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BOOTSTRAPPING TESTS FOR JUMPS WITH AN APPLICATION TO TEST AVERAGING

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Bootstrapping tests for jumps with an application to test averaging *

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Abstract

We propose new bootstrap methods for the Barndorff-Nielsen and Shephard (2006) and Andersen et al. (2012) tests for jumps, as well as for the realized bipower variation and the median realized variation.[1] Both the i.i.d. and the Wild bootstrap are considered. We prove CLT-type results for the couples: realized volatility-realized bipower variation and realized volatility-median realized variation. Based on these results, we build bootstrapped tests for jumps.

We introduce a new jump-testing procedure that uses Fisher (1932)'s method to average p-values from one/ different tests applied at different sampling frequencies. The procedure is proven to be more efficient than applying the asymptotic tests, as we discard less data and extract information from multiple frequencies and/ or procedures. We use a double bootstrap procedure to control the overall size of the test.

Keywords: jumps, nonparametric tests, bootstrap, realized bipower variation, median realized variation

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[1]Further versions of the paper will include consistency results for the realized bipower variation and the median realized variation.

1 Introduction

During the past decade, the increasing availability of high frequency data led to the development of a new class of nonparametric estimators of volatility. The first and most popular such estimator is the realized variance (volatility), introduced by Andersen and Bollerslev (1998). Several other estimator were subsequently proposed to accommodate some other features of the data. One key data characteristic that must be taken into account is the presence of jumps in prices. Thus, several robust to jumps volatility estimators were proposed by Barndorff-Nielsen and Shephard (2004); Mancini (2009); Corsi et al. (2010); Andersen et al. (2012). These contributions led to further developments in the area of testing for jumps based on high frequency data: Andersen et al. (2007), Andersen et al. (2012), Aït-Sahalia and Jacod (2008), Barndorff-Nielsen and Shephard (2006), Corsi et al. (2010), Jiang and Oomen (2008), Lee and Mykland (2008), and Podolskij and Ziggel (2010).

Some of the latest advances in the field of high frequency econometrics concern the development of bootstrap methods for some realized-type estimators. Thus, Gonçalves and Meddahi (2009) propose bootstrap methods for the realized volatility and a class of nonlinear transformations of this estimator. Dovonon et al. (2013) propose bootstrap methods for functions of multivariate high frequency returns such as realized regression coefficients and realized covariances and correlations. Hounyo et al. (2011b) bootstrap estimators of the integrated volatility based on the pre-averaging approach of Jacod et al. (2009). Hounyo et al. (2011a) propose bootstrapping the pre-averaging realized volatility proposed by Podolskij and Vetter (2009). The existing literature lacks in proposing bootstrap methods for robust to jumps volatility estimators and tests for jumps. Podolskij and Ziggel (2007) propose bootstrapping the realized bipower variation by re-sampling the products of adjacent returns in absolute value, $|r_j r_{j-1}|$, where $j = 1 \dots n$, with n the number of equidistant returns in a trading day. This procedure, however, ignores the correlation between the above pairs of returns. Moreover, there are no contributions in the literature on bootstrapping tests for jumps.

In this paper, we propose bootstrap methods for the Barndorff-Nielsen and Shephard (2006) and Andersen et al. (2012)tests for jumps. Moreover, a future version of this paper will provide consistency results for the bootstrapped realized bipower variation of Barndorff-Nielsen and Shephard (2004) and the median realized variation of Andersen et al. (2012). We consider both the i.i.d. and wild bootstrap techