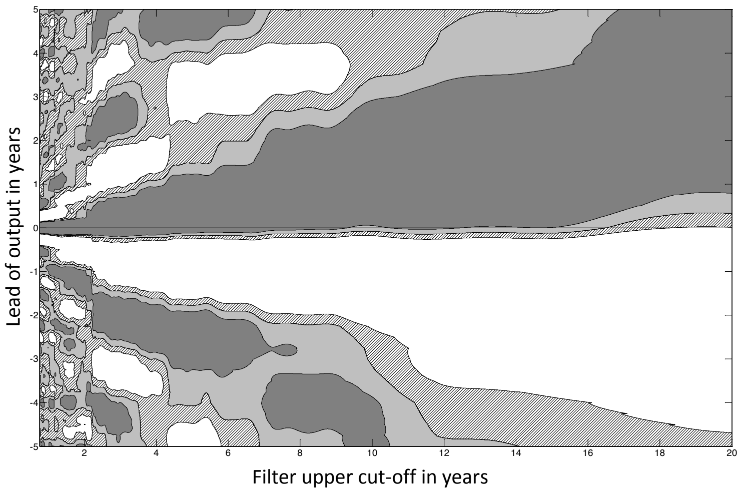

Figure 2: The cross correlation of US output and mark-ups, as a function of filter cut-off.
(Dark grey is a significantly positive correlation (at 5%), light grey is a positive but insignificant one, cross-hatched is a negative but insignificant one and white is a significantly negative one.)

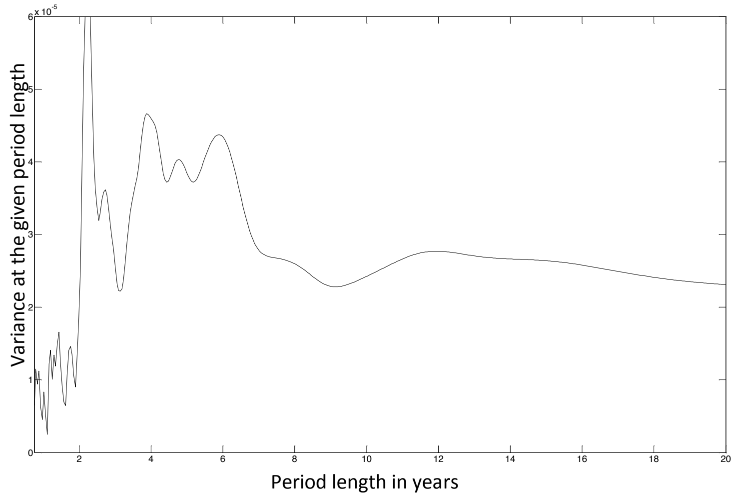

Figure 3: The spectral decomposition of US output growth.

[8] Constructed using an entirely parameter free method. We first filter the data with a Christiano and Fitzgerald (2003) band-pass filter with a lower cut-off of two quarters and a higher cut-off equal to the data length, in order to remove the influence of structural change and ensure stationarity. We then use the Hurvich (1985) cross-validation procedure to choose the bandwidth for the spectral-decomposition of the data, with his Stuetzle-derived estimator of the mean integrated squared error, the standard Blackman-Tukey lag-weights estimate, and the Quadratic Spectral Kernel recommended by Andrews (1991) amongst others.

Variable	**Spec. 1**	**Spec. 2**	**Spec. 3**	**Spec. 4**	**Spec. 5**
Constant	**-2.09811 (0.0300)**	**-2.14048 (0.0206)**	**-1.91285 (0.0180)**	**-2.70784 (0.0000)**	**-2.18372 (0.0009)**
English legal origin[9]	*-0.0506172 (0.8567)*				-0.448554 (0.0810)
French legal origin[9]	*-0.0557074 (0.8394)*				*-0.350747 (0.1653)*
German legal origin[9]	*-0.151587 (0.6364)*				*-0.325196 (0.3154)*
Log GDP per effective adult[10]	*0.0715242 (0.3620)*	*0.0707845 (0.3501)*			
GDP per effective adult growth[10]	*7.39306 (0.1647)*	*7.24517 (0.1606)*			
Socioeconomic Conditions (ICRG)[11]	**-0.224159 (0.0078)**	**-0.229358 (0.0044)**	**-0.170029 (0.0107)**		
Law and order (ICRG)[11]	-0.154013 (0.0856)	-0.150749 (0.0818)	-0.148729 (0.0856)		
Logit overall political risk (ICRG)[11,12]	**0.806772 (0.0013)**	**0.811630 (0.0006)**	**0.823980 (0.0003)**		
Index of patent duration, 1960[13]	**0.357215 (0.0336)**	**0.363052 (0.0242)**	**0.384211 (0.0131)**	**0.395486 (0.0044)**	**0.396382 (0.0060)**
Index of patent duration, 2005[13]	**1.79391 (0.0223)**	**1.79854 (0.0197)**	**1.88715 (0.0140)**	**1.66419 (0.0053)**	**1.50279 (0.0133)**
Observations	100	100	100	111	111
Specification test p-values[14]	*0.50, 0.31, 0.58*	*0.51, 0.20, 0.63*	*0.58,* 0.08, *0.74*	*0.31,* 0.06, **0.05**	*0.32, 0.12,* 0.06

Table 2: The impact of patent duration on the strength of medium frequency cycles.
Coefficients from assorted regression specifications. (P-values in brackets.) In all cases, the dependent variable is a logit transform of the proportion of GDP per effective adult growth variance that is at frequencies with periods greater than eight years[15].

[9] All countries which neither have English, French or German legal origins have Scandinavian legal origin in our sample. Data is from La Porta, Lopez-de-Silanes and Shleifer (2008).

[10] The intercept and the slope from running a regression of log GDP per effective adult on time. Data from the Penn World Tables (Heston, Summers, and Aten 2011), samples identical to those used to construct the dependent variable.

[11] International Country Risk Guide, The PRS Group. Data provided by the Nuffield College Data Library. Variables are means of annual data from 1986-2007 (the largest span available for all countries in the sample).

[12] This is the sum of the two components mentioned above, along with measures of government stability, the investment profile, internal/external conflict, corruption, the military/religion in Politics, ethnic tensions, democratic accountability and bureaucracy quality. The logit transform was taken after the mean. We ran regressions including all components separately and our results were almost identical (p-values on patent duration of 0.0192 and 0.0172 respectively), but to save space here we focus on the components found to be most relevant.

[13] Data kindly provided by Walter Park, updated from Ginarte and Park (1997).

[14] Respectively, a normality test (Doornik and Henrik Hansen 2008), the White heteroskedasticity test (White 1980) and the reset test with squares and cubes (Ramsey 1969).

[15] Data is from the Penn World Tables (Heston, Summers, and Aten 2011) and spans 1950-2009, though many countries have shorter samples. The shortest sample (of growth rates) is 23 years. We ran regressions including the sample length as a regressor, but it consistently came out insignificant. Medium frequency variance shares are constructed from spectral decompositions, following Levy and Dezhbakhsh (2003), where the spectral decomposition is performed using

2.3. GDP variance

Our model predicts that the length of patent-protection should be positively correlated with the observed size of medium-frequency cycles, at least for durations of patent-protection around those we observe in reality. In Table 2, we exploit cross-country variation in effective patent duration to demonstrate the presence of this correlation in the data, even when we control for GDP, legal origins and various measures of political stability and risk. (Full details of the data are given in footnotes to the table.) Patent duration in both 1960 and 2005 has a significantly positive effect (at 5%) on the strength of medium frequency cycles in all our five specifications, and only in in the specification with no controls is there marginal evidence of misspecification (at 5%). Concerns about endogeneity mean some restraint must be exerted in interpreting these results, but they are nonetheless suggestive of a role for patent protection in the mechanism generating medium frequency cycles in the data.

3. The model

Our base model is a standard quarterly real business cycle (RBC) model without capital, augmented by the addition of models of endogenous competition, research, appropriation and invention. The lack of capital means the underlying RBC model has no endogenous propagation mechanism, making clearer the contribution of our additions.

Our model has a continuum of narrow industries, each of which contains finitely many firms producing a unique product. The measure of industries is increased by the invention of new products, which start their life patent-protected. However, we assume that product inventors lack the necessary human capital to produce their product at scale themselves, and so they must licence out their patent to manufacturing firms. The duration of patent-protection is given by a geometric distribution, in line with Serrano's (2010) evidence on the large proportion of patents that are allowed to expire early, perhaps because they are challenged in court or perhaps because another new product is a close substitute. An earlier working-paper version of this model (Holden 2011) considered the fixed duration case, which is somewhat less tractable. Allowing for a distribution of protection lengths also allows us to give a broader interpretation to protection within our model. Even in the absence of patent protection, the combination of contractual agreements such as NDAs, and difficulties in reverse engineering, is likely to enable the inventor of a new product to extract rents for a period.

Our model of endogenous competition within each industry is derived from Jaimovich (2007). We chose the Jaimovich model as it is a small departure from the standard Dixit-Stiglitz (1977) set-up, and leads to some particularly neat expressions. Similar results could be attained with Cournot competition, or the Translog form advocated by Bilbiie, Ghironi, and Melitz (2012). One important departure from the Jaimovich model is that in our model entry decisions take place one period in

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the parameter free method outlined in footnote 8, with the initial filter set to accept period lengths between 2 and 59 years (the length of the largest samples).

advance. This is natural as we wish to model research as taking place after entry but before production.

Productivity within a firm is increased by performing research or appropriation. We regard process research as incremental, with regular small changes rather than the unpredictable jumps found in Schumpetarian models (Aghion and Howitt 1992; Wälde 2005; Phillips and Wrase 2006).

Throughout, we assume that only products are patentable,[16] and so by exerting effort firms are able to "appropriate" process innovations from other industries to aid in the production of their own product. This appropriation is costly since technologies for producing other products will not be directly applicable to producing a firm's own product. We assume that technology transfer within an industry is costless however, due to intra-industry labour flows and the fact that all firms in an industry are producing the same product. This is important for preserving the tractability of the model, as it means that without loss of generality we may think of all firms as just existing for two periods, in the first of which they enter and perform research, and in the second of which they produce.

The broad timing of our model is as follows. At the beginning of period t invention takes place, creating new industries. All holders of current patents (including these new inventors) then decide what level of licence fee to charge. Then, based on these licence fees and the level of overhead costs, firms choose whether to enter each industry. Next, firms perform appropriation, raising their next-period productivity towards that of the frontier, then research, further improving their productivity next period. In period $t+1$, they then produce using their newly improved production process. Meanwhile, a new batch of firms will be starting this cycle again.

We now give the detailed structure of the model.

3.1. Households

There is a unit mass of households, each of which contains N_t members in period t. The representative household maximises:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s N_{t+s} \Theta_{t+s} \left[\log \frac{C_{t+s}}{N_{t+s}} - \frac{\Phi_{t+s}}{1+\nu} \left(\frac{L_{t+s}^{\mathrm{S}}}{N_{t+s}} \right)^{1+\nu} \right]$$

where C_t is aggregate period t consumption, L_t^{S} is aggregate period t labour supply, Θ_t is a demand shock, Φ_t is a labour supply shock, β is the discount rate and ν is the inverse of the Frisch elasticity of labour supply to wages, subject to the aggregate budget constraint that $C_t + B_t = L_t^{\mathrm{S}} W_t +$

[16] This is at least broadly in line with the law in most developed countries: ideas that are not embedded in a product (in which category we include machines) generally have at most limited patentability. In the U.S., the most recent Supreme court decision found that the following was "a useful and important clue" to the patentability of processes (*Bilski v. Kappos*, 561 U.S. ___ (2010)): "a method claim is surely patentable subject matter if (1) it is tied to a particular machine or apparatus, or (2) it transforms a particular article into a different state or thing" (*In re Bilski*, 545 F.3d 943, 88 U.S.P.Q.2d 1385 (Fed. Cir. 2008)). This "machine or transformation" test was widely believed at the time to have ended the patentability of business processes (The Associated Press 2008), and this position was only slightly softened by *Bilski v. Kappos*.

$B_{t-1}R_{t-1} + \Pi_t$, where B_t is the aggregate number of (zero net supply) bonds bought by households in period t, W_t is the period t wage, R_{t-1} is the period t sale price of a (unit cost) bond bought in period $t-1$, and Π_t is the households' period t dividend income. In the following, where we refer to preference shocks we mean either a shock to Θ_t or a shock to Φ_t. However, both of these shocks may be interpreted as proxying for real changes in the economy that are independent of preferences. For example, Θ_t will capture changes in government consumption demand coming from wars, and Φ_t will pick up changes in marginal tax rates and in the degree of imperfect competition in labour markets.

Let $\beta\Xi_{t+1}$ be the households' period t stochastic discount factor, then the households' first order conditions imply:

$$\Xi_t = \frac{\Theta_t N_t C_{t-1}}{\Theta_{t-1} N_{t-1} C_t}, \qquad \Phi_t {L_t^S}^\nu = N_t^{1+\nu} \frac{W_t}{C_t}, \qquad \beta R_t \mathbb{E}_t[\Xi_{t+1}] = 1.$$

3.2. Aggregators

The consumption good is produced by a perfectly competitive industry from the aggregated output $Y_t(i)$ of each industry $i \in [0, I_{t-1}]$, using the following Dixit-Stiglitz-Ethier (Dixit and Stiglitz 1977; Ethier 1982) style technology:

$$Y_t = I_{t-1}^{-\lambda} \left[\int_0^{I_{t-1}} Y_t(i)^{\frac{1}{1+\lambda}} \, di \right]^{1+\lambda}$$

where $\frac{1+\lambda}{\lambda}$ is the elasticity of substitution between goods and where the exponent on the measure of industries (I_{t-1})[17] has been chosen to remove the preference for variety in consumption.[18]

Normalising the price of the aggregate consumption good to 1, and writing $P_t(i)$ for the price of the aggregate good from industry i in period t, we have that:

$$Y_t(i) = \frac{Y_t}{I_{t-1}} P_t(i)^{-\frac{1+\lambda}{\lambda}}, \qquad 1 = \left[\frac{1}{I_{t-1}} \int_0^{I_{t-1}} P_t(i)^{-\frac{1}{\lambda}} \, di \right]^{-\lambda}.$$

Similarly, each industry aggregate good $Y_t(i)$ is produced by a perfectly competitive industry from the intermediate goods $Y_t(i,j)$ for $j \in \{1, \dots, J_{t-1}(i)\}$,[19] using the technology:

$$Y_t(i) = J_{t-1}(i)^{-\eta\lambda} \left[\sum_{j=1}^{J_{t-1}(i)} Y_t(i,j)^{\frac{1}{1+\eta\lambda}} \right]^{1+\eta\lambda}$$

where $\eta \in (0,1)$ controls the degree of differentiation between firms, relative to that between industries.

[17] The $t-1$ subscript here reflects the fact that industries are invented one period before their product is available to consumers.

[18] Incorporating a preference for variety would not change the long-run stability of our model.

[19] Again, the $t-1$ subscript reflects the fact that firms enter one period before production.

This means that if $P_t(i,j)$ is the price of intermediate good j in industry i:

$$Y_t(i,j) = \frac{Y_t(i)}{J_{t-1}(i)}\left(\frac{P_t(i,j)}{P_t(i)}\right)^{-\frac{1+\eta\lambda}{\eta\lambda}}, \qquad P_t(i) = \left[\frac{1}{J_{t-1}(i)}\sum_{j=1}^{J_{t-1}(i)} P_t(i,j)^{-\frac{1}{\eta\lambda}}\right]^{-\eta\lambda}.$$

3.3. Intermediate firms

3.3.1. Pricing

Firm j in industry i has access to the linear production technology $Y_t(i,j) = A_t(i,j)L_t^{\mathrm{P}}(i,j)$ for production in period t. As in Jaimovich (2007), strategic profit maximisation then implies that in a symmetric equilibrium $P_t(i) = P_t(i,j) = \left(1+\mu_{t-1}(i)\right)\frac{W_t}{A_t(i,j)} = \left(1+\mu_{t-1}(i)\right)\frac{W_t}{A_t(i)}$, where:

$$\mu_t(i) \coloneqq \lambda\frac{\eta J_t(i)}{J_t(i)-(1-\eta)} \in (\eta\lambda, \lambda]$$

is the industry i mark-up in period $t+1$ and $A_t(i) = A_t(i,j)$ is the productivity shared by all firms in industry i in symmetric equilibrium.

From aggregating across industries we have that $W_t = \frac{A_t}{1+\mu_{t-1}}$ where:

$$\frac{1}{1+\mu_t} = \left[\frac{1}{I_t}\int_0^{I_t}\left[\frac{1}{1+\mu_t(i)}\right]^{\frac{1}{\lambda}} di\right]^{\lambda}$$

determines the aggregate mark-up μ_{t-1} and where:

$$A_t \coloneqq \frac{\left[\frac{1}{I_{t-1}}\int_0^{I_{t-1}}\left[\frac{A_t(i)}{1+\mu_{t-1}(i)}\right]^{\frac{1}{\lambda}} di\right]^{\lambda}}{\left[\frac{1}{I_{t-1}}\int_0^{I_{t-1}}\left[\frac{1}{1+\mu_{t-1}(i)}\right]^{\frac{1}{\lambda}} di\right]^{\lambda}}$$

is a measure of the aggregate productivity level.[20]

3.3.2. Sunk costs: rents, appropriation and research

Following Jaimovich (2007), we assume that the number of firms in an industry is pinned down by the zero profit condition that equates pre-production costs to production period revenues. Firms borrow in order to cover these upfront costs, which come from four sources.

Firstly, firms must pay a fixed operating cost L^{F} that covers things such as bureaucracy, human resources, facility maintenance, training, advertising, shop set-up and capital installation/creation. Asymptotically, the level of fixed costs will not matter, but including it here will help in our explanation of the importance of patent protection for long run growth.

[20] Due to the non-linear aggregation, it will not generically be the case that aggregate output is aggregate labour input times A_t. However, the aggregation chosen here is the unique one under which aggregate mark-ups are known one period in advance, as industry mark-ups are.

Secondly, if the product produced by industry i is currently patent-protected, then firms must pay a rent of $\mathcal{R}_t(i)$ units of the consumption good to the patent-holder for the right to produce in their industry. Since all other sunk costs are paid to labour, for convenience we define $L_t^{\mathcal{R}}(i) \coloneqq \frac{\mathcal{R}_t(i)}{W_t}$, i.e. the labour amount equivalent in cost to the rent.

Thirdly, firms will expand labour effort on appropriating the previous process innovations of the leading industry. We define the level of the leading technology within industry i by $A_t^*(i) \coloneqq \max_{j\in\{1,\dots,J_{t-1}(i)\}} A_t(i,j)$ and the level of the best technology anywhere by $A_t^* \coloneqq \sup_{i\in[0,I_{t-1}]} A_t^*(i)$. Due to free in-industry transfer, even without exerting any appropriation effort, firms in industry i may start their research from $A_t^*(i)$ in period t. By employing appropriation workers, a firm may raise this level towards A_t^*.

We write $A_t^{**}(i,j)$ for the base from which firm $j \in \{1,\dots,J_t(i)\}$ will start research in period t, and we assume that if firm j employs $L_t^{\mathrm{A}}(i,j)$ units of appropriation labour in period t then:

$$A_t^{**}(i,j) = \left[A_t^*(i)^\tau + \left(A_t^{*\tau} - A_t^*(i)^\tau\right)\frac{A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon L_t^{\mathrm{A}}(i,j)}{1 + A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon L_t^{\mathrm{A}}(i,j)}\right]^{\frac{1}{\tau}}, \tag{3.1}$$

where Υ is the productivity of appropriation labour, $\zeta^{\mathrm{A}} > 0$ controls the extent to which appropriation is getting harder over time (due, for example, to the increased complexity of later technologies) and where $\tau > 0$ controls whether the catch-up amount is a proportion of the technology difference in levels ($\tau = 1$), log-levels ($\tau = 0$) or anything in between or beyond. This specification captures the key idea that the further a firm is behind the frontier, the more productive will be appropriation. Allowing for appropriation (and research, and invention) to get harder over time is both realistic, and essential for the tractability of our model, since it will lead our model to have a finite dimensional state vector asymptotically, despite all the heterogeneity across industries.

Fourthly and finally, firms will employ labour in research. If firm $j \in \{1,\dots,J_t(i)\}$ employs $L_t^{\mathrm{R}}(i,j)$ units of research labour in period t, its productivity level in period $t+1$ will be given by:

$$A_{t+1}(i,j) = A_t^{**}(i,j)\left(1 + \gamma Z_{t+1}(i,j) A_t^{**}(i,j)^{-\zeta^{\mathrm{R}}}\Psi L_t^{\mathrm{R}}(i,j)\right)^{\frac{1}{\gamma}},$$

where Ψ is the productivity of research labour, $\zeta^{\mathrm{R}} > \zeta^{\mathrm{A}}$ controls the extent to which research is getting harder over time, $Z_{t+1}(i,j) > 0$ is a shock representing the luck component of research, and $\gamma > 0$ controls the "parallelizability" of research.[21] If $\gamma = 1$, research may be perfectly parallelized, so arbitrarily large quantities may be performed within a given period without loss of productivity, but if γ is large, then the productivity of research declines sharply as the firm attempts to pack more into one period. The restriction that $\zeta^{\mathrm{R}} > \zeta^{\mathrm{A}}$ means that the difficulty of research is increasing over time faster than the difficulty of appropriation. This is made because research is very much specific to the industry in which it is being conducted, whereas appropriation is a similar

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[21] Peretto (1999) also looks at research that drives incremental improvements in productivity, and chooses a similar specification. The particular one used here is inspired by Groth, Koch, and Steger (2009).

task across all industries attempting to appropriate the same technology, and hence is more likely to have been standardised, or to benefit from other positive spillovers.

In the following, we will assume that $Z_t(i,j) \coloneqq Z_t$ so that all firms in all industries receive the same "idea" shock. We make this assumption chiefly for simplicity, but it may be justified by appeal to common inputs to private research, such as university research output or the availability of new tools, or by appeal to in-period labour market movements carrying ideas with them. We will see in the following that allowing for industry-specific shocks has minimal impact on our results, providing there are at least correlations across industries (plausible if they are producing similar products). For concreteness, we assume that $Z_t \coloneqq \exp(\sigma_Z \epsilon_{Z,t})$, where $\sigma_Z > 0$ and $\epsilon_{Z,t} \sim \mathrm{NIIID}(0,1)$.

3.3.3. Research and appropriation effort decisions

Firms are owned by households and so they choose research and appropriation to maximize:

$$\beta \mathbb{E}_t \left[\Xi_{t+1} \left(P_t(i,j) - \frac{W_{t+1}}{A_{t+1}(i,j)} \right) Y_t(i,j) \right] - [L_t^{\mathrm{R}}(i,j) + L_t^{\mathrm{A}}(i,j) + L_t^{\mathcal{R}}(i) + L^{\mathrm{F}}] W_t$$

It may be shown that, for firms in frontier industries (those for which $A_t^*(i) = A_t^*$), if an equilibrium exists, then it is unique and symmetric within an industry; but we cannot rule out the possibility of asymmetric equilibria more generally.[22] However, since the coordination requirements of asymmetric equilibria render them somewhat implausible, we restrict ourselves to the unique equilibrium in which all firms within an industry choose the same levels of research and appropriation. Let us then define effective research performed by firms in industry i by $\mathcal{L}_t^{\mathrm{R}}(i) \coloneqq A_t^{**}(i)^{-\zeta^{\mathrm{R}}} \Psi L_t^{\mathrm{R}}(i,j)$ (valid for any $j \in \{1, \dots, J_{t-1}(i)\}$) and effective appropriation performed by firms in that industry by $\mathcal{L}_t^{\mathrm{A}}(i) \coloneqq A_t^*(i)^{-\zeta^{\mathrm{A}}} \Upsilon L_t^{\mathrm{A}}(i,j)$ (again, valid for any $j \in \{1, \dots, J_{t-1}(i)\}$).

[22] The equilibrium concept we use is that of pure-strategy subgame-perfect *local* Nash equilibria (SP*L*NE) (i.e. only profitable local deviations are ruled out). We have no reason to believe the equilibrium we find is not in fact a subgame-perfect Nash equilibria (SPNE). Indeed, if there is a pure-strategy symmetric SPNE then it will be identical to the unique pure-strategy symmetric SP*L*NE that we find. Furthermore, our numerical investigations suggest that at least in steady-state, at our calibrated parameters, the equilibrium we describe is indeed an SPNE. (Code available on request.) However, due to the analytic intractability of the second stage pricing game when productivities are asymmetric, we cannot guarantee that it remains an equilibrium away from the steady-state, or for other possible calibrations. However, SPLNE's are independently plausible since they only require firms to know the demand curve they face in the local vicinity of an equilibrium, which reduces the riskiness of the experimentation they must perform to find this demand curve (Bonanno 1988). It is arguable that the coordination required to sustain asymmetric equilibria and the computational demands of mixed strategy equilibria render either of these less plausible than our SPLNE.

Providing $\frac{1}{\mu_t(i)} < \min\{\gamma, \tau\}$, $\gamma > \zeta^{\mathrm{R}}$ and $\lambda < 1$ (for the second order conditions[23] and for uniqueness), combining the first order and free entry conditions then gives us that, in the limit as $\sigma_Z \to 0$:[24]

$$\mathcal{L}_t^{\mathrm{R}}(i) = \max\left\{0, \frac{d_t(i) A_t^{**}(i)^{-\zeta^{\mathrm{R}}} \Psi\left(L_t^{\mathrm{A}}(i,j) + L_t^{\mathcal{R}}(i) + L_t^{\mathrm{F}}\right) - \mu_t(i)}{\gamma \mu_t(i) - d_t(i)}\right\} \tag{3.2}$$

and:

$$\mathcal{L}_t^{\mathrm{A}}(i) = \max\left\{0, \mathscr{f}_t(i) + \sqrt{\max\{0, \mathscr{f}_t(i)^2 + \mathscr{g}_t(i)\}}\right\}, \tag{3.3}$$

where $d_t(i) \in (0,1)$[25] is small when firm behaviour is highly distorted by firms' incentives to deviate from choosing the same price as the other firms in their industry, off the equilibrium path (so $d_t(i) \to 1$ as $J_t(i) \to \infty$), and $\mathscr{f}_t(i)$ and $\mathscr{g}_t(i)$ are increasing in an industry's distance from the frontier,[26] as the further behind a firm is, the greater are the returns to appropriation.

Equations (3.2) and (3.3) mean that research and appropriation levels are increasing in the other sunk costs a firm must pay prior to production, but decreasing in mark-ups. They also mean that the strategic distortions caused by there being a small number of firms within an industry tend to reduce research and appropriation levels. Other sunk costs matter for research levels because when other sunk costs are high, entry into the industry is lower, meaning that each firm receives a greater slice of production-period profits, and so has correspondingly amplified research incentives.

Why mark-up increases decrease research incentives is clearest when those mark-up increases are driven by exogenous decreases in the elasticity of substitution. When products are close substitutes, then by performing research (and cutting its price) a firm may significantly expand its market-share, something that will not happen when the firm's good is a poor substitute for its rivals. When $d_t(i) \approx 1$ (i.e. there are a lot of firms in the industry) firms act as if they faced an exogenous elasticity of substitution $\frac{1+\mu_t(i)}{\mu_t(i)}$, and so when mark-ups are high they will want to perform little research. When $d_t(i)$ is small (i.e. there are only a few firms) then firms' behaviour is distorted by strategic considerations. Each firm realises that if they perform extra research today then their competitors will accept lower mark-ups the next period. This reduces the extent to which research allows market-share expansion, depressing research incentives.

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[23] The second order condition for research may be derived most readily by noting that when $d_t(i) \to 1$, (i.e. $J_t(i) \to \infty$) the first order condition for research is identical to the one that would have been derived had there been a continuum of firms in each industry with exogenous elasticity of substitution $\frac{1+\mu_t(i)}{\mu_t(i)}$. That it holds more generally follows by continuity. Since $A_t^{**}(i,j)$ is bounded above, no matter how much appropriation is performed the highest solution of the appropriation first order condition must be at least a local maximum.

[24] The first order and zero profit conditions are reported in an appendix, section 7.1, where we also derive these solutions. We do not assume $\sigma_Z = 0$ when simulating, but it leads here to expressions that are easier to interpret.

[25] Defined in the appendix, section 7.1.

[26] $\mathscr{f}_t(i) \coloneqq \frac{1}{2}\left[1 + \frac{d_t(i)}{\tau \mu_t(i)} \frac{1+(\gamma-\zeta^{\mathrm{R}})\mathcal{L}_t^{\mathrm{R}}(i)}{1+\gamma \mathcal{L}_t^{\mathrm{R}}(i)}\right]\left[1 - \left(\frac{A_t^*(i)}{A_t^*}\right)^{\tau}\right] - 1$, $\mathscr{g}_t(i) \coloneqq \frac{d_t(i)}{\tau \mu_t(i)} \frac{1+(\gamma-\zeta^{\mathrm{R}})\mathcal{L}_t^{\mathrm{R}}(i)}{1+\gamma \mathcal{L}_t^{\mathrm{R}}(i)} A_t^*(i)^{-\zeta^{\mathrm{A}}} \Upsilon[L_t^{\mathrm{R}}(i) + L_t^{\mathcal{R}}(i) + L^{\mathrm{F}}]\left[1 - \left(\frac{A_t^*(i)}{A_t^*}\right)^{\tau}\right] - \left(\frac{A_t^*(i)}{A_t^*}\right)^{\tau}$.

Perhaps counter-intuitively, the minimum value of $d_t(i)$ occurs when there is a strictly positive number of firms in the industry. It is certainly true that if there is a single firm in an industry, then, as you would expect, very little research will be performed (because the firm's only incentive to cut prices comes from competition from other industries, competition which is very weak, since those industries are producing poor substitutes to its own good). However, this drop in research incentives is working entirely through the mark-up channel, and so in fact we also have that $d_t(i) \to 1$ as $J_t(i) \to 1$. One intuition for this is that there can be no strategic behaviour when there is only a single firm.

The key thing to note about (3.2) and (3.3) is that research and appropriation are independent of the level of demand, except insomuch as demand affects mark-ups and the level of the strategic distortion. This is because when demand is high there is greater entry, so each firm still faces roughly the same demand. This is essential for removing the short-run scale effect.

In industries that are no longer patent-protected, rents will be zero (i.e. $L_t^{\mathcal{R}}(i) \equiv 0$). Since research is getting harder at a faster rate than appropriation ($\zeta^{\mathrm{R}} > \zeta^{\mathrm{A}}$), at least asymptotically, no research will be performed in these industries. This is because $A_t^{**}(i)^{-\zeta^{\mathrm{R}}}\Psi[L_t^{\mathrm{A}}(i) + L^{\mathrm{F}}] - \mu_t(i)$ is asymptotically negative since $\mu_t(i) \in (\eta\lambda, \lambda]$. For growth to continue forever in the absence of patent protection, we would require that the overhead cost (L^{F}) was growing over time at exactly the right rate to offset the increasing difficulty of research. This does not seem particularly plausible. However, it will turn out that optimal patent rents grow at exactly this rate, so with patent protection we will be able to sustain long run growth even when overhead costs are asymptotically dominated by the costs of research. In the presence of sufficiently-severe financial frictions of the "pledgibility constraint" form (Hart and Moore 1994), it may be shown that long run growth is sustainable even without patent protection. We leave the details of this for future work.

Appropriation is performed in an industry if and only if $g_t(i) > 0$, which, for a non-patent protected industry no longer performing research, is true if and only if:

$$\frac{A_t^*(i)}{A_t^*} < \left(\frac{A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon L^{\mathrm{F}}}{A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon L^{\mathrm{F}} + \tau\frac{\mu_t(i)}{d_t(i)}}\right)^{\frac{1}{\tau}}.$$

The left hand side of this equation is the relative productivity of the industry compared to the frontier. The right hand side of this equation will be shrinking over time at roughly $\frac{\zeta^{\mathrm{A}}}{\tau}$ times the growth rate of the frontier, meaning the no-appropriation cut-off point is also declining over time. Indeed, we show in an appendix, section 7.2, that asymptotically the relative productivity of non-protected firms shrinks at $\frac{\zeta^{\mathrm{A}}}{\tau}\left[1 + \frac{\zeta^{\mathrm{A}}}{\tau}\right]^{-1}$ times the growth rate of the frontier. This is plausible since productivity differences across industries have been steadily increasing over time,[27] and is

[27] Some indirect evidence for this is provided by the increase in wage inequality, documented in e.g. Autor, Katz, and Kearney (2008). Further evidence is provided by the much higher productivity growth rates experienced in manufacturing, compared to those in services (mostly unpatented and unpatentable), documented in e.g. Duarte and Restuccia (2009).

important for the tractability of our model since it enables us to focus on the asymptotic case in which non-protected firms never perform appropriation. It is also in line with the long delays in the diffusion of technology found by Mansfield (1993) amongst others.

3.4. Inventors

Each new industry is controlled by an inventor who owns the patent rights to the product the industry produces. Until the inventor's product goes on sale, the patent holder can successfully protect their revenue stream through contractual arrangements, such as non-disclosure agreements. This means that even in the absence of patent-protection a patent holder will receive one period of revenues. In this period, and each subsequent one for which they have a patent, the inventor optimally chooses the rent $\mathcal{R}_t(i)$ (or equivalently $L_t^{\mathcal{R}}(i)$) to charge all the firms that wish to produce their product. We are supposing inventors lack the necessary human capital to produce their product at scale themselves.

The inventor of a new product has a probability of $1-q$ of being granted a patent to enable them to extract rents for a second period. After this, if they have a patent at t, then they face a constant probability of $1-q$ of having a patent at $t+1$.

The reader should have a firm such as Apple in mind when thinking about these inventors. Apple has no manufacturing plants and instead maintains its profits by product innovation and tough bargaining with suppliers.

3.4.1. Optimal rent decisions

Inventor's businesses are also owned by households; hence, an inventor's problem is to choose $L_{t+s}^{\mathcal{R}}(i)$ for $s \in \mathbb{N}$ to maximise their expected profits, which are given by:

$$\pi_t \coloneqq \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s (1-q)^s \left[\prod_{k=1}^{s} \Xi_{t+k}\right] L_{t+s}^{\mathcal{R}}(i) W_{t+s} J_{t+s}(i),$$

subject to an enforceability constraint on rents. If the rents charged by a patent-holder go too high, a firm is likely to ignore them completely in the hope that either they will be lucky, and escape having their profits confiscated from them by the courts (since proving patent infringement is often difficult), or that the courts will award damages less than the licence fee. This is plausible since the relevant U.S. statute states that "*upon finding for the claimant the court shall award the claimant damages adequate to compensate for the infringement but in no event less than a reasonable royalty for the use made of the invention by the infringer, together with interest and costs as fixed by the court*".[28,29] The established legal definition of a "reasonable royalty" is set at the outcome of

[28] 35 U.S.C. § 284 Damages.

[29] The reasonable royalty condition is indeed the relevant one for us since our assumption that the patent-holder lacks the necessary human capital to produce at scale themself means it would be legally debatable if they had truly "lost profits" following an infringement (Pincus 1991).

a hypothetical bargaining process that took place immediately before production,[30] so patent-holders may just as well undertake precisely this bargaining process before production begins.[31]

This leads patent-holders to set:

$$L_t^{\mathcal{R}}(i) = \frac{1-\wp}{\wp}\left[L_t^{\mathrm{R}}(i) + L_t^{\mathrm{A}}(i) + L^{\mathrm{F}}\right], \tag{3.4}$$

at least for sufficiently large t, where $\wp \in (0,1)$ is the bargaining power of the firm, in the sense of the generalized Nash bargaining solution. The simple form of this expression comes from the fact that a firm's production period revenues (which is what is being bargained over) are precisely equal to the costs they face prior to production, thanks to the free entry condition. A full description of the legally motivated bargaining process is contained in an appendix, section 7.3, along with a discussion of some technical complications pertaining to off equilibrium play.

From combining (3.2) and (3.4) then, at least for sufficiently large t, in the limit as as $\sigma_Z \to 0$, we have that:

$$\mathcal{L}_t^{\mathrm{R}}(i) = \frac{\wp\mu_t(i) - d_t(i)A_t^{**}(i)^{-\zeta^{\mathrm{R}}}\Psi\left(L_t^{\mathrm{A}}(i) + L^{\mathrm{F}}\right)}{d_t(i) - \gamma\wp\mu_t(i)}.$$

For there to be growth in the long run then, we require $d_t(i) > \gamma\wp\mu_t(i)$, which together with the second order and appropriation uniqueness conditions means that it must at least be true that $\wp\gamma < \frac{1}{\mu_t(i)} < \min\{\gamma, \tau\}$.[32] We see that, once optimal rents are allowed for, research is no longer decreasing in mark-ups within an industry, at least for firms at the frontier. Mathematically, this is because the patent-holder sets rents as such a steeply increasing function of research levels. More intuitively, you may think of the patent-holder as effectively controlling how much research is performed by firms in their industry, and as taking most of the rewards from this research. It is then unsurprising that we reach these Schumpeterian conclusions.

The empirical evidence (Scott 1984; Richard C. Levin, Cohen, and Mowery 1985; Aghion et al. 2005; Tingvall and Poldahl 2006) suggests that the cross-industry relationship between competition and research takes the form of an inverted-U. Based on the fact that strategic distortions are maximised (i.e. $d_t(i)$ is minimised) when there is a small finite number of firms, one might perhaps hope that

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[30] *Georgia-Pacific,* 318 F. Supp. at 1120 (S.D.N.Y. 1970), *modified on other grounds,* 446 F.2d 295 (2d Cir.), *cert. denied,* 404 U.S. 870 (1971), cited in Pincus (1991), defines a reasonable royalty as "*the amount that a licensor (such as the patentee) and a licensee (such as the infringer) would have agreed upon (at the time the infringement began) if both had been reasonably and voluntarily trying to reach an agreement; that is, the amount which a prudent licensee—who desired, as a business proposition, to obtain the licence to manufacture and sell a particular article embodying the patented invention—would have been willing to pay as a royalty and yet be able to make a reasonable profit and which amount would have been acceptable by a prudent patentee who was willing to grant a licence.*"

[31] In any case, if we allow for idiosyncratic "idea shocks" firms will wish to delay bargaining until this point anyway, since with a bad shock they will be less inclined to accept high rents. Patent-holders also wish to delay till this point because the more sunk costs the firms have already expended before bargaining begins, the greater the size of the "pie" they are bargaining over.

[32] If the number of firms in protected industries is growing over time then $d_t(i) \to 1$, so asymptotically these conditions are equivalent.

this holds in our model too. Unfortunately, the maximum of $\frac{\mu_t(i)}{d_t(i)}$ (and hence of research) as a function of $J_t(i)$ may be shown to always occur at some $J_t(i) < 1$. While fractional entry may be a legitimate way of modelling niche products that are never fully commercialised, we prefer to explain the inverted-U in the data with reference to the cross-sectional distribution of industries. New industries will start with a production process behind that of the frontier, and thus firms in them will wish to perform large amounts of appropriation and relatively small amounts of research, since appropriation is a cheaper means of increasing productivity for a firm behind the frontier. In the presence of a luck component to appropriation (not included above, for simplicity) this leads new industries to have the highest degree of productivity dispersion, as older industries remain close to the frontier. As a result of this high productivity dispersion, there will be firms in new industries setting both very high, and very low mark-ups, which, combined with the fact they are performing less research than more mature patent-protected industries, would generate an inverted-U.

3.4.2. Invention and long-run stability

We consider invention as a costly process undertaken by inventors until the expected profits from inventing a new product fall to zero. New products appear at the end of the product spectrum. Additionally, once a product has been invented, it cannot be "un-invented". Therefore, the product index i always refers to the same product, once it has been invented.

There is, however, no reason to think that newly invented products will start off with a competitive production process. A newly invented product may be thought of as akin to a prototype: yes, identical prototypes could be produced by the same method, but doing this is highly unlikely to be commercially viable. Instead, there will be rapid investment in improving the product's production process until it may be produced as efficiently as its rivals can be. In our model, this investment in the production process is performed not by the inventor but by the manufacturers. Prototyping technology has certainly improved over time;[33] in light of this, we assume that a new product i is invented with a production process of level $A_t^*(i) = E_t A_t^*$, where $E_t \in (0,1)$ controls initial relative productivity.

Just as we expect process research to be getting harder over time, as all the obvious process innovations have already been discovered, so too we may expect product invention to be getting harder over time, as all the obvious products have already been invented. In addition, the necessity of actually finding a way to produce a prototype will result in the cost of product invention also being increasing in $A_t^*(i)$, the initial productivity level of the process for producing the new product. As a result of these considerations, we assume that the labour cost is given by $\mathcal{L}_t^{\mathrm{I}} I_{t-1}^{\chi} A_t^*(i)^{\zeta^{\mathrm{I}}}$, where $\mathcal{L}^{\mathrm{I}} > 0$ determines the difficulty of invention and where $\chi \in \mathbb{R}$ and $\zeta^{\mathrm{I}} > 0$ control the rate at which inventing a new product gets more difficult because of, respectively, an increased number of existing products or an increased level of productivity.

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[33] Examples of recent technologies that have raised the efficiency of prototype production include 3D printing and computer scripting languages such as Python.

We are assuming there is free entry of new inventions, so the marginal entrant must not make a positive profit from entering. That is, $I_t \geq I_{t-1}$ must be as small as possible such that:

$$\mathcal{L}^{\mathrm{I}} I_{t-1}^{\chi} A_t^*(i)^{\zeta^{\mathrm{I}}} W_t \geq \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s (1-q)^s \left[\prod_{k=1}^{s} \Xi_{t+k}\right] L_{t+s}^{\mathcal{R}}(I_t) W_{t+s} J_{t+s}(I_t).$$

If, after a shock, invention can satisfy this equation with equality without the growth rate of the stock of products turning negative, then the measure of firms will not have to adjust significantly. However, if the $I_t \geq I_{t-1}$ constraint binds, then the measure of firms will have to adjust instead, meaning there may be an asymmetry in the response of mark-ups to certain shocks.

It may be shown that, in the long run, $g_I = \frac{1}{1+\chi}(g_N - \zeta^{\mathrm{I}} g_{A^*})$ (where g_V is the asymptotic growth rate of the variable V_t). Therefore, if $\chi = \zeta^{\mathrm{I}} = 0$ the stock of products will grow at exactly the same rate as population, and away from this special case it will be growing more slowly. If invention were to stop asymptotically, eventually there would be no protected industries, and hence no productivity growth. Therefore, for long-run growth, we either require that $g_N \geq \zeta^{\mathrm{I}} g_{A^*}$ (which will hold providing research is getting more difficult sufficiently slowly, as long as population growth continues), or that there is sufficiently fast depreciation of the stock of products.[34] Even without product depreciation, productivity growth may be sustained indefinitely in the presence of a declining population if the government offers infinitely renewable patent-protection.

The existence of a solution for our model, at all time periods, requires the number of firms in a protected industry to be bounded below asymptotically. The previous result on the growth rate of the stock of products implies it is sufficient that $\left(\zeta^{\mathrm{R}} - \frac{\zeta^{\mathrm{I}}}{1+\chi}\right) g_{A^*} \leq \frac{\chi}{1+\chi} g_N$ for this to hold. This inequality is guaranteed to be satisfied providing $\zeta^{\mathrm{R}} - \frac{\zeta^{\mathrm{I}}}{1+\chi}$ is sufficiently small. To do this while also ensuring that $g_I > 0$ requires that $\max\left\{\zeta^{\mathrm{I}}, \zeta^{\mathrm{R}} + \frac{1}{\chi}(\zeta^{\mathrm{R}} - \zeta^{\mathrm{I}})\right\} < \frac{g_N}{g_{A^*}}$, which will hold for a positive measure of parameter values providing population growth is strictly positive.[35]

Assuming this condition holds, we may show[36] that providing the growth rate of the productivity of newly invented products is sufficiently close to the frontier growth rate (i.e. E_t does not decline too

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[34] Bilbiie, Ghironi, and Melitz (2012) include such product depreciation in their model. We have chosen not to model it here.

[35] More generally, when population is stable, providing there is sufficiently fast (proportional) depreciation of the stock of products, we just require that $\zeta^{\mathrm{R}} < \frac{\zeta^{\mathrm{I}}}{1+\chi}$.

[36] Suppose $(i_t)_{t=0}^{\infty}$ is a sequence of industries, all protected at t, whose productivity grows at rate $\tilde{g} \leq g_{A^*}$ asymptotically. We conjecture that $\lim_{t\to\infty} A_t^{**}(i_t)^{-\zeta^{\mathrm{R}}} \Psi L_t^{\mathrm{A}}(i_t) = 0$ and verify. This assumption implies that effective research is asymptotically bounded, since mark-ups are. Hence from (3.3), since $\zeta^{\mathrm{R}} > \zeta^{\mathrm{A}}$, effective appropriation is growing at a rate in the interval $\left(\frac{\zeta^{\mathrm{R}}\tilde{g} - \zeta^{\mathrm{A}}\tilde{g}}{2}, \frac{\zeta^{\mathrm{R}} g_{A^*} - \zeta^{\mathrm{A}}\tilde{g}}{2}\right) \subseteq (0, \infty)$. Therefore $A_t^{**}(i)^{-\zeta^{\mathrm{R}}} \Psi L_t^{\mathrm{A}}(i_t)$ is growing at a rate in the interval $\left(-\zeta^{\mathrm{R}} g_{A^*} + \zeta^{\mathrm{A}}\tilde{g} + \frac{\zeta^{\mathrm{R}}\tilde{g} - \zeta^{\mathrm{A}}\tilde{g}}{2}, -\zeta^{\mathrm{R}}\tilde{g} + \zeta^{\mathrm{A}}\tilde{g} + \frac{\zeta^{\mathrm{R}} g_{A^*} - \zeta^{\mathrm{A}}\tilde{g}}{2}\right)$. For our claim to be verified we then just need that $\frac{\zeta^{\mathrm{R}}}{2\zeta^{\mathrm{R}} - \zeta^{\mathrm{A}}} g_{A^*} < \tilde{g}$, which certainly holds when $\tilde{g} = g_{A^*}$ as $\zeta^{\mathrm{R}} > \zeta^{\mathrm{A}}$.

quickly[37]), asymptotically catch-up to the frontier is instantaneous in protected industries, and the frontier growth rate is stationary. This instantaneous catch-up to the frontier means that, had we allowed for industry-specific shocks, all other protected industries would “inherit” the best industry shock, the period after it arrived. This justifies our focus on aggregate “idea” shocks. Additionally, instantaneous catch-up to the frontier means that providing there is population growth or product depreciation, long-run growth may be sustained even in the absence of patent-protection (i.e. when $q = 0$), as the one period in which the inventor has a first mover advantage is sufficient for their industry to surpass the existing frontier.

If the number of firms in protected industries were asymptotically infinite, then our simulations would tell us nothing about the consequences of the variations in this number that we might see non-asymptotically. Therefore, it will be helpful if it is additionally the case that this number is asymptotically finite. To guarantee this will, unfortunately, require a knife-edge assumption, namely that $\left(\zeta^{\mathrm{R}} - \frac{\zeta^{\mathrm{I}}}{1+\chi}\right) g_{A^*} = \frac{\chi}{1+\chi} g_N$. To satisfy this without restricting population growth rates means $\chi = 0$ (so invention is not made more difficult by the number of existing products) and $\zeta^{\mathrm{R}} = \zeta^{\mathrm{I}}$ (so prototype production is increasing in difficulty at the same rate as research). The former assumption may be justified by noting that many situations in which invention is apparently getting harder over time because of congestion effects may equally well by explained by production-process-difficulty effects. The latter assumption is immediately plausible, since both parameters are measuring the complexity of working with a given production process. However, unlike with knife-edge growth models whereby relatively slight departures from the stable parameter values results in growth that could not possibly explain our observed stable exponential growth, here, away from the knife-edge case we will have slowly decreasing mark-ups, consistent with Ellis’s (2006) evidence of a persistent decline in UK whole economy mark-ups over the last thirty years and Kim’s (2010) evidence of non-stationarity in mark-ups.

We assume then that $0 = \chi < \zeta^{\mathrm{A}} < \zeta^{\mathrm{R}} = \zeta^{\mathrm{I}}$. Since asymptotically non-protected industries perform no research or appropriation under these assumptions, their entry cost to post-entry industry profits ratio is tending to zero, meaning their number of firms will tend to infinity as $t \to \infty$. This is in line with our motivating intuition that excess entry in non-protected industries kills research and appropriation incentives.

3.5. Simulations

With $0 = \chi < \zeta^{\mathrm{A}} < \zeta^{\mathrm{R}} = \zeta^{\mathrm{I}}$, as $t \to \infty$ the behaviour our model tends towards stationarity in the key variables. It is this asymptotically stationary model that we simulate. For convenience we define $\zeta \coloneqq \zeta^{\mathrm{R}} = \zeta^{\mathrm{I}}$. The full set of equations of the de-trended model are given in an appendix, section 7.4. The definition of equilibrium here is entirely standard.

When $\lambda = \nu = \gamma = 1$, it may be shown analytically that the equations determining the model’s steady-state have at most two solutions with more than one firm in each industry. However, only one of these two solutions exists for large values of $\mathcal{L}^{\mathrm{I}}$, i.e. when invention is costly. Since we think

[37] As $\zeta^{\mathrm{A}} \to 0$ it is sufficient that E_t is declining at less than half the rate that A_t^* is growing.

that in reality invention is getting harder over time due to congestion effects (i.e. $\chi > 0$), any solution that only exists for small values of $\mathcal{L}^{\mathrm{I}}$ is non-feasible. Our numerical investigations suggest that the model always has at most these two equilibria, and that always at most one of them exists for large values of $\mathcal{L}^{\mathrm{I}}$.[38] However, since the existence of multiple-equilibria is indicative that linear approximations may be inaccurate in that region, rather than just picking the solution that exists for arbitrarily large $\mathcal{L}^{\mathrm{I}}$, we instead restrict the parameter space to regions in which there is a unique solution. This ensures that the value of $\mathcal{L}^{\mathrm{I}}$ we use is indeed large, in this sense.

Since $\Psi E^{\zeta} \mathcal{L}_t^{\mathrm{I}}$ always occurs as a group, without loss of generality we may make the normalization $\Psi \coloneqq E \coloneqq 1$. We fix all of the model's other parameters, except $\mathcal{L}^{\mathrm{I}}$, to the values estimated for our extended model in section 4. $\mathcal{L}^{\mathrm{I}}$ is set such that the number of firms in patent-protected industries in this model is equal to that of the estimated extended model. The full parameterisation is reported in an appendix (section 7.7). At the chosen parameters, the model has a unique solution, which will exist for arbitrarily high values of $\mathcal{L}^{\mathrm{I}}$.

3.5.1. Simulation method

We take a first-order perturbation approximation around the non-stochastic steady state, perturbing in the variance of shocks, and solve for the rational expectations solution of the linearized model.[39] As we have previously mentioned, the zero lower bound on net product creation (i.e. on $g_{I,t}$) means there may be an asymmetric response to sufficiently large shocks, but in fact we do not find that the bound is hit with shocks of the magnitudes we consider.

3.5.2. Impulse responses

In Figure 4 we present the impulse responses that result from 0.1% IID (hence non-persistent) shocks to "ideas" (Z_t), labour supply (Φ_t), demand (Θ_t) and population growth ($G_{N,t}$). Each graph is given in terms of per cent deviations from the value the variable would have taken had the shock never arrived, and the horizontal axis shows time in years, though this remains a quarterly model. Each shock is in a different column, and the key response variables are in rows.

The principle mechanism of our paper is shown most clearly by the population growth rate shock, shown in the final column. (We do not wish to advance population shocks as a key driver of business cycles though, since real rigidities will significantly reduce their impact.) Following a permanent increase in population, demand is permanently higher, so, in the long run, the number of industries must grow to balance this out. Given sufficiently inelastic labour supply, this long-run increase in the measure of industries requires a short-run substitution of labour from production to invention, pushing down consumption and pushing up wages, and so moderating the rate at which invention will grow. Consequently, in the short run some of the additional demand is absorbed by fluctuations in the number of firms in each industry. Without this additional margin of adjustment,

[38] It may be shown analytically that the complete model may always be solved by solving a single nonlinear equation, which was always concave for all the parameters we examined.

[39] This was performed using Dynare (Adjemian et al. 2011).

this shock would have led to a large increase in average firm sizes, with a consequent increase in the frontier growth rate and counter-factually large unit root in output.

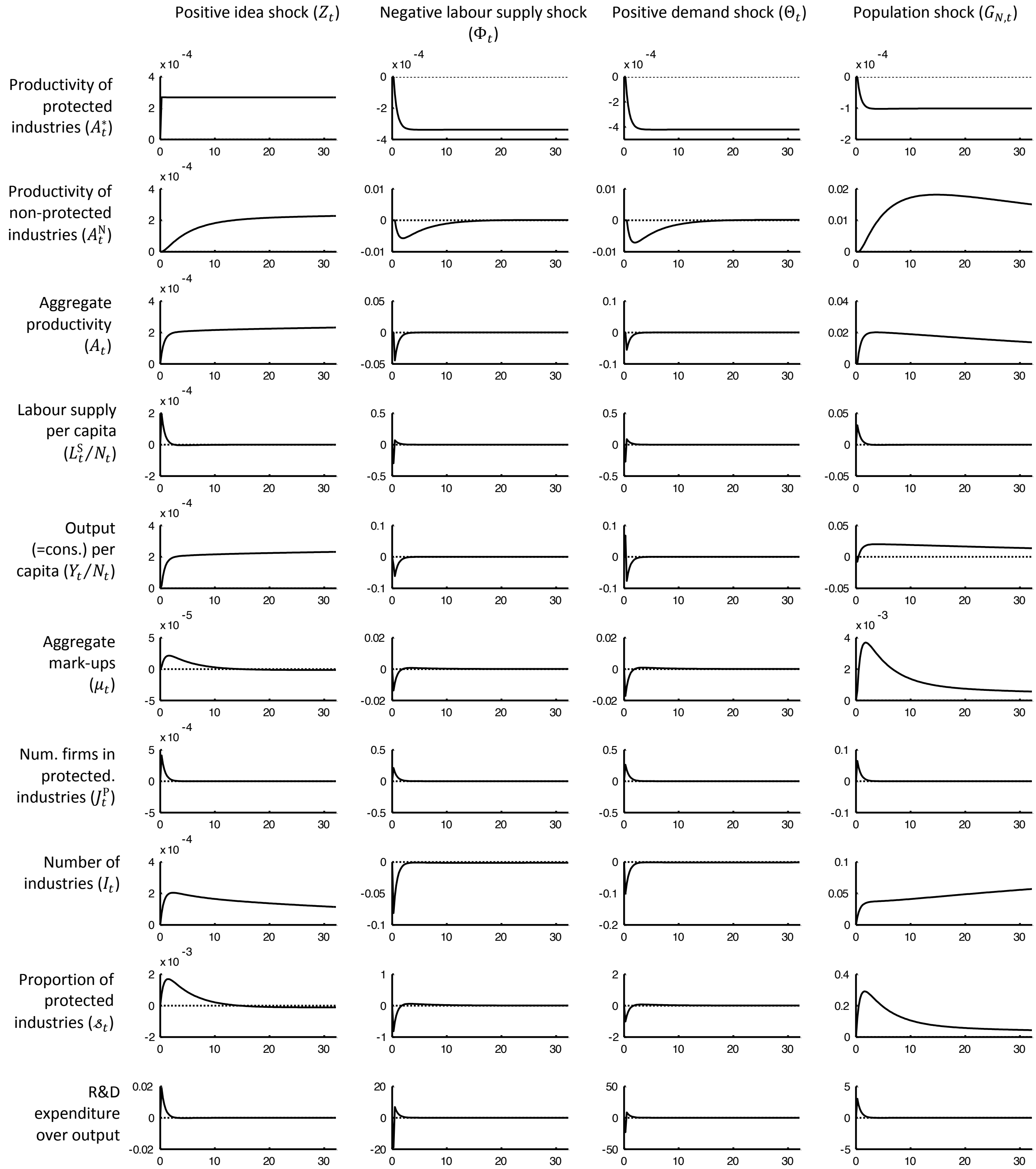

Figure 4: Impulse responses from the core model.
(Vertical axes are in percent, horizontal axes are in years.)

Despite the tiny movement in frontier productivity (less than 0.01%), there is still however a substantial movement in aggregate productivity in the medium-term. Following the shock, more new products are being invented each period, meaning that a greater proportion of industries are relatively new, and so a greater proportion are patent-protected. But because patent-protected industries have such strong incentives to catch-up to the frontier, patent-protected industries are more productive than non-protected ones, so an increase in the proportion of industries that are patent protected means an increase in aggregate productivity. Patent-protected industries also have higher mark-ups due to the cost of paying licence fees, and so we also see a rise in mark-ups over the medium-term. It is this mechanism that generates medium-frequency pro-cyclical mark-ups in our model.[40]

This mechanism also underlies our model's response to the other shocks we consider. Following a negative labour supply shock, invention is temporarily more expensive, meaning fewer new industries and consequently lower productivity and mark-ups.[41] Following a demand shock labour is transferred away from research and invention towards production, in order to satisfy the temporary higher demand. This drop in invention on impact means that demand shocks actually reduce output in subsequent periods. This is no longer the case when the shock has some persistence, or when there are real rigidities.

An idea shock permanently increases the productivity of patent-protected industries. Over time, these industries fall out of patent-protection, carrying their higher productivity with them, and thus increasing the average productivity of non-protected firms too. Consequently, aggregate productivity slowly rises towards its permanently higher long run level. However, since the magnitude of the original shock was very small, this will not result in a large unit root in output. Following the shock, patent-protected industries are relatively more productive than normal, and so they are also relatively more profitable. This means patent holders can extract higher rents, and so we see an increase in invention with a corresponding increase in mark-ups over the medium-term.

4. Extended model and empirical tests

In order to compare our model to the data seriously, we incorporate habits, capital, and imperfect competition in labour markets. We allow for the possibility of stochastic movements in the key parameters $\mathcal{L}^{\mathrm{I}}$, γ and η,[42] (though it turns out that the data favours constant values for these parameters), and we specify an $\mathrm{AR}(1)$ form for these and all other shocks, with the exception of Z_t, the true technology shock which remains uncorrelated across time. The data will be allowed to choose which, if any, of these shocks might be important drivers of business cycles, at high, or medium frequencies.

Additionally, we include intermediate goods as a factor of production, which may be necessary in order to reconcile the low mark-ups found in micro-evidence with the higher mark-ups implied by the inverse labour share. The presence of intermediates in production will amplify shocks in our economy, as it implies that an increase in the proportion of industries that are patent-protected

[40] Pavlov and Weder (2012) also develop a business cycle model capable of generating pro-cyclical mark-ups, via the changing importance of different types of *buyers* over the business cycle. The properties of these buyers are exogenous in their model however, whereas the properties of the different types of *sellers* that drive our results are endogenous.

[41] Were the number of firms per protected industry to absorb the cost-cut instead, then next-period mark-ups would rise and so future wages would fall. However, an expected fall in wages will increase invention today, since inventor returns are increasing in the expected future wage. Hence, invention must fall in the period of the shock.

[42] These parameters are assumed to be known before the entry decision at t, for production in period $t+1$.

means intermediate inputs are cheaper for non-protected industries, increasing their output too. To potentially dampen our model's overly powerful amplification mechanism, we include some spill-overs from frontier productivity growth; these mean that the variance of TFP may be less than that of A_t.

We also allow for sticky nominal wages in line with the micro-evidence of Barattieri, Basu, and Gottschalk (2010), and to enable us to make preliminary remarks about the possible medium-term impact of monetary policy. In all of the impulse responses presented below though, we will show the model's performance both with and without this feature. We do not include sticky prices for several reasons. Firstly, it is hard to reconcile the highly sophisticated behaviour of firms in our model with the naïve behaviour of firms in the Calvo (1983) model. Secondly, introducing sticky prices would make solving for firm behaviour very complicated, unless the sticky prices were only introduced to a separate retail sector, further increasing the size of our model. Finally, as is well known, introducing sticky prices results in counter-cyclical mark-ups, contrary to the evidence of Nekarda and Ramey (2010). The observed frequency of price adjustment can perhaps be reconciled with pro-cyclical mark-ups using a consumer search model as in Head et al. (2011). We do not pursue this avenue here.

4.1. Model changes

We assume that firm i in industry j has access to the production technology:

$$Y_t(i,j) = A_t(i,j) X_t^{\mathrm{P}}(i,j)^{\iota_{\mathrm{P}}} [K_t^{\mathrm{P}}(i,j)^{\alpha_{\mathrm{P}}} L_t^{\mathrm{P}}(i,j)^{1-\alpha_{\mathrm{P}}}]^{1-\iota_{\mathrm{P}}}$$

where $X_t^{\mathrm{P}}(i,j)$ is their level of intermediate good input and $K_t^{\mathrm{P}}(i,j)$ is the quantity of capital they hire from households, at a cost of R_t^{KP} per unit. We use a Hicks-neutral specification here since it minimises the changes necessary to the model without capital. (In particular, profits take the same form, and so research incentives are identical.)

Research, appropriation and invention will also use capital, but we assume that the capital they use is from a different stock. This research/invention capital may be thought of as capturing (variously) education, creativity, ideas, knowledge and advanced physical capital. Rather than the input to the research function for firm j in industry i being $L_t^{\mathrm{R}}(i,j)$, as it was originally, it is now:

$$X_t^{\mathrm{R}}(i,j)^{\iota_R} [K_t^{\mathrm{R}}(i,j)^{\alpha_R} L_t^{\mathrm{R}}(i,j)^{1-\alpha_R}]^{1-\iota_R},$$

where $X_t^{\mathrm{R}}(i,j)$ are the intermediates they use in research, and $K_t^{\mathrm{R}}(i,j)$ is the research/invention capital they use. This will not significantly change research incentives as we can decompose the research problem into a research level one and a cost minimisation one. Additionally, rather that invention requiring a stochastic amount of invention labour, it now requires a stochastic amount of invention output, which is produced using the same production function as research (chiefly for simplicity).

Households' preferences are now given by:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s N_{t+s} \Theta_{t+s} \left[\log \tilde{C}_{t+s}(h) - \frac{\Phi_{t+s}}{1+\nu} \tilde{L}_t^{\mathrm{S}}(h)^{1+\nu} \right]$$

for each household $h \in [0,1]$, where

$$\tilde{C}_t(h) \coloneqq \frac{C_t(h)}{N_t} - \hbar \frac{(1-\hbar^{\mathrm{INT}})C_{t-1} + \hbar^{\mathrm{INT}} C_{t-1}(h)}{N_{t-1}}$$

is habit adjusted consumption per head, (with $\hbar \in [0,1)$ controlling the strength of consumption habits and $\hbar^{\mathrm{INT}} \in [0,1]$ controlling whether consumption habits are internal or external), and where:

$$\tilde{L}_t^{\mathrm{S}}(h) \coloneqq \frac{L_{t+s}^{\mathrm{S}}(h)}{N_{t+s}} - \hbar^{\mathrm{LS}} \frac{L_{t+s-1}^{\mathrm{S}}}{N_{t+s-1}}$$

is habit adjusted labour supply per head (with $\hbar^{\mathrm{LS}}$ determining the strength of these external labour habits). Each household now supplies a different type of labour $L_t^{\mathrm{S}}(h)$ and potentially receives a different real wage, $W_t(h)$. They face the budget constraint: $C_t + I_t^{\mathrm{KP}} + I_t^{\mathrm{KR}} + B_t = L_t^{\mathrm{S}}(h) W_t(h) + R_t^{\mathrm{KP}} u_t^{\mathrm{P}} K_{t-1}^{\mathrm{P}} + R_t^{\mathrm{KR}} u_t^{\mathrm{R}} K_{t-1}^{\mathrm{R}} + B_{t-1} R_{t-1} + \Pi_t$, where I_t^{KP} and I_t^{KR} is investment in the two capital stocks,[43] and $u_t^{\mathrm{P}} K_{t-1}^{\mathrm{P}}$ and $u_t^{\mathrm{R}} K_{t-1}^{\mathrm{R}}$ are the quantities of these stocks that households make available to firms, with u_t^{P} and u_t^{R} their chosen utilisation rates and K_{t-1}^{P} and K_{t-1}^{R} the level of the capital stocks at the end of period $t-1$. The utilisation of research/invention decision may be thought of as capturing the incentives to bunch the implementation of ideas, as stressed by Francois and Lloyd-Ellis (2008; 2009).

Following Schmitt-Grohé and Uribe (2011), investment goods of type $\mathrm{V} \in \{\mathrm{P},\mathrm{R}\}$ are produced from consumption goods using the technology:

$$I_t^{\mathrm{KV}*} = A_t^{*\xi_{\mathrm{KV}}} E_t^{\mathrm{KV}} I_t^{\mathrm{KV}}$$

where I_t^{KV} is investment in units of consumption goods and $A_t^{*\xi_{\mathrm{KV}}} E_t^{\mathrm{KV}}$ captures investment specific technological change, as a short-cut alternative to modelling separate endogenous growth processes in a multi-sector model. As in Schmitt-Grohé and Uribe (2011), the productivity of the frontier (i.e. the underlying trend in A_t) enters into this expression in order to capture the cointegration between the relative price of investment and productivity that is observed in the data. It may be justified as reflecting improvements in installation technologies, or improvements to the allocation of new capital across firms, both of which come as a side-effect of the increase in general knowledge following an increase in A_t^*. Explicitly modelling a role for human capital in physical capital production would generate very similar results, while adding unnecessary complications.

[43] We assume a complete set of nominal state contingent securities, meaning C_t, I_t^{KP} and I_t^{KR} will not differ across households.

Both capital stocks evolve according to:

$$K_t^{\mathrm{V}} = \left(1 - \delta_t^{\mathrm{V}}\left(u_t^{\mathrm{V}}\right)\right) K_{t-1}^{\mathrm{V}} + \Gamma_t I_t^{\mathrm{KV}*} \left[1 - Q^{\mathrm{KV}}\left(\frac{I_t^{\mathrm{KV}*}}{I_{t-1}^{\mathrm{KV}*}}\right)\right]$$

for $\mathrm{V} \in \{\mathrm{P,R}\}$, where $\delta_t^{\mathrm{V}}(\cdot)$ for $\mathrm{V} \in \{\mathrm{P,R}\}$ are increasing functions capturing the effect of utilisation on depreciation, locally convex at the steady-state, $Q^{\mathrm{KV}}(\cdot)$ for $\mathrm{V} \in \{\mathrm{P,R}\}$ are convex functions capturing adjustment costs to the rate of investment (following Christiano, Eichenbaum, and Evans (2005)), which attain their minimum value of zero at the steady state rate of growth of investment, and where Γ_t is a shock to the marginal efficiency of investment, which, following Justiniano, Primiceri, and Tambalotti (2011), we will identify with a decreasing function of Moody's BAA-AAA bond spreads.[44] (The difference between Γ_t and E_t^{KV} is that only the latter will appear in the measured relative price of investment, and only the former is common to both processes.) $\delta_t^{\mathrm{V}}(\cdot)$ has a time subscript since we allow for a shock to depreciation to capture some of the volatility in depreciation shares we observe in the data.[45] There is a single shock across both capital types, which we call $\tilde{\delta}_t$, and it is constrained to weakly increase both the levels and the first derivatives of $\delta_t^{\mathrm{P}}(\cdot)$ and $\delta_t^{\mathrm{R}}(\cdot)$.[46] Depreciation shocks have been shown to be important by Dueker, Fischer, and Dittmar (2007), Liu, Waggoner, and Zha (2011) and Furlanetto and Seneca (2011) amongst others, and will turn out to be important here too. As these authors note, they may be interpreted as proxying for a combination of product specific capital, heterogeneity in capital quality across products, and changes in consumer preferences across these products. With this interpretation allowing depreciation shocks to affect the first derivative of $\delta_t^{\mathrm{V}}(\cdot)$ as well as its level is natural, since low quality capital will both break faster on average, and be more sensitive to heavy usage. This will also aid us in matching the negative correlation between depreciation and utilisation that is observed in the data.

Aggregate labour services to firms are now provided by a competitive industry of labour packers using the technology $L_t^{\mathrm{T}} = A_t^{*\xi_{\mathrm{L}}} E_t^{\mathrm{L}} \left[\int_0^1 L_t^{\mathrm{S}}(h)^{\frac{1}{1+\lambda_{\mathrm{L}}}} \, dh\right]^{1+\lambda_{\mathrm{L}}}$, where E_t^{L} is an exogenous stationary labour productivity shock. (In the absence of research and development, this E_t^{L} shock would act exactly like a classical TFP shock.) The productivity of the frontier enters our expression for labour services in order to capture the improvements in labour productivity that arise from the higher knowledge levels after an increase in frontier productivity. Again, explicitly modelling human capital evolution would add little to our model's performance. However, following Jaimovich and Rebelo

[44] Justiniano, Primiceri, and Tambalotti (2011) used the high yield to AAA spread. We choose the BAA-AAA one due to increased data availability.

[45] Our measure of depreciation is the consumption of fixed capital from NIPA. If anything, this will underestimate the true variance of depreciation, since the NIPA measure omits variation in depreciation rates within individual product categories. We thank Martin Seneca for this observation.

[46] We additionally constrain the response of $\delta_t^{\mathrm{V}}(\cdot)$ to the shock such that in its linearised version, with utilisation at its steady-state level, both $\delta_t^{\mathrm{V}}(\cdot)$ and $\delta_t^{\mathrm{V}\prime}(\cdot)$ are positive with at least 95% probability. This is true automatically in the source non-linear specification in which $\delta_t^{\mathrm{V}}(\cdot)$ and $\delta_t^{\mathrm{V}\prime}(\cdot)$ are log-linear in $\tilde{\delta}_t$ when utilisation is at its steady-state, but in preliminary estimates the linearised $\delta_t^{\mathrm{P}\prime}(\cdot)$ turned negative a high proportion of the time, in the absence of this additional constraint.

(2008) we do include labour adjustment costs. In particular, we assume that in sector $\mathrm{V} \in \{\mathrm{P,R}\}$ there is a perfectly competitive industry that transforms aggregate labour services into sector specific labour services using the technology $L_t^{\mathrm{EV}} = L_t^{\mathrm{TV}}\left[1 - Q^{\mathrm{LV}}\left(\frac{L_t^{\mathrm{TV}}}{L_{t-1}^{\mathrm{TV}}}\right)\right]$, where $Q^{\mathrm{LV}}(\cdot)$ is a monotone increasing function that is zero at the steady state rate of growth of L_t^{TV}. The aggregate labour market clearing condition is then $L_t^{\mathrm{T}} = L_t^{\mathrm{TP}} + L_t^{\mathrm{TR}}$. In the absence of labour adjustment costs, there is a risk that the capital share of R&D would be biased upwards since there are adjustment costs to capital. Labour adjustment costs also help generate plausible business cycles in response to news about future productivity (Jaimovich and Rebelo 2008), which may be important here due to the endogenous movements in future productivity that our model generates.

The two positive spillovers from frontier productivity growth mean that the steady-state growth rate of real output per capita is given by $\frac{g_A}{(1-\alpha_{\mathrm{P}})(1-\iota_{\mathrm{P}})} + \left(\xi_{\mathrm{L}} + \frac{\alpha_{\mathrm{P}}}{1-\alpha_{\mathrm{P}}}\xi_{\mathrm{KP}}\right)g_{A^*}$. If ξ_{L} and ξ_{KP} are positive then g_{A^*} will not need to be as high, meaning the variance of g_A (and hence that of output) will be lower. Providing the technology for producing overheads takes the same form as that for producing the input to research and invention, neither these spillovers nor the presence of capital and intermediate goods in the production function will change the criterion for no appropriation to be performed asymptotically in non-protected industries. (Away from this special case the lower bound on ζ would be non-zero, and possibly negative.)

We model sticky nominal wages in the standard Calvo (1983) fashion, following Erceg, Henderson and Levin (2000). Each household is able to set its wage optimally with probability $1 - \mathcal{v}$. We assume that those households that cannot adjust their wage optimally will fully index their wage to its steady-state growth rate.

Monetary policy takes an augmented Taylor rule form. We allow the central bank to respond to all prices in the economy (i.e. the price of consumption, production investment, research investment and labour), four proxies for the real interest rate (the return on production and research investment, the demand shock and the depreciation shock), as well as both output's deviation from trend and its growth rate.

In particular:

$$\frac{R_t^{\mathrm{NOM}}}{R^{\mathrm{NOM}}} = \left(\frac{R_{t-1}^{\mathrm{NOM}}}{R^{\mathrm{NOM}}}\right)^{\rho_{R^{\mathrm{NOM}}}}\left[\left(\frac{G_{P,t}}{G_{P,t}^*}\right)^{\mathcal{M}_{\mathrm{P}}}\left(\frac{E_{t-1}^{\mathrm{KP}}G_{A^*}^{\xi_{\mathrm{KP}}}}{E_t^{\mathrm{KP}}G_{A^*,t}^{\xi_{\mathrm{KP}}}}\right)^{\mathcal{M}_{\mathrm{PKP}}}\left(\frac{E_{t-1}^{\mathrm{KR}}G_{A^*}^{\xi_{\mathrm{KR}}}}{E_t^{\mathrm{KR}}G_{A^*,t}^{\xi_{\mathrm{KR}}}}\right)^{\mathcal{M}_{\mathrm{PKR}}}\left(\frac{G_{W,t}}{G_W}\right)^{\mathcal{M}_{\mathrm{W}}}\right]^{1-\rho_{R^{\mathrm{NOM}}}}$$
$$\cdot\left[\left(\frac{R_t^{\mathrm{KP}}}{A_t^{*\xi_{\mathrm{KP}}}}\right)^{\mathcal{M}_{\mathrm{RKP}}}\left(\frac{R_t^{\mathrm{KR}}}{A_t^{*\xi_{\mathrm{KR}}}}\right)^{\mathcal{M}_{\mathrm{RKR}}}\Theta_t^{\mathcal{M}_\Theta}\tilde{\delta}_t^{-\mathcal{M}_{\tilde{\delta}}}\right]^{1-\rho_{R^{\mathrm{NOM}}}}$$
$$\cdot\left[\left(\frac{Y_t}{N_t A_t^a A_t^{*e}}\right)^{\mathcal{M}_{\mathrm{Y}}}\left(\frac{G_{Y,t}/G_{N,t}}{G_Y/G_N}\right)^{\mathcal{M}_{\mathrm{G}}}\right]^{1-\rho_{R^{\mathrm{NOM}}}}\exp\epsilon_{R^{\mathrm{NOM}},t},$$

where R_t^{NOM} is the gross nominal interest rate, $G_{P,t}$ is the (gross) growth rate of the nominal price of the consumption good, $G_{P,t}^*$ is the stochastic target for this growth rate, $\frac{E_{t-1}^{\mathrm{KV}}}{E_t^{\mathrm{KV}}G_{A^*,t}^{\xi_{\mathrm{KV}}}}$ is the growth rate of the *real* price of investment goods of type $\mathrm{V} \in \{\mathrm{P,R}\}$, $G_{W,t}$ is the growth rate of the real wage, Y_t/N_t is log real GDP, $G_{Y,t}/G_{N,t}$ is the real per capita GDP growth rate and

$R_t^{\mathrm{SHOCK}} \coloneqq \exp \sigma_{R^{\mathrm{NOM}}} \epsilon_{R^{\mathrm{NOM}},t}$ is a monetary policy shock. Variables without time subscripts are steady-state values, and the constants a and e are defined in the appendix, section 7.5. In the absence of endogenous productivity, the optimal policy would fully stabilise nominal wages, completely removing the Calvo distortion, thus it is important to allow wages to enter the Taylor rule. There is no guarantee though that this prescription carries over into our model with endogenous productivity. (We intend to investigate optimal policy in this model in future work.) It turns out however that the only significant terms in the Taylor rule are the lag, the price response, and the response to the depreciation shock and the rental rate of production capital (which is tightly correlated with the Wicksellian real interest rate (Woodford 2001)), so the estimated rule takes a more standard form.

The model's full set of de-trended equations is given in an appendix (section 7.5).

4.2. Data and estimation

The model is estimated on logs of quarterly U.S. series for nominal output growth,[47] consumption price inflation, investment price inflation, population growth, labour supply per capita, the R&D share, the consumption share, the labour share, the depreciation share, nominal interest rates, capacity utilisation and the BAA-AAA spread. The longest samples are from 1947Q1 to 2011Q2, though some series are shorter. (Our estimation method can cope with an uneven sample.) Most series comes from NIPA or the FRB. Full details of the sources and construction methods of the data are given in an appendix, section 7.6, and the full data set is available from the author on request.

In order to remove any structural change, we filter the data before estimation, with a high-pass filter that allows frequencies with periods below the sample length (258 quarters). We adjust the level of the filtered data so that the mean of the filtered series matches that of the original data. (Broadly) following Canova (2009) we also include IID, $\mathrm{AR}(1)$ and repeated-root $\mathrm{AR}(2)$[48] "measurement error" shocks in each observation equation, to prevent our model from being contorted to fit the data. (Canova (2009) advocates the inclusion of IID, $\mathrm{I}(1)$ and $\mathrm{I}(2)$ shocks.)

In standard DSGE models, there are usually enough degrees of freedom that almost any set of first moments may be matched without impacting the model's ability to match second moments. The presence of endogenous growth in our model, though, means this is no longer true for us. In our model, almost all first moments are tightly coupled both to each other (e.g. the labour-share, mark-ups and growth) and to the model's dynamics. This raises the possibility that our model's inevitable misspecification may mean it is impossible for our model to match simultaneously all first moments without grossly compromising its dynamics. The Canova (2009) approach is to discard all information about first moments, and to assume the "measurement error" has a unit root, but this

[47] We use nominal output as there should be less measurement error in the nominal series than in the real series.

[48] Our justification for going up to a repeated-root $\mathrm{AR}(2)$ process is that as the auto-regressive parameter of such a process tends to one, the process becomes an $\mathrm{I}(2)$ trend, which is exactly the type of trend removed by the widely used HP-filter (Hodrick and Prescott 1997). In order to avoid implicitly removing an $\mathrm{I}(3)$ trend from the series in differences (nominal output growth, consumption price inflation, investment price inflation and population growth) we suppose that the measurement error enters the observation equations for these series with the over-differenced moving average form $\mathrm{me}_t - \mathrm{me}_{t-1}$.

necessitates the use of strong priors, something that is infeasible here since the dimensionality of our model rules out MCMC based estimation. Additionally, allowing unit roots in measurement error would prevent us using the variance share of measurement error as a measure of the quality of our model. Instead, we allow for a mean term in the measurement error to prevent misspecification of the kind described from severely biasing other parameters. However, to ensure the means of the data series remain informative, we follow Lee et al. (2010) and Candès, Wakin, and Boyd (2008) in imposing a sparsity inducing "adaptive lasso" (generalized t) prior on these mean measurement error terms.[49]

Since we want our model to rely on its internal persistence mechanism, rather than the persistence of shocks, and since we want all shocks to be stationary, we impose a prior on all the "ρ" parameters of our model (these include the persistence of shocks, the persistence of $\mathrm{AR}(1)$ and repeated-root $\mathrm{AR}(2)$ measurement errors, and the persistence of monetary policy). We use a logit-normal distribution that is scaled to $[-1,1]$ then truncated to $[0,1]$ (i.e. if Z is normally distributed, $\frac{1-\exp(-Z)}{1+\exp(-Z)}\Big| Z > \frac{1}{2}$ has our distribution). We set the mean of the underlying normal distribution to 0 and its variance to 2, which are the unique values which result in a density which has zero first, second and third derivatives at the origin, ensuring small to medium values of ρ are not biased.

We fix the discount factor (β) at 0.99 following standard practice. We also bound the inverse-Frisch elasticity (ν) to be above 0.25, which is a lower bound on standard macro calibrations as reported by Peterman (2011). All the other parameters of our model are given flat priors. We then estimate by the "maximum a posteriori" method (which is very close to maximum-likelihood since the majority of parameters have flat priors), subject to:

- all variables being stationary,
- a unique (determinate) solution existing for both the simple model and this extended one, (with an identical number of firms per industry in both, and with all parameters identical except possibly $\mathcal{L}^{\mathrm{I}}$),
- all parameters being in the region in which the model is well behaved asymptotically,[50]
- the steady-state value of the average mark-up (μ_t) equalling 0.056 (to 3 decimal places), in line with the micro-evidence of Boulhol (2007),[51]
- patent protected industries being 17% (to 0 decimal places) more productive than non-protected industries in steady-state, in line with the micro-evidence of Balasubramanian and Sivadasan (2011),[52]

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[49] In the notation of Lee et al. (2010), in this prior we set a_j to the length of the data to the power of $^1/_3$ (to ensure the method possesses the oracle property), and b_j is chosen so that the expected absolute measurement error mean term is 1%. To reduce the dimensionality of the state space, we force these measurement error mean terms to the level at which the model's steady state for observable variables exactly matches their mean in the data.

[50] $\wp\lambda\mathbb{E}\gamma_t < 1$, $\lambda\mathbb{E}\eta_t\gamma_t \geq 1$, $\mathbb{E}d_t > \wp\mathbb{E}\gamma_t\mu_t^{\mathrm{P}}$, $\mathbb{E}g_{I,t} > 0$ and $\mathbb{E}J_t^P > 1$.

[51] This is implemented by adding the steady-state mark-up as an additional observation variable to the model, with an $\mathrm{NIID}(0,0.0005)$ shock (added both to the data and to the model, with known standard deviation).

[52] Similarly, this is implemented by adding the steady-state value of $\log \hat{A}_t^{\mathrm{N}}$ as an additional observation variable, with an $\mathrm{NIID}\big(0, 1/2\,(\log(1/1.165) - \log(1/1.175))\big)$ shock (as before).

- the correlation of log mark-ups (as measured by the inverse labour share) and log output, being positive when the data is filtered by a cut-off of one, five or eleven years and *negative* when the data is filtered by a filter with a cut-off of twenty years,[53]
- the share of medium frequency variance[54] decreasing when the mean length of patent protection is reduced by one quarter.

By disciplining mark-ups and relative-productivity from micro-evidence, we hope to go some way to answering the concerns about the introduction of free-parameters raised by Chari, Kehoe and McGrattan (2009).

For technical reasons, we ignore the positivity constraint on $g_{\mathrm{I},t}$ during estimation.

The maximisation is carried out using the CMA-ES algorithm (N. Hansen et al. 2009), which is known to have good global search performance, particularly when run with large populations, as we do. However, although the dimensionality of our model is much smaller than that of a $\mathrm{VAR}(1)$ run on the same series, we still cannot absolutely guarantee that a global maximum has been found. This is a standard problem in estimating large models.

4.3. Estimation results

The full list of estimated parameters is given in an appendix, section 7.7. We briefly discuss a few key parameters here however. In the below, approximate posterior standard errors are given in brackets. (These are generated from the optimisation algorithm, which gives the inverse hessian of a robust quadratic approximation to the upper envelope of the maximand. Our Monte Carlo experiments indicate that the resulting standard errors are moderately biased upwards, meaning that parameters may be estimated more precisely than they appear to be.[55])

$\mathcal{p}$ was estimated at $0.0427\ (0.00021)$, implying that manufacturing firms have very little bargaining power in dealing with patent holders. The large bargaining power of patent holders suggests that they may be bargaining simultaneously with all firms keen to licence their product, rather than bargaining with each independently as in our model. In future work we intend to study the strategic interactions in this simultaneous bargaining and entry process more rigorously.

$\mathcal{q}$ was estimated at $0.0374\ (0.00030)$, which implies that only 4.9% of patents last twenty years. This is consistent with some patented products not being commercialised until long after their patent was granted, and others having their patent challenged in court prior to their expiry. It is

[53] More specifically, we begin by generating 2^{10} simulated runs from the model, each the same length as the data, using the same random seed for each set of runs, for the sake of variance reduction. We then take the correlation of the given variables at each filter cut-off, for each of the runs. We require that the proportion of the runs for which these are of the correct sign is both greater than one-half and significantly different from one-half at 5%. (We use a two-sided test in order to preserve comparability with Figure 2.)

[54] As measured by applying a perfect filter to the spectral density generated by the transition matrices, with accepted band between 8 and 60 years.

[55] Our estimate of the Hessian of the maximand may be affected by the inclusion of exact bound constraints, since these will tend to reduce the variance of parameters that lead the bound constraint to be violated. However, our procedure estimates the scale of the hessian separately, so still on average over all parameters we expect posterior standard errors to be upward biased.

also consistent with a broader interpretation of "patent protection" within the model, since some inventors are able to exclude entry to their industry for a while, even in the absence of patent protection, via obfuscation or contractual arrangements.

The inverse Frisch elasticity of labour supply was driven to its lower bound of $\nu = 0.25$ by the estimation procedure.[56] While older studies suggested that such highly elastic labour supplies were difficult to reconcile with the micro-data, recent studies (e.g. Peterman (2011) and Keane & Rogerson (2012)) have concluded that highly elastic labour supplies are consistent with the micro evidence when that data includes a broad range of individuals, and is interpreted in light of e.g. human capital accumulation. Our model also includes labour adjustment costs, which make aggregate labour supply appear less elastic. Consequently, a standard RBC calibration of the Frisch elasticity based on simulated data from our model would produce a much lower Frisch elasticity than 4. In light of this, we do not consider our estimated elasticity to be implausible. Nonetheless, in future work we intend to investigate the performance of our model when it is augmented by employment search and participation decisions.

α_{P} was estimated to be $0.201\ (0.00040)$, much lower than the traditional value for the capital share of around 0.3. In line with this low value, the consumption share generated by our model was about 10.9% higher than the true value, and the labour share was around 34.5% higher. The treatment here of net exports as investment may be one factor that is biasing down the capital share, due to the US's persistent trade deficit. Another explanation is the existence of some missing heterogeneity across sectors in the real world, with the sectors that are driving growth (e.g. services) tending to be more labour intensive. There is further evidence of missing sectoral heterogeneity in the estimated intermediate goods share in production of $0.0534\ (0.0026)$, (standard estimates are around 0.4), however, this is most likely just a function of the absence of a retail sector in our model. Allowing for the possibility that consumption of intermediate goods in R&D is measured as investment, rather than intermediate consumption, would also help fix these shares as it would decrease the numerators and increase the denominators ($\iota_{\mathrm{R}} = 0.178\ (0.0032)$).

However, the low value for the capital share of output is at least partially balanced by a very high estimated value for the capital share of R&D ($\alpha_{\mathrm{R}} = 0.996\ (7.4 \times 10^{-6})$). Further insight into the nature of this research-capital comes from the very high adjustment costs to increasing the growth rate of its stock ($Q^{\mathrm{R}''}\left(G_{I^{\mathrm{KR}*}}\right) = 62.6\ (4.0)$, in comparison, $Q^{\mathrm{P}''}\left(G_{I^{\mathrm{KP}*}}\right) = 0.00533\ (0.0012)$). These values suggest our interpretation of research-capital as being an external "idea-stock" may be correct. Additional evidence for this comes from the fact that depreciation shocks knock large amounts off the level of the research capital stock (ideas we thought were good turned out to be not so great), whereas they only affect the sensitivity of production-capital depreciation to utilisation (machines we thought to be reliable turned out to be quite sensitive).

In estimating our model, we allowed the data to specify whether investment in R&D capital was measured in the standard national accounts, or whether it was only measured in the R&D satellite account data, since it was not obvious a priori that those producing the accounts can distinguish

[56] When this bound was not imposed, the estimated value was below 0.01.

investment to help future R&D from investment to help future production. Our estimates suggests that 49.4% (1.3%) of all R&D investment is actually captured by the standard national accounts, with the rest measured in the satellite accounts. This level of mis-measurement seems plausible given the difficulties in ascertaining for what a piece of physical capital will be used.

The frictions in our model take plausible values, with households able to update their wage optimally in 17.4% (0.42%) of quarters, which is not statistically different (at 5%) from the probability of a wage change for hourly workers found in micro data by Barattieri et al. (2010) (18%). Recall, too, that when households in our model cannot optimally update their wage, they instead index to steady-state inflation, so the welfare costs of this friction are likely to be small. As observed previously, there is virtually no adjustment cost on production capital, however we find a substantial adjustment cost to production labour ($Q^{\mathrm{LP}'}(G_{L^{\mathrm{TP}}}) = 0.0875\ (0.0047)$). As shown by Jaimovich and Rebelo (2009), this enables the model to produce co-movement in response to news about future productivity, which is provided in our model by almost any standard shock, thanks to the endogenous growth mechanism. Consumption habits are estimated as being predominately external ($\hbar^{\mathrm{INT}} = 0.0151\ (0.0032)$), and much less strong than in many DSGE models ($\hbar = 0.253\ (0.0041)$). Estimated habits in labour are negligible. This lesser role for habits of both kinds stems from the much stronger persistence mechanism in our model.

We now turn to the estimated sources of growth. Core (Hicks-neutral) frontier productivity is estimated to grow at 1.11% per year, which is further scaled up by the roles of intermediates and capital, along with the various spillovers, to arrive at an aggregate real growth rate (in units of the consumption good) of 1.57% per year, only slightly lower than that found in the data (1.76% per year[57]). The importance of spillovers for growth has been stressed extensively in the empirical literature before (Griliches 1998; Eaton and Kortum 1999; Forni and Paba 2002; Klenow and Rodríguez-Clare 2005). It is likely that there is some downwards bias in real GDP growth rate estimates, due to the difficulty of valuing new products (Broda and Weinstein 2010), so in future work we intend to examine the robustness of our results to correcting for this in the data, at least approximately.

Finally, on the sources of cycles, we find that all variables are primarily driven by the depreciation shock, with lesser contributions from the labour supply shock and the population shock. The monetary policy shock plays an even smaller role (contributing to less than 1% of each variable's non measurement error variance), and all other shocks make a negligible contribution. (The full variance decomposition is given in Table 6 in the appendix.) Of note is the fact that all shocks have a persistence parameter of less than 0.9, suggesting that the model is able to generate the observed persistence in macroeconomic time series on its own.

The depreciation shock is estimated as having two distinct effects here. Firstly, it increases the sensitivity of the production-capital depreciation rate to increased utilisation. Since the derivative of the depreciation rate with respect to utilisation enters directly into the investment and utilisation equations, even under a first order approximation this can have a large effect on

[57] The low figure comes from deflating by the consumption price, rather than by a consumption-investment price aggregate.

investment and utilisation, by increasing the costs of using capital. Secondly, it increases the depreciation rate of the stock of research-capital, independent of utilisation. The natural interpretation for the shock then is as a proxy for the financial wedge. Indeed, the correlation between the estimated series for $\tilde{\delta}_t$ and the BAA-AAA spread is 0.296 (with a p-value of less than 0.00001), confirming this interpretation.

In a time of great uncertainty, or low asset values, such as the aftermath of the recent crisis, if capital is "put to work" there is a risk it will disappear completely. This is in the spirit of the Kiyotaki-Moore model (Kiyotaki and Moore 1997), and captures the first of these two effects. (For an example that makes clear the effect is on the sensitivity of the depreciation rate to utilisation, consider the incentives of a mortgage-holder in negative-equity to maintain their house.) That financial shocks should result in an increase in the depreciation rate of the stock of ideas is equally clear. In the absence of sufficiently valuable collateral, inventors may be unable to finance the commercialisation of their invention, and by the time asset values recover, it may no longer be "timely" enough to warrant that expense. Obviously, this calls for the inclusion of structurally modelled financial frictions in our model. We intend to pursue this avenue in future work.

4.4. Model evaluation

As previously mentioned, we use the estimated amount of measurement error to quantify the model's performance. Aside from the two series previously discussed (the labour and capital shares), all of our series had mean levels of measurement error below 0.05%, implying the model is well able to capture the rest of the data's first moments. This leaves the data's second moments to discuss. Since our model is designed to explain cycles at business and medium frequencies, but is unlikely to be able to match either very high frequency noise, or low-frequency structural change, we report measurement error variance in a range of frequency bands. (These are produced by applying perfect filters to the measurement error and observation variable series.) The results of this may be seen in Table 3 below.

Data series	**High frequency 0-1 years**	**Business cycles 1-8 years**	**Medium frequency 8-50 years**	**Low frequency >50 years**
Nominal output growth	2.2%	9.8%	44.1%	1.3%
Consumption price inflation	89.0%	94.0%	66.1%	2.4%
Investment price inflation	97.6%	99.0%	93.2%	17.7%
Population growth	6.6%	37.2%	89.9%	80.1%
Labour supply per capita	44.6%	24.7%	48.8%	82.6%
R&D share	0.0%	0.0%	0.0%	0.0%
Consumption share	67.3%	22.3%	16.7%	35.2%
Labour share	100.0%	100.0%	99.6%	99.1%
Depreciation share	5.5%	37.4%	83.1%	89.9%
Nominal interest rates	86.6%	89.2%	54.3%	15.0%
Capacity utilisation	47.2%	87.8%	89.1%	87.7%
BAA-AAA Spread	100.0%	100.0%	100.0%	100.0%

Table 3: Proportion of variance attributed to measurement error in the unconstrained model.

Significantly, our model explains much of the variance in nominal GDP, labour supply, and the R&D and consumption shares, suggesting it is capturing well the linkages between research and the business cycle. Indeed, from summing the percentages our model explains (i.e. 100% minus the measurement error share), we see that the model is fully explaining the equivalent of 5.0 variables at business cycle frequencies and 4.2 variables at medium frequencies. Given there are only four shocks given any weight by the estimation procedure (with one of those given a tiny weight), the model is fully explaining more variables than there are driving shocks. Note too that the interpretation of these percentages is somewhat different to the percentages of explained variance given in traditional business cycle analysis. Whereas for us explaining a high percentage of the variance means that the model's response is preferred by the data to the general measurement error process (i.e. it is a claim about the full covariance structure of the model), the claim in the business cycle literature is really only about the variance of each variable, and covariances across variables or time need not be plausible.

Nonetheless, the model's poor performance along other axes deserves comment. Its difficulties matching inflation rates and nominal interest rates at business cycle frequencies most likely reflect the absence of short run price-rigidity in our model. The model also does spectacularly poorly in matching the variance of the labour share. However, we will see below that the labour share our model generates has a similar correlation structure with GDP across frequencies as we observe in the data. This suggests that the pro-cyclical movements in mark-ups generated by our model are too small relative to those in the data, which is not too surprising given that at the estimated parameters, there are 6.47 firms even in patent protected industries, meaning even these industries will have quite low mark-ups. Now, certainly our model can generate larger swings in mark-ups over the cycle with alternative parameterizations, but these parameterizations will imply even larger movements in productivity. One way of dampening down these excessively large movements in productivity would be to consider the non-asymptotic version of our model in which it takes several periods for new firms to catch-up to the frontier. Producing a non-asymptotic version of the model that may be feasibly simulated is left for future work.

As an additional test of the model, we re-estimated the model under the constraint that $q_t = 0$, (and without the constraint on the effect of increasing q_t on the share of medium-frequency variance). Doing this reduced the log posterior density by 14.14[58] which with flat priors would mean we could reject the null of the validity of the $q_t = 0$ constraint at even 0.01% significance. Now, with $q_t = 0$, patent protection is indefinite, so there cannot be any of the movement in the share of patent protected industries that was previously seen to drive our model's behaviour, and the model collapses to a medium scale variant of the Jaimovich (2007) model. Hence, our ability to reject the null of $q_t = 0$ provides strong evidence of the macroeconomic importance of our key mechanism.

We can further statistically test our model by looking for evidence of misspecification. Under the null hypothesis of no misspecification, the estimated shock residuals should be $\mathrm{NIID}(0,1)$. In Table 5 in the appendix, we report the p-values of LM tests for the presence of auto-correlation in these residuals. We are unable to reject the null of no auto-correlation (at 1%) for six shocks, including the depreciation shock, the population shock and the monetary policy shock. Given these last three shocks together explain more than 50% of the non-measurement-error variance in ten out of the

[58] The log posterior density decreased from 13462.01 to 13447.86.

twelve variables (including output and prices), and given that the estimated shocks from DSGE models tend to be highly auto-correlated, this is a further strong vindication of our model.

A final natural test of the model is its ability to replicate the results of section 2.

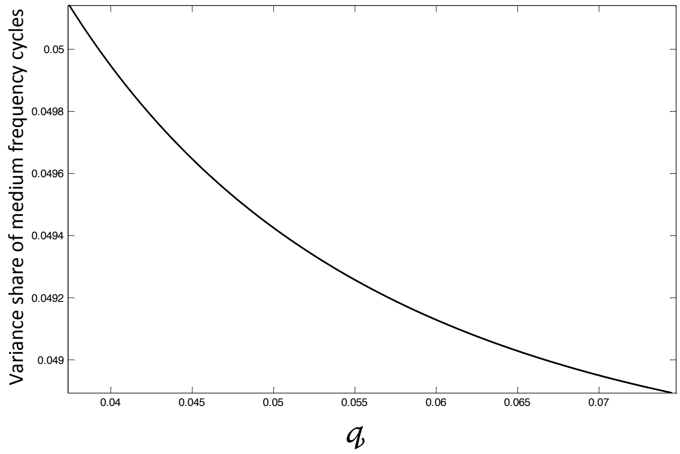

Figure 5: The effect of patent duration on the importance of medium-frequency cycles.

In Figure 5, we verify that the model does indeed predict that increasing the duration of patent-protection increases the share of variance attributable to medium-frequency cycles. Each dot represents an estimated variance share using the spectral density implied by the transition matrices. With longer patent-protection (i.e. a smaller value of q_p), following a boom in invention the share of patent-protected industries will be above its steady-state level for longer, implying that productivity too will be above trend for longer. Consequently, we see in Figure 3 that increasing patent duration (reducing q_p) does indeed increase the share of medium-frequency variance. The left hand axis of this graph corresponds to the estimated value of q_p, so of course at that point it was constrained to have negative slope, but its continual decrease across the range was not a product of a constraint imposed in estimation.

Additionally, output per capita is near trend stationary in our model, just as in the data. By construction, there is only one potential source of non-stationarity in output per capita: the non-stationarity of A_t^*. However, the standard deviation of g_{A^*} is only 0.00186%, meaning that A_t^* is very close to being deterministic. Thus in the long run in our model, log-output will always return to its original linear trend. The low variance of g_{A^*} comes from the fact that fluctuations in the number of industries and the number of firms absorb almost all demand variations in the long and short runs, meaning each individual firm faces roughly constant incentives to perform research. Despite this long-run return to trend however, our model still generates sizeable medium-frequency cycles, as may be seen in the impulse responses shown in the next section.

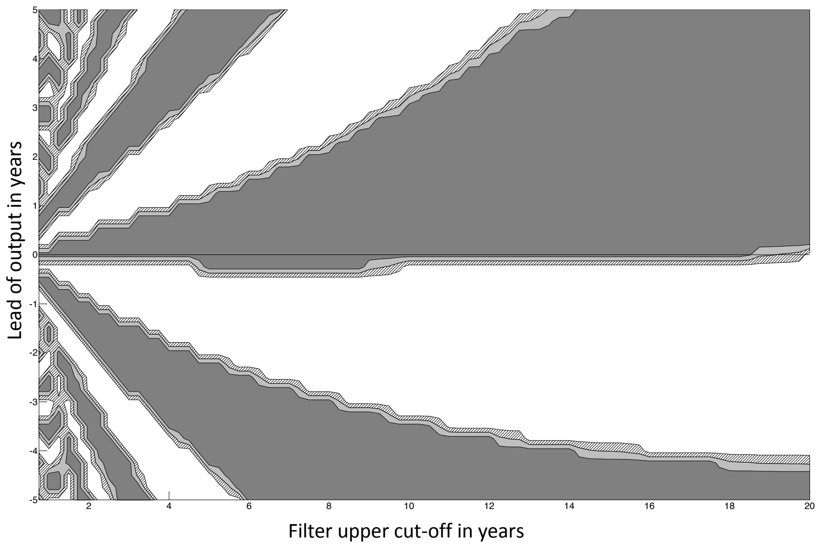

Figure 6: The cross correlation of model output and mark-ups, as a function of filter cut-off.
(Dark gray is a significantly positive correlation (at 5%), light grey is a positive but insignificant one, cross-hatched is a negative but insignificant one and white is a significantly negative one.)

Finally, although our estimation procedure guarantees that mark-ups (inverse labour-shares) are pro-cyclical when the model's output is filtered with a cut-off of one, five or eleven years and counter-cyclical when the output is filtered by a filter with a cut-off of twenty years, the estimation procedure does not impose anything about the cross-correlation of output and mark-ups at lags or leads. In Figure 6, we replicate Figure 2 with simulated data from the estimated model. Immediately, we see that only the bound at twenty years is actually binding, meaning our model is not being contorted in order to produce pro-cyclicality. Indeed, the similarity between the figures is remarkable. Just as in reality, the model predicts that mark-ups are pro-cyclical for small lags or leads, unless the data is filtered with a very low frequency lower cut-off. Again, as in reality, the model predicts that mark-ups are positively correlated with leads of output, and negatively correlated with its lags.

This pro-cyclicality is not driven by sticky wages. Indeed, with fully flexible wages we get pro-cyclicality whatever our filter cut-off. Instead, the pro-cyclicality is driven by the fact that increases in the proportion of industries producing patent protected products both increase mark-ups and productivity. This also explains why mark-ups should lead output; the increase in mark-ups is instant, however due to the assorted real rigidities in our model, the increase in output will only occur gradually.

4.5. Impulse responses

In Figure 7, we present the impulse responses to the four key driving shocks. As in the previous section, each graph is given in terms of per cent deviations from the value the variable would have taken had the shock never arrived, and the horizontal axis shows time in years, though this is a quarterly model. For no shocks was there an asymmetric positive and negative response, so the lower bound on invention is irrelevant. Each shock is in a different column, and the key response variables are in rows. Solid lines show the response with the estimated degree of wage stickiness, dashed lines show responses under flexible wages.

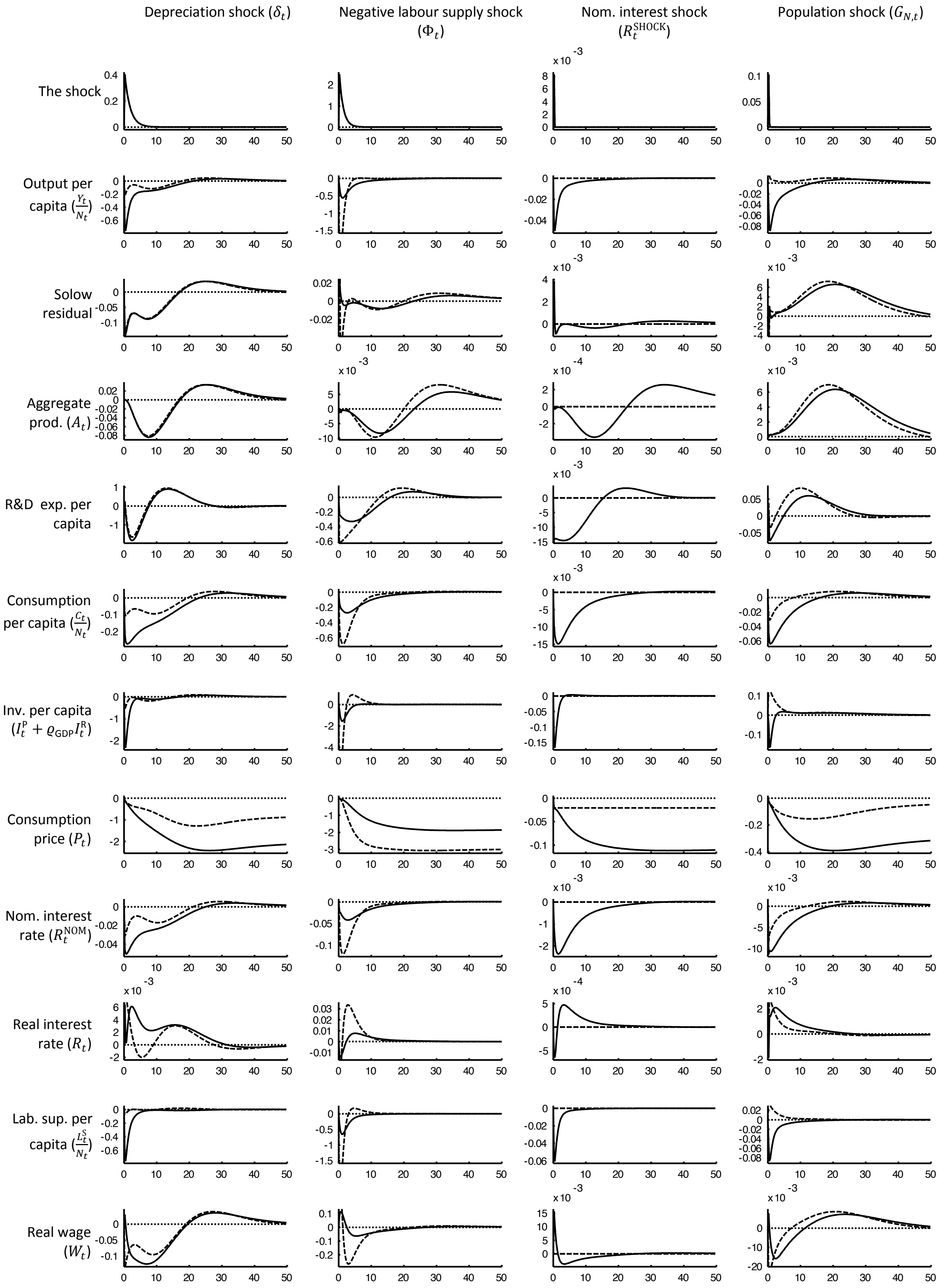

Figure 7: Impulse responses from the core model.

(Vertical axes are in percent, horizontal axes are in years. Solid lines are with nominal wage rigidity, dashed lines are with flexible wages.)

To show the magnitude of the effects of these shocks on productivity, we include the implied Solow residual[59] in the third row. Our chief driving shock, that to depreciation, has both a direct effect on the Solow residual through reduced utilisation, and an indirect one through the consequent reduction in invention and transfer away from new, highly productive industries, both of which operate in the same direction initially. However, the indirect effect far outlasts the direct one, with aggregate productivity still negative nearly twenty years after the original shock. It then slightly overshoots due to our model's real rigidities, producing a medium frequency cycle in productivity.

In fact, thanks to the model's endogenous growth component, the Solow residual moves following each of the four shocks, so in a sense all shocks are TFP shocks. Most interesting of these is our monetary policy shock, as a large medium term impact of monetary policy on productivity would substantially alter prescriptions for optimal monetary policy. However, at the estimated parameters the movement in productivity following a monetary policy shock is miniscule, so (perhaps unsurprisingly) the medium term impacts of monetary policy on productivity are not something that policy makers need to factor in to their decisions.

5. Conclusion

Many have expressed the worry that "the apparent fit of the DSGE model [has] more to do with the inclusion of suitable exogenous driving processes than with the realism of the model structure itself"[60]. In this paper, we have demonstrated that if productivity is endogenized through research, appropriation and invention then even a frictionless RBC model is capable of generating rich persistent dynamics from uncorrelated shocks, and a full medium-scale model is capable of accurately matching key moments, providing a statistically significant improvement in model fit.

We showed that all shocks lead to changes in the rate of product invention that have significant consequences for aggregate productivity and mark-ups at medium-frequency, due to fluctuations in the proportion of industries that are producing patent-protected products. Our model's propagation mechanisms thus lend persistence to all shocks, not just shocks to the invention or research process. Furthermore, this improvement in the model's propagation mechanism does not come at the expense of counter-factual movements in mark-ups, implausibly large unit roots in output, or the use of a growth model that we can reject thanks to the absence of strong scale effects in the data. In all of these respects, then, our model presents a substantial advance on the prior literature.

The fact we are able to combine a plausible growth model with a business cycle model also enables us to get much tighter estimates of the strength of externalities (for example) than is possible from traditional growth models, since these parameters have an impact on the dynamics as well as on the long run growth rate. This will enable the testing of hypotheses about the mechanics of endogenous growth that were previously near impossible to test.

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[59] The Solow residual is given by $\frac{Y_t}{K_{t-1}^{\alpha_{\mathrm{P}}} L_t^{1-\alpha_{\mathrm{P}}}} = \frac{\hat{Y}_t A_t^{\frac{1}{1-\iota_{\mathrm{P}}}} A_t^{*(1-\alpha_{\mathrm{P}})\xi_{\mathrm{L}}}}{\hat{K}_{t-1}^{\alpha_{\mathrm{P}}} \hat{L}_t^{1-\alpha_{\mathrm{P}}}}$ in the notation of the appendix, section 7.5.

[60] Del Negro et al. (2007) paraphrasing Kilian (2007).

Our model suggests that a switch to indefinite patent protection would result in significant welfare improvements. Such a switch would both permanently increase the level of aggregate productivity, and substantially lessen its variance and persistence, while only slightly increasing mark-ups and efficiency losses due to research duplication. Indeed, it may be shown that in our model increasing patent protection even slightly increases growth rates, as industry profits are decreasing in aggregate productivity, and so with indefinite patent protection each (protected) industry has fewer firms meaning higher mark-ups and higher research. However, it is clear that the structure of our model has "stacked-the-deck" in favour of finding a beneficial role for patent protection. Patents in our model are less broad than in the real world, and they do not hinder future research or invention. One minimal conclusion we can draw on patent protection is that product patents should at least be long enough that by the end of patent protection, production process have reached frontier productivity. In our model, this time goes to zero asymptotically. A less radical policy change might be to grant temporary extensions to patents that would otherwise expire during a recession. We intend to explore the full policy implications of this model in future work.

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7. Online appendices

7.1. The free-entry and first order conditions

When deciding how much research and appropriation to perform, firms realise that if they perform a non-equilibrium amount then in the next period they will have an incentive to set a different mark-up to the other firms in their industry. (The clearest example of this is when we have perfect competition, in which case the most productive firm would want to price just below the second most productive firms' marginal cost.) It may be seen that in non-symmetric equilibrium the optimal price satisfies:

$$P_t(i,j) = \frac{W_t}{A_t(i,j)}\left[1 + \frac{\eta\lambda}{1-(1-\eta)\frac{1}{J_{t-1}(i)}\left(\frac{P_t(i,j)}{P_t(i)}\right)^{-\frac{1}{\eta\lambda}}}\right].$$

Since we are looking for a symmetric equilibrium, it is sufficient to approximate this locally around $P_t(i) = P_t(i,j)$ in order to calculate firms' research and appropriation incentives. Taking a log-linear approximation of $\log P_t(i,j)$ in $\frac{P_t(i,j)}{P_t(i)}$ gives us that:

$$P_t(i,j) \approx \frac{W_t}{A_t(i,j)}\left(1+\mu_{t-1}(i)\right)\left(\frac{P_t(i,j)}{P_t(i)}\right)^{-\omega_{t-1}(i)}$$

where $\omega_t(i) \coloneqq \frac{J_t(i)(1-\eta)}{\left(J_t(i)-(1-\eta)\right)^2\left(1+\mu_t(i)\right)}$ captures the strength of these incentives to deviate from setting the same mark-up as all other firms in their industry. Therefore $P_t(i) \approx \frac{W_t}{A_t(i)}\left(1+\mu_{t-1}(i)\right)$ and $P_t(i,j) \approx \frac{W_t}{A_t(i,j)}\left(1+\mu_{t-1}(i)\right)\left(\frac{A_t(i,j)}{A_t(i)}\right)^{\frac{\omega_{t-1}(i)}{1+\omega_{t-1}(i)}}$ where:

$$A_t(i) \coloneqq \left[\frac{1}{J_{t-1}(i)}\sum_{j=1}^{J_{t-1}(i)} A_t(i,j)^{\frac{1}{\eta\lambda\left(1+\omega_{t-1}(i)\right)}}\right]^{\eta\lambda\left(1+\omega_{t-1}(i)\right)}.$$

Therefore, up to a first order approximation around the symmetric solution, profits are given by:

$$\beta\frac{1}{I_t J_t(i)}\left(\frac{1+\mu_t}{1+\mu_t(i)}\right)^{\frac{1}{\lambda}}\mathbb{E}_t\Xi_{t+1}Y_{t+1}\left[\left(\frac{A_{t+1}(i,j)}{A_{t+1}(i)}\right)^{\frac{\omega_t(i)}{1+\omega_t(i)}} - \frac{1}{1+\mu_t(i)}\right]\left(\frac{A_{t+1}(i,j)}{A_{t+1}(i)}\right)^{\frac{1-\eta\lambda\omega_t(i)}{\eta\lambda\left(1+\omega_t(i)\right)}}\left(\frac{A_{t+1}(i)}{A_{t+1}}\right)^{\frac{1}{\lambda}}$$
$$- \left[L_t^{\mathrm{R}}(i,j) + L_t^{\mathrm{A}}(i,j) + L_t^{\mathcal{R}}(i) + L^{\mathrm{F}}\right]W_t.$$

Note that if $J_t(i) > \frac{2\sqrt{2}(3-\sqrt{2})}{1+2\sqrt{2}} \approx 1.17$, then $1-\eta\lambda\omega_t(i) > 0$ (by tedious algebra), so providing there are at least two firms in the industry, this expression is guaranteed to be increasing and concave in $A_{t+1}(i,j)$.

Let $m_t^{\mathrm{R}}(i,j)W_t$ be the Lagrange multiplier on research's positivity constraint and $m_t^{\mathrm{A}}(i,j)W_t$ be the Lagrange multiplier on appropriation's positivity constraint. Then in a symmetric equilibrium the two first order conditions and the free entry condition (respectively) mean:

$$\beta\frac{1}{I_tJ_t(i)}\frac{\mu_t(i)}{1+\mu_t(i)}\left(\frac{1+\mu_t}{1+\mu_t(i)}\right)^{\frac{1}{\lambda}}\mathbb{E}_t\Xi_{t+1}Y_{t+1}\left(\frac{A_{t+1}(i)}{A_{t+1}}\right)^{\frac{1}{\lambda}}\frac{d_t(i)}{\mu_t(i)}\frac{Z_{t+1}A_t^{**}(i)^{-\zeta^{\mathrm{R}}}\Psi}{1+\gamma Z_{t+1}A_t^{**}(i)^{-\zeta^{\mathrm{R}}}\Psi L_t^{\mathrm{R}}(i)}$$
$$=W_t\left(1-m_t^{\mathrm{R}}(i)\right)$$

$$\beta\frac{1}{I_tJ_t(i)}\frac{\mu_t(i)}{1+\mu_t(i)}\left(\frac{1+\mu_t}{1+\mu_t(i)}\right)^{\frac{1}{\lambda}}\mathbb{E}_t\Xi_{t+1}Y_{t+1}\left(\frac{A_{t+1}(i)}{A_{t+1}}\right)^{\frac{1}{\lambda}}\frac{d_t(i)}{\mu_t(i)}\frac{1+(\gamma-\zeta^{\mathrm{R}})Z_{t+1}A_t^{**}(i)^{-\zeta^{\mathrm{R}}}\Psi L_t^{\mathrm{R}}(i)}{1+\gamma Z_{t+1}A_t^{**}(i)^{-\zeta^{\mathrm{R}}}\Psi L_t^{\mathrm{R}}(i)}$$
$$\cdot\frac{1}{\tau}\frac{A_t(i)^{-\zeta^{\mathrm{A}}}\Upsilon(A_t^{*\tau}-A_t(i)^{\tau})}{A_t^{**}(i)^{\tau}\left(1+A_t(i)^{-\zeta^{\mathrm{A}}}\Upsilon L_t^{\mathrm{A}}(i)\right)^2}=W_t\left(1-m_t^{\mathrm{A}}(i)\right)$$

$$\beta\frac{1}{I_tJ_t(i)}\frac{\mu_t(i)}{1+\mu_t(i)}\left(\frac{1+\mu_t}{1+\mu_t(i)}\right)^{\frac{1}{\lambda}}\mathbb{E}_t\Xi_{t+1}Y_{t+1}\left(\frac{A_{t+1}(i)}{A_{t+1}}\right)^{\frac{1}{\lambda}}=[L_t^{\mathrm{R}}(i,j)+L_t^{\mathrm{A}}(i,j)+L_t^{\mathcal{R}}(i)+L_t^{\mathrm{F}}]W_t$$

where:

$$d_t(i)\coloneqq 1-\frac{\omega_t(i)}{1+\omega_t(i)}\frac{\left(\lambda-\mu_t(i)\right)(\mu_t(i)-\eta\lambda)}{\lambda(1-\eta)\mu_t(i)}<1$$

and where we have dropped j indices on variables which are the same across the industry.

We also have that:

$$\frac{\left(\lambda-\mu_t(i)\right)(\mu_t(i)-\eta\lambda)}{\lambda(1-\eta)\mu_t(i)}\le\frac{\lambda\left(1-\sqrt{\eta}\right)\left(\sqrt{\eta}-\eta\right)}{\sqrt{\eta}}<\lambda$$

so providing $\lambda<1$, $d_t(i)>0$.

That the solution for research when $Z_{t+1}\equiv 1$ is given by equation (3.2) is a trivial consequence of the complementary slackness condition and the facts that $\frac{1}{\mu_t(i)}<\gamma$ and $d_t(i)<1$. Deriving (3.3) is less trivial though.

Begin by defining $k_t(i)\coloneqq\frac{1+(\gamma-\zeta^{\mathrm{R}})\mathcal{L}_t^{\mathrm{R}}(i)}{1+\gamma\mathcal{L}_t^{\mathrm{R}}(i)}$, and note that since we are assuming $\gamma>\zeta^{\mathrm{R}}\ge 0$, we have that $0<k_t(i)\le 1$.

Also define:

$$n_t(i)\coloneqq\frac{d_t(i)k_t(i)}{\tau\mu_t(i)}A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon\left[\left(\frac{A_t^*}{A_t^*(i)}\right)^{\tau}-1\right][L_t^{\mathrm{R}}(i)+L_t^{\mathcal{R}}(i)+L_t^{\mathrm{F}}]\ge 0,$$

which is not a function of $L_t^{\mathrm{A}}(i)$, given $L_t^{\mathrm{R}}(i)$.

We can then combine the appropriation first order condition with the free entry condition to obtain:

$$\frac{1}{\left(1+\mathcal{L}_t^{\mathrm{A}}(i)\right)^2}\left(\frac{A_t^*(i)}{A_t^{**}(i)}\right)^{\tau}\left[\frac{d_t(i)k_t(i)}{\tau\mu_t(i)}\left[\left(\frac{A_t^*}{A_t^*(i)}\right)^{\tau}-1\right]\mathcal{L}_t^{\mathrm{A}}(i)+n_t(i)\right]=1-m_t^{\mathrm{A}}(i).$$

Since the left hand side is weakly positive, from the dual feasibility condition we know $\mathcal{m}_t^{\mathrm{A}}(i) \in [0,1]$. Now when $L_t^{\mathrm{A}}(i) = 0$, this becomes:

$$\mathcal{n}_t(i) = 1 - \mathcal{m}_t^{\mathrm{A}}(i),$$

since in this case $A_t^*(i) = A_t^{**}(i)$. Therefore when $L_t^{\mathrm{A}}(i) = 0$, $\mathcal{n}_t(i) \leq 1$.

We now prove the converse. Suppose then for a contradiction that $L_t^{\mathrm{A}}(i) > 0$, but $\mathcal{n}_t(i) \leq 1$. By complementary slackness, we must have $\mathcal{m}_t^{\mathrm{A}}(i) = 0$, hence:

$$\begin{aligned}
1 \geq \mathcal{n}_t(i) &= \left(1 + \mathcal{L}_t^{\mathrm{A}}(i)\right)^2 \left(\frac{A_t^{**}(i)}{A_t^*(i)}\right)^\tau - \frac{d_t(i)\mathcal{k}_t(i)}{\tau\mu_t(i)}\left[\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau - 1\right]\mathcal{L}_t^{\mathrm{A}}(i) \\
&\geq \left(1 + \mathcal{L}_t^{\mathrm{A}}(i)\right)^2 \left(\frac{A_t^{**}(i)}{A_t^*(i)}\right)^\tau - \left[\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau - 1\right]\mathcal{L}_t^{\mathrm{A}}(i) \\
&= \left(1 + \mathcal{L}_t^{\mathrm{A}}(i)\right)\left[\left(1 + \mathcal{L}_t^{\mathrm{A}}(i)\right) + \mathcal{L}_t^{\mathrm{A}}(i)\left[\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau - 1\right]\right] - \left[\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau - 1\right]\mathcal{L}_t^{\mathrm{A}}(i),
\end{aligned}$$

where we have used the facts that $d_t(i)\mathcal{k}_t(i) \leq 1$ and $\frac{1}{\mu_t(i)} < \tau$ to derive the second inequality.

Expanding the brackets then gives that:

$$1 \geq 1 + 2\mathcal{L}_t^{\mathrm{A}}(i) + \left(\frac{A_t^*}{A_t^*(i)}\right)^\tau \mathcal{L}_t^{\mathrm{A}}(i)^2,$$

i.e. that $0 \geq 2 + \left(\frac{A_t^*}{A_t^*(i)}\right)^\tau \mathcal{L}_t^{\mathrm{A}}(i)$ which is a contradiction as $\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau \mathcal{L}_t^{\mathrm{A}}(i) \geq 0$.

We have proven then that providing $\frac{1}{\mu_t(i)} < \tau$, $L_t^{\mathrm{A}}(i) = 0$ if and only if $\mathcal{n}_t(i) \leq 1$. It just remains for us to solve for $L_t^{\mathrm{A}}(i)$ when it is strictly positive. From the above, we have that, in this case:

$$\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau [\mathcal{n}_t(i) - 1] = 2\left[1 - \frac{1}{2}\left[1 + \frac{d_t(i)\mathcal{k}_t(i)}{\tau\mu_t(i)}\right]\left[1 - \left(\frac{A_t^*(i)}{A_t^*}\right)^\tau\right]\right]\mathcal{L}_t^{\mathrm{A}}(i) + \mathcal{L}_t^{\mathrm{A}}(i)^2.$$

Hence:

$$\begin{aligned}
\mathcal{L}_t^{\mathrm{A}}(i) = &-\left[1 - \frac{1}{2}\left[1 + \frac{d_t(i)\mathcal{k}_t(i)}{\tau\mu_t(i)}\right]\left[1 - \left(\frac{A_t^*(i)}{A_t^*}\right)^\tau\right]\right] \\
&+ \sqrt{\left[1 - \frac{1}{2}\left[1 + \frac{d_t(i)\mathcal{k}_t(i)}{\tau\mu_t(i)}\right]\left[1 - \left(\frac{A_t^*(i)}{A_t^*}\right)^\tau\right]\right]^2 + \left(\frac{A_t^*(i)}{A_t^*}\right)^\tau [\mathcal{n}_t(i) - 1]},
\end{aligned}$$

since the lower solution is guaranteed to be negative as $\mathcal{n}_t(i) > 1$ when $L_t^{\mathrm{A}}(i) > 0$.

7.2. The steady state for non-patent-protected industries

In an industry i which is not patent-protected and in which appropriation, but no research, is performed, from (3.1) and (3.3):

$$\mathscr{f}_t(i)+\sqrt{\mathscr{f}_t(i)^2+\mathscr{g}_t(i)}=\mathcal{L}_t^{\mathrm{A}}(i)=\left[1-\frac{\left(\frac{A_{t+1}^*(i)}{A_t^*(i)}\right)^\tau-1}{1-\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau}\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau\right]^{-1}-1.$$

If we treat $\mathfrak{p}_1:=\tau\frac{\mu_t(i)}{d_t(i)}-1\approx 0$, $\mathfrak{p}_2:=A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon_t L_t^{\mathrm{F}}\approx 0$ and $\mathfrak{p}_3:=\left(\frac{A_{t+1}^*(i)}{A_t^*(i)}\right)^\tau-1\approx 0$ as fixed, this leaves us with a cubic in $\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau$, for which only one solution will be feasible (i.e. strictly less than 1). Taking a second order Taylor approximation of this solution in $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$, reveals (after some messy computation), that:

$$\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau\approx\mathfrak{p}_2\big(1-(\mathfrak{p}_1+\mathfrak{p}_2)\big)=A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon_t L_t^{\mathrm{F}}\left(2-\tau\frac{\mu_t(i)}{d_t(i)}-A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon_t L_t^{\mathrm{F}}\right)$$

(The effect of $\mathfrak{p}_3$ on $\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau$ is third order and hence it does not appear in this expression.)

From this approximate solution for $\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau$ then, we have that the relative productivity of a non-protected industry is decreasing in its mark-up. Furthermore, from dropping to a first order approximation, we have that $A_t^*(i)^{1+\frac{\zeta^{\mathrm{A}}}{\tau}}\approx A_t^*(\Upsilon_t L_t^{\mathrm{F}})^{\frac{1}{\tau}}$, so asymptotically non-protected industries are growing at $\left[1+\frac{\zeta^{\mathrm{A}}}{\tau}\right]^{-1}$ times the growth rate of the frontier.

7.3. The inventor-firm bargaining process

We model the entire process of setting and paying rents as follows:

1) Firms enter, paying the fixed cost.
2) Firms who have entered conduct appropriation, then research.
3) The "idea shock" for next period's production, Z_{t+1}, is realised and firms and patent holders learn its level.
4) Finally, firms arrive at the patent-holder to conduct bargaining, with these arrivals taking place sequentially but in a random order. (For example, all firms phone the patent-holder sometime in the week before production is to begin.) In this bargaining we suppose that the patent-holder has greater bargaining power, since they have a longer outlook[61] and since they lose nothing if bargaining collapses[62]. We also suppose that neither patent-holders nor firms are able to

[61] Consider what happens as the time gap between offers increases. When this gap is large enough only one offer would be made per-period, meaning the patent-holder would make a take-it-or-leave-it offer giving (almost) nothing to the firm, which the firm would then accept.

[62] The firm owner may, for example, face restrictions from starting businesses in future if as a result of the bargaining collapse they are unable to repay their creditors.

observe or verify either how many (other) firms paid the fixed cost, or what research and appropriation levels they chose. This is plausible because until production begins it is relatively easy to keep such things hidden (for example, by purchasing the licence under a spin-off company), and because it is hard to ascertain ahead of production exactly what product a firm will be producing. We assume bargaining takes an alternating offer form, (Rubinstein 1982) but that it happens arbitrarily quickly (i.e. in the no discounting limit).

5) Firms pay the agreed rents if bargaining was successful. Since this cost is expended before production, we continue to suppose firms have to borrow in the period before production in order to cover it. Firms will treat it as a fixed cost, sunk upon entry, since our unobservability assumptions mean bargaining's outcome will not be a function of research and appropriation levels.
6) The next period starts, other aggregate shocks are realised and production takes place.
7) The patent-holder brings court cases against any firms who produced but decided not to pay the rent. For simplicity, we assume the court always orders the violating firm to pay damages to the patent-holder, which are given as follows:
    a) When the courts believe rents were not reasonable (i.e. $L_t^{\mathcal{R}}(i) > L_t^{\mathcal{R}*}(i)$, where $L_t^{\mathcal{R}*}(i)W_t$ is the level courts determine to be "reasonable royalties"), they set damages greater than $L_t^{\mathcal{R}*}(i)W_t$, as "*the infringer would have nothing to lose, and everything to gain if he could count on paying only the normal, routine royalty non-infringers might have paid*"[63]. We assume excess damages over $L_t^{\mathcal{R}*}(i)W_t$ are less than the patent-holder's legal costs however.
    b) When the courts consider the charged rent to have been reasonable (i.e. $L_t^{\mathcal{R}}(i) \leq L_t^{\mathcal{R}*}(i)$) the courts award punitive damages of more than $\max\left\{ L_t^{\mathcal{R}*}(i)W_t, \left(\frac{1}{1-\wp}\right) L_t^{\mathcal{R}}(i)W_t, \right\}$, where $\wp$ is the bargaining power of the firm, in the sense of the generalized Nash bargaining solution.[64]

Under this specification:

$$L_t^{\mathcal{R}}(i) = \min\{L_t^{\mathcal{R}*}(i), (1-\wp)[L_t^{\mathrm{R}}(i) + L_t^{\mathrm{A}}(i) + L_t^{\mathcal{R}}(i) + L^{\mathrm{F}}]\}$$

since entry is fixed when bargaining takes place, since patent-holders know that bargaining to a rent level any higher than $L_t^{\mathcal{R}*}(i)W_t$ will just result in them having to pay legal costs,[65] and since

[63] Panduit Corp. v. Stahlin Brothers Fibre Works, Inc., 575 F.2d 1152, 1158 (6th Circuit 1978), cited in Pincus (1991).

[64] The level $\left(\frac{1}{1-\wp}\right) L_t^{\mathfrak{R}}(i)W_t$ is chosen to ensure that, with equilibrium rents, firms prefer not to produce at all rather than to produce without paying rents.

[65] The disagreement point is zero since it is guaranteed that $L_t^{\mathfrak{R}}(i) \leq L_t^{\mathfrak{R}*}(i)$ and so punitive damages would be awarded were the firm to produce without paying rents, which, by construction, leaves them worse off than not producing.

$[L_t^{\mathrm{R}}(i) + L_t^{\mathrm{A}}(i) + L_t^{\mathcal{R}}(i) + L^{\mathrm{F}}]W_t$ is equal to the production period profits of each firm in industry i, by the free entry condition.[66] Therefore, in equilibrium:

$$L_t^{\mathcal{R}}(i) = \min\{L_t^{\mathcal{R}*}(i), L_t^{\mathcal{R}\dagger}(i)\}, \tag{7.1}$$

where $L_t^{\mathcal{R}\dagger}(i)$ is a solution to equations (3.2), (3.3) along with equation (3.4), (i.e. $L_t^{\mathcal{R}}(i) = \frac{1-\wp}{\wp}[L_t^{\mathrm{R}}(i) + L_t^{\mathrm{A}}(i) + L^{\mathrm{F}}]$) if one exists, or $+\infty$ otherwise. Because damages are always greater than $L_t^{\mathcal{R}*}(i)W_t$, these rents will be sufficiently low to ensure firms are always prepared to licence the patent at the bargained price in equilibrium.

Now suppose we are out of equilibrium and fewer firms than expected have entered. Since neither the patent-holder nor firms can observe how many firms have entered, and since firms arrive at the patent-holder sequentially, both sides will continue to believe that the equilibrium number of firms has entered and so rents will not adjust. On the other hand, suppose that (out of equilibrium) too many firms enter. When the first unexpected firm arrives at the patent-holder to negotiate, the patent-holder will indeed realise that too many firms have entered. However, since the firm they are bargaining with has no way of knowing this,[67] the patent-holder can bargain for the same rents as in equilibrium. Therefore, even out of equilibrium:

$$L_t^{\mathcal{R}}(i) = \min\{L_t^{\mathcal{R}*}(i), L_t^{\mathcal{R}\dagger}(i)\}$$

where we stress $L_t^{\mathcal{R}\dagger}(i)$ is not a function of the decisions any firm happened to take. This ensures that any solution of equations (3.2), (3.3) and (7.1) for research, appropriation and rents will also be an equilibrium, even allowing for the additional condition that the derivative of firm profits with respect to the number of firms must be negative at an optimum.

We now just have to pin down "reasonable royalties", $L_t^{\mathcal{R}*}(i)W_t$. Certainly it must be the case that $L_t^{\mathcal{R}*}(i) \le L_t^{\bar{\mathcal{R}}}(i)$, where $L_t^{\bar{\mathcal{R}}}(i)$ is the level of rents at which $J_t(i) = 1$, since rents so high that no one is prepared to pay them must fall foul of the courts' desire to ensure licensees can make a profit.[68] However, since when $J_t(i) = 1$ the sole entering firm (almost) may as well be the patent-holder themselves, where possible the courts will set $L_t^{\mathcal{R}*}(i)$ sufficiently low to ensure that $J_t(i) > 1$ in equilibrium, again following the idea that licensees ought to be able to make a profit. When there is a $J_t(i) > 1$ solution to equations (3.2), (3.3) and (3.4) already (i.e. $L_t^{\mathcal{R}\dagger}(i) < \infty$), the courts will just set $L_t^{\mathcal{R}*}(i)$ at the rent level that would obtain in that solution, thus preventing the possibility of

[66] A similar expression can also be derived if we assume instead that courts guarantee infringers a fraction $\wp$ of production profits, or if we assume courts always award punitive damages but firms are able to hide a fraction $\wp$ of their production profits.

[67] Either they are a firm that thinks the equilibrium number of firms has entered, or they are a firm that thinks more than the equilibrium number of firms has entered, but that does not know whether the patent-holder has yet realised this.

[68] "*…the very definition of a reasonable royalty assumes that, after payment, the infringer will be left with a profit*." Georgia-Pacific Corp. v. U.S. Plywood-Champion Papers Corp., 446 F.2d 295, 299 & n.1 (2d Cir.), *cert. denied,* 404 U.S. 870 (1971), cited in Pincus (1991).

$J_t(i) = 1$ being an equilibrium. It may be shown that for sufficiently large t such a solution is guaranteed to exist, so in this case $L_t^{\mathcal{R}*}(i) = L_t^{\mathcal{R}\dagger}(i) = L_t^{\mathcal{R}}(i)$.[69]

7.4. The de-trended model

Below we give the equations of the stationary model to which the model described in section 3 converges as $t \to \infty$.

7.4.1. Households

- **Stochastic discount factor:** $\Xi_t = \frac{\Theta_t \hat{C}_{t-1}}{\Theta_{t-1}\hat{C}_t G_{A,t}}$, where $\hat{C}_t \coloneqq \frac{C_t}{N_t A_t}$ is consumption per person in labour supply units and $G_{V,t}$ is the exponent of the growth rate of the variable V_t at t.
- **Labour supply:** $\Phi_t {\hat{L}_t^{\mathrm{S}}}^{\nu} = \frac{\widehat{W}_t}{\hat{C}_t}$, where $\hat{L}_t^{\mathrm{S}} \coloneqq \frac{L_t^{\mathrm{S}}}{N_t}$ is labour supply per person and $\widehat{W}_t \coloneqq \frac{W_t}{A_t}$ is the wage per effective unit of labour supply.
- **Euler equation:** $\beta R_t \mathbb{E}_t[\Xi_{t+1}] = 1$, where R_t is the real interest rate.

7.4.2. Aggregate relationships

- **Aggregate mark-up pricing:** $\widehat{W}_t = \frac{1}{1+\mu_{t-1}}$ where μ_{t-1} is the aggregate mark-up in period t.
- **Mark-up aggregation:** $\left(\frac{1}{1+\mu_t}\right)^{\frac{1}{\lambda}} = \left(\frac{1}{1+\mu_t^{\mathrm{P}}}\right)^{\frac{1}{\lambda}} \mathscr{s}_t + \left(\frac{1}{1+\eta\lambda}\right)^{\frac{1}{\lambda}}(1-\mathscr{s}_t)$, where $\mu_t^{\mathrm{P}} = \mu_t(I_t)$ is the mark-up in any protected industry at $t+1$, and $\mathscr{s}_t \coloneqq (1-\mathscr{q})\frac{\mathscr{s}_{t-1}}{G_{I,t}} + 1 - \frac{1}{G_{I,t}}$ is the proportion of industries that will produce a patent protected product in period $t+1$.
- **Productivity aggregation:** $\left(\frac{\hat{A}_t}{1+\mu_{t-1}}\right)^{\frac{1}{\lambda}} = \left(\frac{1}{1+\mu_{t-1}^{\mathrm{P}}}\right)^{\frac{1}{\lambda}} \mathscr{s}_{t-1} + \left(\frac{\hat{A}_t^{\mathrm{N}}}{1+\eta\lambda}\right)^{\frac{1}{\lambda}}(1-\mathscr{s}_{t-1})$, where $\hat{A}_t \coloneqq \frac{A_t}{A_t^*}$ is aggregate productivity relative to the frontier[70] and
$\hat{A}_t^{\mathrm{N}} \coloneqq \left[\left(\frac{1}{G_{A^*,t}}\right)^{\frac{1}{\lambda}}\left(\frac{\mathscr{q}}{1/\mathscr{s}_{t-2}-(1-\mathscr{q})}\right) + \left(\frac{\hat{A}_{t-1}^{\mathrm{N}}}{G_{A^*,t}}\right)^{\frac{1}{\lambda}}\left(1-\frac{\mathscr{q}}{1/\mathscr{s}_{t-2}-(1-\mathscr{q})}\right)\right]^{\lambda}$ is the aggregate relative productivity of non-protected industries.

7.4.3. Firm decisions

- **Strategic in-industry pricing:** $\mu_t^{\mathrm{P}} = \lambda\frac{\eta \hat{J}_t^{\mathrm{P}}}{\hat{J}_t^{\mathrm{P}}-(1-\eta)}$, where $\hat{J}_t^{\mathrm{P}} \coloneqq J_t(I_t)$ is the number of firms in a protected industry performing research at t.

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[69] There may still be multiple solutions for rents (as (3.2), (3.3) and (3.4) might have multiple solutions), but of these only the one with minimal entry is really plausible, since this is both weakly Pareto dominant (firms always make zero profits and it may be shown that the patent-holder prefers minimal entry) and less risky for entering firms (if entering firms are unsure if the patent-holder will play the high rent or the low rent equilibrium, they are always better off assuming the high rent one since if that assumption is wrong they make strict profits, whereas had they assumed low rents but rents were in fact high they would make a strict loss).

[70] As a consequence, we have that $G_{A,t} = \frac{\hat{A}_t}{\hat{A}_{t-1}} G_{A^*,t}$.

- **Firm research decisions:** $\frac{d_t}{\wp\mu_t^{\mathrm{P}}}\mathbb{E}_t\Xi_{t+1}G_{Y,t+1}\hat{A}_{t+1}^{-\frac{1}{\lambda}}\frac{Z_{t+1}\hat{\mathcal{L}}_t^{\mathrm{R}}}{1+\gamma Z_{t+1}\hat{\mathcal{L}}_t^{\mathrm{R}}} = (1-\mathcal{m}_t^{\mathrm{R}})\mathbb{E}_t\Xi_{t+1}G_{Y,t+1}\hat{A}_{t+1}^{-\frac{1}{\lambda}}$, where $\hat{\mathcal{L}}_t^{\mathrm{R}} \coloneqq A_t^{*-\zeta}\Psi L_t^{\mathrm{R}}$ is the amount of effective research conducted by firms in protected industries d_t is the value of $d_t(i)$ in protected industries and Z_t is the aggregate research-return shock. (This equation means that $\hat{\mathcal{L}}_t^{\mathrm{R}} \approx \frac{\wp\mu_t^{\mathrm{P}}}{d_t-\wp\gamma\mu_t^{\mathrm{P}}}$.)
- **Research and appropriation payoff:** $G_{A^*,t} = \left(1+\gamma Z_t\hat{\mathcal{L}}_{t-1}^{\mathrm{R}}\right)^{\frac{1}{\gamma}}$.
- **Free entry of firms:** $\beta\frac{1}{\hat{I}_t\hat{J}_t^{\mathrm{P}}}\frac{\mu_t^{\mathrm{P}}}{1+\mu_t^{\mathrm{P}}}\left(\frac{1+\mu_t}{1+\mu_t^{\mathrm{P}}}\right)^{\frac{1}{\lambda}}\mathbb{E}_t\Xi_{t+1}G_{Y,t+1}\hat{A}_{t+1}^{-\frac{1}{\lambda}} = \frac{1}{\wp}\hat{\mathcal{L}}_t^{\mathrm{R}}\frac{\widehat{W}_t}{\hat{Y}_t}$, where $\hat{I}_t \coloneqq \frac{I_t}{N_tA_t^{*-\zeta}\Psi}$ is the measure of products relative to its trend,[71] and $\hat{Y}_t \coloneqq \frac{Y_t}{N_tA_t}$ is output per person in labour supply units.

7.4.4. Inventor decisions

- **Inventor profits:** are given recursively by:

$$\hat{\pi}_t = \frac{1-\wp}{\wp}\hat{\mathcal{L}}_t^{\mathrm{R}}\widehat{W}_t\hat{J}_t^{\mathrm{P}} + \beta(1-\mathcal{q})\mathbb{E}_t\Xi_{t+1}G_{A,t+1}G_{A^*,t+1}^{\zeta}\hat{\pi}_{t+1},\text{ where } \hat{\pi}_t \coloneqq \frac{\pi_t}{A_tA_t^{*\zeta}}.$$

- **Free entry of inventors:** Either $G_{I,t} \geq 1$ binds or $\Psi E^{\zeta}\mathcal{L}^{\mathrm{I}}\widehat{W}_t \geq \hat{\pi}_t$ does.

7.4.5. Market clearing

- **Labour market clearing:** $\hat{L}_t^{\mathrm{S}} = \Psi E^{\zeta}\mathcal{L}_t^{\mathrm{I}}\hat{I}_t\left(1-\frac{1}{G_{I,t}}\right) + \hat{I}_t\mathcal{s}_t\hat{J}_t^{\mathrm{P}}\hat{\mathcal{L}}_t^{\mathrm{R}} + \hat{Y}_t\left[\left(\frac{1}{\hat{A}_t}\right)^{\frac{1}{\lambda_t}}\left(\frac{1+\mu_{t-1}}{1+\mu_{t-1}^{\mathrm{P}}}\right)^{\frac{1+\lambda}{\lambda}}\mathcal{s}_{t-1} + \left(\frac{\hat{A}_t^{\mathrm{N}}}{\hat{A}_t}\right)^{\frac{1}{\lambda_t}}\left(\frac{1+\mu_{t-1}}{1+\eta\lambda}\right)^{\frac{1+\lambda}{\lambda}}(1-\mathcal{s}_{t-1})\right]$.
- **Goods market clearing:** $\hat{Y}_t = \hat{C}_t$.

7.5. The extended de-trended model

Define $\mathcal{a} \coloneqq \frac{1}{(1-\alpha_{\mathrm{P}})(1-\iota_{\mathrm{P}})}$, $\mathcal{b} \coloneqq (1-\alpha_{\mathrm{R}})(1-\iota_{\mathrm{R}})$, $\mathcal{c} \coloneqq \left(\frac{1-\alpha_{\mathrm{R}}}{1-\alpha_{\mathrm{P}}}\alpha_{\mathrm{P}}\xi_{\mathrm{KP}} - \alpha_{\mathrm{R}}\xi_{\mathrm{KR}}\right)(1-\iota_{\mathrm{R}})$, $\mathcal{e} \coloneqq \xi_{\mathrm{L}} + \frac{\alpha_{\mathrm{P}}}{1-\alpha_{\mathrm{P}}}\xi_{\mathrm{KP}}$ and make the normalisation $\Psi = E = 1$.

7.5.1. Households

- **Budget constraint Lagrange multiplier:** $\frac{1}{\hat{\tilde{C}}_t} = \widehat{\mathcal{m}}_t^{\mathrm{C}} + \beta\mathcal{h}\mathcal{h}^{\mathrm{INT}}\mathbb{E}_t\frac{N_{t+1}\Theta_{t+1}}{N_t\Theta_t}\frac{1}{G_{A,t+1}^{\mathcal{a}}\hat{\tilde{C}}_{t+1}}$, where $\frac{\widehat{\mathcal{m}}_t^{\mathrm{C}}}{A_t^{\mathcal{a}}A_t^{*\mathcal{e}}N_t}$ is the Lagrange multiplier on the budget constraint and $\hat{\tilde{C}}_t \coloneqq \frac{\tilde{C}_t}{A_t^{\mathcal{a}}A_t^{*\mathcal{e}}} = \hat{C}_t - \mathcal{h}\frac{\hat{C}_{t-1}}{G_{A,t}^{\mathcal{a}}G_{A^*,t}^{\mathcal{e}}}$.
- **Stochastic discount factor:** $\Xi_t = \frac{\Theta_t\widehat{\mathcal{m}}_t^{\mathrm{C}}}{\Theta_{t-1}\widehat{\mathcal{m}}_{t-1}^{\mathrm{C}}G_{A,t}^{\mathcal{a}}G_{A^*,t}^{\mathcal{e}}}$.

[71] This means $G_{I,t} = G_{N,t}G_{A^*,t}^{-\zeta}\frac{\hat{I}_t}{\hat{I}_{t-1}}$.

- **Labour supply:** $(1+\lambda_{\mathrm{L}})\hat{\mathcal{w}}_{1,t} = \hat{\widetilde{W}}_t^{1+\nu\frac{1+\lambda_{\mathrm{L}}}{\lambda_{\mathrm{L}}}}\hat{\mathcal{w}}_{2,t}$, where $\hat{\widetilde{W}}_t := \left[\frac{\widehat{W}_t^{-\frac{1}{\lambda_{\mathrm{L}}}} - \upsilon\left(\frac{G_P}{G_{P,t}}\frac{G_W}{G_{A,t}^{a}G_{A^*,t}^{e}}\widehat{W}_{t-1}\right)^{-\frac{1}{\lambda_{\mathrm{L}}}}}{1-\upsilon}\right]^{-\lambda_{\mathrm{L}}}$ ($\hat{\widetilde{W}}_t A_t^a A_t^{*e}$ is the real wage set by a household that updates its wage at t), $\widehat{W}_t := \frac{W_t}{A_t^a A_t^{*e}}$, and where $\hat{\mathcal{w}}_{1,t}$ and $\hat{\mathcal{w}}_{2,t}$ are the sums of costs and benefits respectively from the wage setting first order conditions.[72]
- **Euler equation:** $\beta R_t \mathbb{E}_t[\Xi_{t+1}] = 1$.
- **Investment decisions:** for $\mathrm{V} \in \{\mathrm{P,R}\}$:
$\frac{1}{E_t^{\mathrm{KV}}} = \Gamma_t \frac{\hat{R}_t^{\mathrm{KV}}}{\delta^{\mathrm{V}\prime}(u_t^{\mathrm{V}})}\left[1 - Q^{\mathrm{KV}}\left(G_{I^{\mathrm{KV}*},t}\right) - G_{I^{\mathrm{KV}},t}Q^{\mathrm{KV}\prime}\left(G_{I^{\mathrm{KV}*},t}\right)\right] + \beta\mathbb{E}_t\Xi_{t+1}\Gamma_{t+1}\frac{\hat{R}_{t+1}^{\mathrm{KV}}}{G_{A^*,t}^{\xi_{\mathrm{KV}}}\delta^{\mathrm{V}\prime}(u_{t+1}^{\mathrm{V}})}G_{I^{\mathrm{KV}*},t+1}^2 Q^{\mathrm{KV}\prime}\left(G_{I^{\mathrm{KV}*},t+1}\right)$, where $\hat{R}_t^{\mathrm{KV}} := R_t^{\mathrm{KV}}A_t^{*\xi_{\mathrm{KV}}}$ and $G_{I^{\mathrm{KV}*},t} = G_{A^*,t}^{\xi_{\mathrm{KV}}}\frac{E_t^{\mathrm{KV}}}{E_{t-1}^{\mathrm{KV}}}G_{I^{\mathrm{KV}},t}$
- **Utilisation decisions:** for $\mathrm{V} \in \{\mathrm{P,R}\}$: $\frac{\hat{R}_t^{\mathrm{KV}}}{\delta^{\mathrm{V}\prime}(u_t^{\mathrm{V}})} = \beta\mathbb{E}_t\Xi_{t+1}\frac{\hat{R}_{t+1}^{\mathrm{KV}}}{G_{A^*,t}^{\xi_{\mathrm{KV}}}}\left[u_{t+1}^{\mathrm{V}} + \frac{1-\delta^{\mathrm{V}}(u_{t+1}^{\mathrm{V}})}{\delta^{\mathrm{V}\prime}(u_{t+1}^{\mathrm{V}})}\right]$.
- **Capital accumulation:** for $\mathrm{V} \in \{\mathrm{P,R}\}$: $\widehat{K}_t^{\mathrm{V}} = \left(1-\delta^{\mathrm{V}}\left(u_t^{\mathrm{V}}\right)\right)\frac{\widehat{K}_{t-1}^{\mathrm{V}}}{G_{N,t}G_{A,t}^{a}G_{A^*,t}^{e+\xi_{\mathrm{KV}}}} + \Gamma_t E_t^{\mathrm{KV}}\hat{I}_t^{\mathrm{KV}}\left[1 - Q^{\mathrm{KV}}\left(G_{I^{\mathrm{KV}},t}\right)\right]$, where $\widehat{K}_t^{\mathrm{V}} := \frac{K_t^{\mathrm{V}}}{N_t A_t^a A_t^{*e+\xi_{\mathrm{KV}}}}$ and $\hat{I}_t^{\mathrm{KV}} = \frac{I_t^{\mathrm{KV}}}{N_t A_t^a A_t^{*e}}$ (hence $G_{I^{\mathrm{KV}},t} = G_{N,t}G_{A,t}^{a}G_{A^*,t}^{e}\frac{\hat{I}_t^{\mathrm{KV}}}{\hat{I}_{t-1}^{\mathrm{KV}}}$).

7.5.2. Aggregate relationships

- **Aggregate mark-up pricing:** $\frac{\left[R_t^{\mathrm{KP}\,\alpha_{\mathrm{P}}}\widehat{W}_t^{\mathrm{EP}\,1-\alpha_{\mathrm{P}}}\right]^{1-\iota_{\mathrm{P}}}}{\iota_{\mathrm{P}}^{\iota_{\mathrm{P}}}(1-\iota_{\mathrm{P}})^{1-\iota_{\mathrm{P}}}\left[\alpha_{\mathrm{P}}^{\alpha_{\mathrm{P}}}(1-\alpha_{\mathrm{P}})^{1-\alpha_{\mathrm{P}}}\right]^{1-\iota_{\mathrm{P}}}} = \frac{1}{1+\mu_{t-1}}$ where $\widehat{W}_t^{\mathrm{EP}} := \frac{\widehat{W}_t}{E_t^{\mathrm{L}}\left[1-Q^{\mathrm{LP}}\left(\frac{\hat{L}_t^{\mathrm{TP}}}{\hat{L}_{t-1}^{\mathrm{TP}}}G_{N,t}G_{A^*,t}^{\xi_L}\right)\right]}$ and $\hat{L}_t^{\mathrm{TP}} = \frac{L_t^{\mathrm{TP}}}{N_t A_t^{*\xi_L}}$, where $L_t^{\mathrm{T}} := A_t^{*\xi_{\mathrm{L}}}E_t^{\mathrm{L}}L_t^{\mathrm{S}}$.
- **Mark-up aggregation:** $\left(\frac{1}{1+\mu_t}\right)^{\frac{1}{\lambda}} = \left(\frac{1}{1+\mu_t^{\mathrm{P}}}\right)^{\frac{1}{\lambda}}\mathcal{s}_t + \left(\frac{1}{1+\eta\lambda}\right)^{\frac{1}{\lambda}}(1-\mathcal{s}_t)$, where $\mu_t^{\mathrm{P}} = \mu_t(I_t)$ and $\mathcal{s}_t := (1-\mathcal{q})\frac{\mathcal{s}_{t-1}}{G_{I,t}} + 1 - \frac{1}{G_{I,t}}$.
- **Productivity aggregation:** $\left(\frac{\hat{A}_t}{1+\mu_{t-1}}\right)^{\frac{1}{\lambda}} = \left(\frac{1}{1+\mu_{t-1}^{\mathrm{P}}}\right)^{\frac{1}{\lambda}}\mathcal{s}_{t-1} + \left(\frac{\hat{A}_t^{\mathrm{N}}}{1+\eta\lambda}\right)^{\frac{1}{\lambda}}(1-\mathcal{s}_{t-1})$, where $\hat{A}_t := \frac{A_t}{A_t^*}$ and $\hat{A}_t^{\mathrm{N}} := \left[\left(\frac{1}{G_{A^*,t}}\right)^{\frac{1}{\lambda}}\left(\frac{\mathcal{q}}{1/\mathcal{s}_{t-2}-(1-\mathcal{q})}\right) + \left(\frac{\hat{A}_{t-1}^{\mathrm{N}}}{G_{A^*,t}}\right)^{\frac{1}{\lambda}}\left(1 - \frac{\mathcal{q}}{1/\mathcal{s}_{t-2}-(1-\mathcal{q})}\right)\right]^{\lambda}$.

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[72] $\hat{\mathcal{w}}_{1,t} = \Phi_t\left(\widehat{W}_t^{\frac{1+\lambda_{\mathrm{L}}}{\lambda_{\mathrm{L}}}}\tilde{L}_t^{\mathrm{S}}\right)^{\nu} + \beta\upsilon\mathbb{E}_t\frac{\Theta_{t+1}N_{t+1}}{\Theta_t N_t}\left(\frac{G_P}{G_{P,t+1}}\frac{G_W}{G_{W,t+1}}\right)^{-\frac{1+\lambda_{\mathrm{L}}}{\lambda_{\mathrm{L}}}}\frac{\tilde{L}_{t+1}^{\mathrm{S}}}{\tilde{L}_t^{\mathrm{S}}}\left(\frac{G_P}{G_{P,t+1}}\frac{G_W}{G_{A,t+1}^{a}G_{A^*,t+1}^{e}}\right)^{-\nu\frac{1+\lambda_{\mathrm{L}}}{\lambda_{\mathrm{L}}}}\hat{\mathcal{w}}_{1,t+1}$, $\hat{\mathcal{w}}_{2,t} = \widehat{mc}_t^{\mathrm{C}} + \beta\upsilon\mathbb{E}_t\frac{\Theta_{t+1}N_{t+1}}{\Theta_t N_t}\left(\frac{G_P}{G_{P,t+1}}\frac{G_W}{G_{W,t+1}}\right)^{-\frac{1+\lambda_{\mathrm{L}}}{\lambda_{\mathrm{L}}}}\frac{\tilde{L}_{t+1}^{\mathrm{S}}}{\tilde{L}_t^{\mathrm{S}}}\frac{G_P}{G_{P,t+1}}\frac{G_W}{G_{A,t+1}^{a}G_{A^*,t+1}^{e}}\hat{\mathcal{w}}_{2,t+1}$. This formulation avoids any explicit log-linearization and allows us to compute arbitrarily high order approximations to the model, for robustness checks. A similar formulation is used in Schmitt-Grohé and Uribe (2006).

7.5.3. Firm decisions

- **Strategic in-industry pricing:** $\mu_t^{\mathrm{P}} = \lambda \frac{\eta \hat{J}_t^{\mathrm{P}}}{\hat{J}_t^{\mathrm{P}} - (1-\eta)}$, where $\hat{J}_t^{\mathrm{P}} = J_t(I_t)$.
- **Firm research decisions:** $\frac{d_t}{\mathscr{p} \mu_t^{\mathrm{P}}} \mathbb{E}_t \Xi_{t+1} G_{Y,t+1} \hat{A}_{t+1}^{-\frac{1}{\lambda}} \frac{Z_{t+1} \hat{\mathcal{L}}_t^{\mathrm{R}}}{1+\gamma Z_{t+1} \hat{\mathcal{L}}_t^{\mathrm{R}}} = (1 - \mathscr{m}_t^{\mathrm{R}}) \mathbb{E}_t \Xi_{t+1} G_{Y,t+1} \hat{A}_{t+1}^{-\frac{1}{\lambda}}$, where $\hat{\mathcal{L}}_t^{\mathrm{R}} \coloneqq A_t^{*-\zeta} X_t^{\mathrm{R}\,\iota_R} \left[K_t^{\mathrm{R}\,\alpha_R} L_t^{\mathrm{R}\,1-\alpha_R} \right]^{1-\iota_R}$ is the amount of effective research conducted by firms in protected industries.
- **Research and appropriation payoff:** $G_{A^*,t} = \left(1 + \gamma Z_t \hat{\mathcal{L}}_{t-1}^{\mathrm{R}}\right)^{\frac{1}{\gamma}}$.
- **Free entry of firms:** $\beta \frac{1}{\hat{I}_t \hat{J}_t^{\mathrm{P}}} \frac{\mu_t^{\mathrm{P}}}{1+\mu_t^{\mathrm{P}}} \left(\frac{1+\mu_t}{1+\mu_t^{\mathrm{P}}} \right)^{\frac{1}{\lambda}} \mathbb{E}_t \Xi_{t+1} G_{Y,t+1} \hat{A}_{t+1}^{-\frac{1}{\lambda}} = \frac{1}{\mathscr{p}} \hat{\mathcal{L}}_t^{\mathrm{R}} \frac{\hat{\mathcal{C}}_t}{\hat{Y}_t}$, where $\hat{I}_t \coloneqq \frac{I_t}{N_t A_t^{\mathscr{a}(1-\mathscr{b})} A_t^{*\mathscr{e}-(\mathscr{c}+\zeta)}}$ is the measure of products relative to its trend,[73] $\hat{Y}_t^{\mathrm{GROSS}} \coloneqq \frac{Y_t^{\mathrm{GROSS}}}{N_t A_t^{\mathscr{a}} A_t^{*\mathscr{e}}}$ is *gross* output relative to trend and $\hat{\mathcal{C}}_t \coloneqq \frac{\left[R_t^{\mathrm{KR}\,\alpha_{\mathrm{R}}} \widehat{W}_t^{\mathrm{ER}\,1-\alpha_{\mathrm{R}}} \right]^{1-\iota_{\mathrm{R}}}}{\iota_{\mathrm{R}}^{\iota_{\mathrm{R}}} (1-\iota_{\mathrm{R}})^{1-\iota_{\mathrm{R}}} \left[\alpha_{\mathrm{R}}^{\alpha_{\mathrm{R}}} (1-\alpha_{\mathrm{R}})^{1-\alpha_{\mathrm{R}}} \right]^{1-\iota_{\mathrm{R}}}}$ is the marginal cost of a unit of research or invention, divided by $A_t^{\mathscr{a}\mathscr{b}} A_t^{*\mathscr{c}}$ (where $\widehat{W}_t^{\mathrm{ER}} \coloneqq \frac{\widehat{W}_t}{E_t^{\mathrm{L}} \left[1 - Q^{\mathrm{LR}} \left(\frac{\hat{L}_t^{\mathrm{TR}}}{\hat{L}_{t-1}^{\mathrm{TR}}} G_{N,t} G_{A^*,t}^{\xi_L} \right) \right]}$ and $\hat{L}_t^{\mathrm{TR}} = \frac{L_t^{\mathrm{TR}}}{N_t A_t^{*\xi_L}}$).

7.5.4. Inventor decisions

- **Inventor profits:** are given recursively by:
  $\hat{\pi}_t = \frac{1-\mathscr{p}}{\mathscr{p}} \hat{\mathcal{L}}_t^{\mathrm{R}} \hat{\mathcal{C}}_t \hat{J}_t^{\mathrm{P}} + \beta (1 - \mathscr{q}) \mathbb{E}_t \Xi_{t+1} G_{A,t+1}^{\mathscr{a}\mathscr{b}} G_{A^*,t+1}^{\mathscr{c}+\zeta} \hat{\pi}_{t+1}$, where $\hat{\pi}_t \coloneqq \frac{\pi_t}{A_t^{\mathscr{a}\mathscr{b}} A_t^{*\mathscr{c}+\zeta}}$.
- **Free entry of inventors:** Either $G_{I,t} \geq 1$ binds or $\mathcal{L}_t^{\mathrm{I}} \hat{\mathcal{C}}_t \geq \hat{\pi}_t$ does.

7.5.5. Market clearing

- **R&D expenditure:** $\mathrm{RND}_t \coloneqq \hat{\mathcal{C}}_t \hat{I}_t \left[\mathcal{L}_t^{\mathrm{I}} \left(1 - \frac{1}{G_{I,t}} \right) + \hat{\mathcal{L}}_t^{\mathrm{R}} \mathscr{s}_t \hat{J}_t^{\mathrm{P}} \right]$.
- **Labour market clearing:** $E_t^{\mathrm{L}} \hat{L}_t^{\mathrm{S}} = \hat{L}_t^{\mathrm{TY}} + \hat{L}_t^{\mathrm{TR}}$, where $\hat{L}_t^{\mathrm{S}} \coloneqq \frac{L_t^{\mathrm{T}}}{A_t^{*\xi_{\mathrm{L}}} N_t E_t^{\mathrm{L}}}$.
- **Production labour market clearing:** $\widehat{W}_t \hat{L}_t^{\mathrm{TY}} = E_t^{\mathrm{L}} (1 - \alpha_{\mathrm{P}})(1 - \iota_{\mathrm{P}}) \mathcal{J}_t \hat{Y}_t^{\mathrm{GROSS}}$ where $\mathcal{J}_t \coloneqq \frac{\mathscr{s}_{t-1}}{1+\mu_{t-1}^{\mathrm{P}}} \left(\frac{1}{\hat{A}_t} \frac{1+\mu_{t-1}}{1+\mu_{t-1}^{\mathrm{P}}} \right)^{\frac{1}{\lambda}} + \frac{1-\mathscr{s}_{t-1}}{1+\eta\lambda} \left(\frac{\hat{A}_t^{\mathrm{N}}}{\hat{A}_t} \frac{1+\mu_{t-1}}{1+\eta\lambda} \right)^{\frac{1}{\lambda}}$ is a weighted measure of average inverse gross mark-ups.
- **R&D labour market clearing:** $\widehat{W}_t \hat{L}_t^{\mathrm{TR}} = E_t^{\mathrm{L}} (1 - \alpha_{\mathrm{R}})(1 - \iota_{\mathrm{R}}) \mathrm{RND}_t$.
- **Capital markets clearing:** $u_t^{\mathrm{P}} \widehat{K}_{t-1}^{\mathrm{P}} \hat{R}_t^{\mathrm{KP}} = \alpha_{\mathrm{P}} (1 - \iota_{\mathrm{P}}) \mathcal{J}_t \hat{Y}_t^{\mathrm{GROSS}}$, $u_t^{\mathrm{R}} \widehat{K}_{t-1}^{\mathrm{R}} \hat{R}_t^{\mathrm{KR}} = \alpha_{\mathrm{R}} (1 - \iota_{\mathrm{R}}) \mathrm{RND}_t$
- **Goods market clearing:** $\hat{Y}_t = \hat{Y}_t^{\mathrm{GROSS}} (1 - \iota_{\mathrm{P}} \mathcal{J}_t) - \iota_{\mathrm{R}} \mathrm{RND}_t - (1 - \varrho_{\mathrm{GDP}}) \hat{I}_t^{\mathrm{R}} = \hat{C}_t + \hat{I}_t^{\mathrm{P}} + \varrho_{\mathrm{GDP}} \hat{I}_t^{\mathrm{R}}$, where $\hat{Y}_t$ is GDP over $N_t A_t^{\mathscr{a}} A_t^{*\mathscr{e}}$ and ϱ_{GDP} specifies the proportion of R&D capital investment that is measured in GDP. (Given R&D itself is not measured in GDP it is not obvious that this equals 1.)

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[73] This means $G_{I,t} = G_{N,t} G_{A,t}^{\mathscr{a}(1-\mathscr{b})} G_{A^*,t}^{\mathscr{e}-(\mathscr{c}+\zeta)} \frac{\hat{I}_t}{\hat{I}_{t-1}}$.

- **Monetary rule:**

$$\frac{R_t^{\text{NOM}}}{R^{\text{NOM}}} = \left(\frac{R_t^{\text{NOM}}}{R^{\text{NOM}}}\right)^{\rho_{R^{\text{NOM}}}} \left[\left(\frac{G_{P,t}}{G_{P,t}^*}\right)^{\mathcal{M}_{\text{P}}} \left(\frac{E_{t-1}^{\text{KP}} G_{A^*}^{\xi_{\text{KP}}}}{E_t^{\text{KP}} G_{A^*,t}^{\xi_{\text{KP}}}}\right)^{\mathcal{M}_{\text{PKP}}} \left(\frac{E_{t-1}^{\text{KR}} G_{A^*}^{\xi_{\text{KR}}}}{E_t^{\text{KR}} G_{A^*,t}^{\xi_{\text{KR}}}}\right)^{\mathcal{M}_{\text{PKR}}} \left(\frac{G_{W,t}}{G_W}\right)^{\mathcal{M}_{\text{W}}} \left(\frac{\hat{R}_t^{\text{KP}}}{\hat{R}^{\text{KP}}}\right)^{\mathcal{M}_{\text{RKP}}} \left(\frac{\hat{R}_t^{\text{KR}}}{\hat{R}^{\text{ KR}}}\right)^{\mathcal{M}_{\text{RKR}}} \Theta_t^{\mathcal{M}_\Theta} \tilde{\delta}_t^{-\mathcal{M}_{\tilde{\delta}}}\right]^{1-\rho_{R^{\text{NOM}}}} \cdot \left[\left(\frac{\hat{Y}_t}{\hat{Y}}\right)^{\mathcal{M}_{\text{Y}}} \left(\frac{G_{Y,t}/G_{N,t}}{G_Y/G_N}\right)^{\mathcal{M}_{\text{G}}}\right]^{1-\rho_{R^{\text{NOM}}}} \exp \epsilon_{R^{\text{NOM}},t}.$$

7.5.6. Observation equations

- **Nominal output growth:** $g_{Y,t} + g_{P,t} + \text{me}_{\text{Y},t} - \text{me}_{\text{Y},t-1}$, where $g_{Y,t} = \log\left(\frac{\hat{Y}_t}{\hat{Y}_{t-1}} G_{N,t} G_{A,t}^{a} G_{A^*,t}^{e}\right)$.
- **Consumption price inflation:** $g_{P,t} + \text{me}_{\text{PC},t} - \text{me}_{\text{PC},t-1}$.
- **Investment price inflation:** $g_{P,t} + g_{P^{\text{I}},t} + \text{me}_{\text{PI},t} - \text{me}_{\text{PI},t-1}$, where:

$$G_{P^{\text{I}},t} = \sqrt{\frac{\left(\frac{E_{t-1}^{\text{KP}} \hat{I}_{t-1}^{\text{KP}}}{E_t^{\text{KP}} G_{A^*,t}^{\xi_{\text{KP}}}} + \varrho_{\text{GDP}} \frac{E_{t-1}^{\text{KR}} \hat{I}_{t-1}^{\text{KR}}}{E_t^{\text{KR}} G_{A^*,t}^{\xi_{\text{KR}}}}\right)}{\left(\hat{I}_{t-1}^{\text{KP}} + \varrho_{\text{GDP}} \hat{I}_{t-1}^{\text{KR}}\right)} \frac{\left(\hat{I}_t^{\text{KP}} + \varrho_{\text{GDP}} \hat{I}_t^{\text{KR}}\right)}{\left(\frac{E_t^{\text{KP}} \hat{I}_t^{\text{KP}} G_{A^*,t}^{\xi_{\text{KP}}}}{E_{t-1}^{\text{KP}}} + \varrho_{\text{GDP}} \frac{E_t^{\text{KR}} \hat{I}_t^{\text{KR}} G_{A^*,t}^{\xi_{\text{KR}}}}{E_{t-1}^{\text{KR}}}\right)}}.$$

- **Population growth:** $g_{N,t} + \text{me}_{\text{N},t} - \text{me}_{\text{N},t-1}$.
- **Demeaned labour supply**: $l_t^{\text{S}} + \text{me}_{\text{LS},t}$.
- **R&D share**: $\log\left(\frac{\text{RND}_t + \varrho_{\text{RND}} \hat{I}_t^R}{\hat{Y}_t}\right) + \text{me}_{\text{RND},t}$, where ϱ_{RND} is the proportion of R&D capital investment that is measured in the NIPA R&D measure. ($\varrho_{\text{GDP}} + \varrho_{\text{RND}} \leq 1$).
- **Consumption share:** $\log\left(\frac{\hat{C}_t}{\hat{Y}_t}\right) + \text{me}_{\text{C},t}$.
- **Labour share:** $\log\left(\frac{\hat{W}_t \hat{L}_t^{\text{S}}}{\hat{Y}_t}\right) + \text{me}_{\text{L},t}$.
- **Depreciation share:** $\log\left(\frac{\delta^{\text{Y}}(u_t^{\text{Y}}) \hat{K}_{t-1}^{\text{Y}}}{\hat{Y}_t\left(G_{N,t} G_{A,t}^{a} G_{A^*,t}^{e+\xi_{\text{KY}}} E_t^{\text{KY}}\right)} + \varrho_{\text{GDP}} \frac{\delta^{\text{R}}(u_t^{\text{R}}) \hat{K}_{t-1}^{\text{R}}}{\hat{Y}_t\left(G_{N,t} G_{A,t}^{a} G_{A^*,t}^{e+\xi_{\text{KR}}} E_t^{\text{KR}}\right)}\right) + \text{me}_{\text{D},t}$.
- **Demeaned nominal interest rates:** $\log\left(\frac{R_t^{\text{NOM}}}{R^{\text{NOM}}}\right) + \text{me}_{\text{R},t}$.
- **Capacity utilisation:** $\dfrac{u_t^{\text{Y}} \frac{\hat{K}_{t-1}^{\text{Y}}}{G_{A^*,t}^{\xi_{\text{KY}}} E_t^{\text{KY}}} + u_t^{\text{R}} \varrho_{\text{GDP}} \frac{\hat{K}_{t-1}^{\text{R}}}{G_{A^*,t}^{\xi_{\text{KR}}} E_t^{\text{KR}}}}{\frac{\hat{K}_{t-1}^{\text{Y}}}{G_{A^*,t}^{\xi_{\text{KY}}} E_t^{\text{KY}}} + \varrho_{\text{GDP}} \frac{\hat{K}_{t-1}^{\text{R}}}{G_{A^*,t}^{\xi_{\text{KR}}} E_t^{\text{KR}}}} + \text{me}_{\text{U},t}$. (The capital stocks enter here in order to correctly weight to produce the average utilisation.)
- **BAA-AAA Spread:** $\varsigma_0 - \varsigma_1 \log \Gamma_t + \text{me}_{\text{S},t}$.

7.6. Data details

- **Nominal output growth** (1947Q2 – 2011Q2), from NIPA table 1.1.5.

- **Consumption price inflation** (1947Q2 – 2011Q2), including non-durables and durables (from NIPA table 1.1.4) and government consumption[74] (from NIPA table 3.9.4) and excluding education[75] (from NIPA tables 2.4.4[76] and 3.15.4[77]).
- **Investment price inflation** (1947Q2 – 2011Q2), including education (data sources as for consumption price inflation).
- **Population growth** (1948Q2 – 2011Q2), X-12 seasonally adjusted, from the BLS's Civilian Non-institutional Population Over 16 series.
- **Labour supply per capita** (1948Q1 – 2011Q2), from NIPA table 6.9, interpolated to quarterly using the Litterman (1983) method, with "Business Sector: Hours of All Persons" from the BEA as a high frequency indicator.
- **R&D share** (1959Q1 – 2007Q4), given by R&D expenditure from NIPA R&D Satellite Account (1959-2007) table 2.1, over GDP from NIPA table 1.1.5, interpolated to quarterly using the Litterman (1983) method with GDP as the high frequency indicator.
- **Consumption share** (1947Q1 – 2011Q2), given by consumption of durables and non-durables (from NIPA table 1.1.5) plus government consumption (from NIPA table 3.9.5) minus education expenditure (from NIPA table 2.4.5[78] and NIPA table 3.15.5[79]) all over GDP (from NIPA table 1.1.5).[80]
- **Labour share** (1947Q1 – 2011Q2), given by compensation of employees paid from NIPA table 1.10, over GDP (from NIPA table 1.1.5).
- **Depreciation share** (1947Q1 – 2011Q2), given by consumption of fixed capital from NIPA table 1.10, over GDP (from NIPA table 1.1.5).
- **Nominal interest rates** (1947Q1 – 2011Q2), in particular, the 3-month Treasury bill secondary market rate, from the FRB, release H.15.
- **Capacity utilisation** (1967Q1 – 2011Q2), (total industry) from the FRB, release G.17, table 7.
- **BAA-AAA Spread** (1947Q1 – 2011Q2), from the FRB, release H.15.

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[74] We are implicitly making the optimistic assumption that government consumption is a perfect substitute for private consumption. This is a simplifying shortcut to save us modelling government consumption.

[75] Removing education from the consumption share brings it substantially closer to stationarity, so it is important to do the same for the price level too. The price disaggregation necessary to remove education was performed by inverting the Fisher formula, which, due to its approximate aggregation property (Diewert 1978) is sufficiently accurate.

[76] Interpolated to quarterly using the Litterman (1983) method, with consumption and investment prices as indicators (from NIPA table 1.1.4).

[77] Extrapolated back to 1947 using the Litterman (1983) method, with government consumption and investment prices (from NIPA table 3.9.4) and private education prices (from NIPA table 2.4.4) as indicators, then interpolated to quarterly using the same method with government consumption and investment prices (from NIPA table 3.9.4) as high frequency indicators.

[78] Interpolated to quarterly using the Litterman (1983) method, with consumption and investment as indicators (from NIPA table 1.1.5).

[79] Extrapolated back to 1947 using the Litterman (1983) method with log-linearly interpolated data from the National Centre for Education Statistics, Digest of Education Statistics 2010, table 29 as an indicator, along with government consumption and investment (from NIPA table 3.9.5) and private education expenditure (from NIPA table 2.4.5). Then interpolated using the same method with government consumption and investment (from NIPA table 3.9.5) as high frequency indicators.

[80] In fitting this to the model, we are implicitly treating net exports as investment.

7.7. Estimated parameters

Any parameters in bold are fixed rather than estimated. All values are reported to three significant figures, except those below 10^{-4} which are rounded down to zero, those which are of the form $1 + x$, with $|x| < 0.1$ in which case we give x to three significant figures, percentages, which are given to one decimal place, and approximate standard errors (in brackets) which are given to two significant figures.

Variable	Value	Variable	Value
ν	0.250 (0.0056)	β	$\mathbf{0.99}$
$\hbar$	0.253 (0.0041)	$\hbar^{\mathrm{INT}}$	0.0151 (0.0032)
$\hbar^{\mathrm{LS}}$	0 (0)	$\mathcal{v}$	0.826 (0.0042)
λ	0.320 (0.00054)	λ_{L}	0.170 (0.0041)
$\mathcal{p}$	0.0427 (0.00021)	$\mathcal{q}$	0.0374 (0.00030)
$\rho_{R^{\mathrm{NOM}}}$	0.615 (0.013)	$\mathcal{M}_{\mathrm{P}}$	1.0275 (0.0059)
$\mathcal{M}_{\mathrm{PKP}}$	0 (0)	$\mathcal{M}_{\mathrm{PKR}}$	0 (0)
$\mathcal{M}_{\mathrm{RKP}}$	0.0509 (0.0016)	$\mathcal{M}_{\mathrm{RKR}}$	0 (0)
$\mathcal{M}_{\Theta}$	0 (0)	$\mathcal{M}_{\tilde{\delta}}$	0.0108 (0.0074)
$\mathcal{M}_{\mathrm{Y}}$	0 (0)	$\mathcal{M}_{\mathrm{G}}$	0 (0)
$\mathcal{M}_{\mathrm{W}}$	0 (0)		
$\exp \varsigma_0$	2.57 (2.9×10^{-5})	ς_1	872 (880)
ϱ_{GDP}	0.494 (0.013)	ϱ_{RND}	0.506 (0.013)
ζ	0 (0)	ξ_{L}	0.0859 (0.0012)
ξ_{KP}	0.0828 (0.00053)	ξ_{KR}	2.73 (0.0094)
α_{P}	0.201 (0.00040)	α_{R}	0.996 (7.4×10^{-6})
ι_{P}	0.0427 (0.0011)	ι_{R}	0.178 (0.0032)
$\delta^{\mathrm{P}}(u^{\mathrm{P}})$	0.0189 (7.5×10^{-5})	$\delta^{\mathrm{R}}(u^{\mathrm{R}})$	0.0284 (0.00062)
$\delta^{\mathrm{P}\prime}(u^{\mathrm{P}})$	0.0413 (0.00011)	$\delta^{\mathrm{R}\prime}(u^{\mathrm{R}})$	0.0501 (0.00063)
$\delta^{\mathrm{P}\prime\prime}(u^{\mathrm{P}})$	1.64 (0.035)	$\delta^{\mathrm{R}\prime\prime}(u^{\mathrm{R}})$	133 (9.4)
$\frac{d}{d\tilde{\delta}} \log \delta^{\mathrm{P}}(u^{\mathrm{P}})$	$\mathbf{1}$	$\frac{d}{d\tilde{\delta}} \log \delta^{\mathrm{R}}(u^{\mathrm{R}})$	64.2 (1.5)
$\frac{d}{d\tilde{\delta}} \log \delta^{\mathrm{P}\prime}(u^{\mathrm{P}})$	64.2 (1.5)	$\frac{d}{d\tilde{\delta}} \log \delta^{\mathrm{R}\prime}(u^{\mathrm{R}})$	0 (0)
$Q^{\mathrm{P}\prime\prime}(G_{I^{\mathrm{KP}*}})$	0.00533 (0.0012)	$Q^{\mathrm{R}\prime\prime}(G_{I^{\mathrm{KR}*}})$	62.6 (4.0)
$Q^{\mathrm{LP}\prime}(G_{L^{\mathrm{TP}}})$	0.0875 (0.0047)	$Q^{\mathrm{LR}\prime}(G_{L^{\mathrm{TR}}})$	0 (0)

Table 4: Estimated parameters, excluding shocks.

Variable	V (i.e. steady-state)	ρ_V	$100\sigma_V$	p-value on 1 lag LM-test[81]
Φ	1.0349 (0.0047)	0.815 (0.010)	2.46 (0.16)	0
Θ	**1**	0.443 (0.0056)	0.0231 (0.0114)	**0.0318**
G_{N}	1.00372 (1.4×10^{-5})	0.0675 (0.019)	0.103 (0.0021)	**0.146**
$\mathcal{L}^{\mathrm{I}}$	7.26 (0.034)	0 (0)	0 (0)	0
Z	**1**	**0**	0 (0)	0
Γ	**1**	0 (0)	0 (0)	0.000926
E^{L}	**1**	0.614 (0.0056)	0 (0)	**0.725**
E^{KP}	**1**	0 (0)	0 (0)	0
E^{KR}	**1**	0.664 (0.0071)	0.000360 (0.00012)	0.000148
$G^{*}_{P,t}$	1.00851 (6.1×10^{-6})	0.887 (0.00027)	0 (0)	**0.161**
η	0.169 (0.00024)	0.0605 (0.2)	0.0147 (0.012)	0
γ	18.6 (0.054)	0 (0)	0 (0)	0
$\exp\tilde{\delta}$	**1**	0.862 (0.0027)	0.403 (0.011)	**0.958**
R_t^{SHOCK}	**1**	**0**	0.00824 (0.00075)	**0.464**

Table 5: Estimated parameters from non-m.e. shocks, tests of misspecification of their residuals. Each shock takes the form $\log V_t = (1-\rho_V)\log V + \rho_V \log V_{t-1} + \sigma_V \epsilon_{V,t}$, where $\epsilon_{v,t} \sim \mathrm{NIID}(0,1)$.

Variable	Φ	Θ	G_{N}	$\mathcal{L}^{\mathrm{I}}$	Z	Γ	E^{L}	E^{KP}	E^{KR}	$G^{*}_{P,t}$	η	γ	R_t^{SHOCK}	$\exp\tilde{\delta}$
Nom. output growth	**17.9**	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	**81.1**
Con. price inflation	**37.5**	0.0	**2.5**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.4	**59.6**
Inv. price inflation	**37.1**	0.0	**2.4**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.4	**60.1**
Population growth	0.0	0.0	**100.0**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Lab. supply per capita	**60.4**	0.0	0.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	**38.8**
R&D share	**2.1**	0.0	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**97.6**
Consumption share	**45.0**	0.0	0.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.3	**54.3**
Labour share	**1.8**	0.0	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**97.9**
Depreciation share	**45.3**	0.0	**1.8**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	**52.7**
Nominal interest rates	**41.5**	0.0	**2.0**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	**56.4**
Capacity utilisation	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**99.9**
BAA-AAA Spread	0.0	0.0	0.0	0.0	0.0	**100.0**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 6: Percentage non-m.e. variance decomposition of the observation variables.[82]

[81] Bold values indicate the cases in which we cannot reject the null hypothesis of no auto-correlation at 1%. The test uses heteroskedasticity robust standard errors. The lag length of 1 was preferred by the AIC, AICc and BIC criterions for all variables.

[82] Bold values are larger than 1%.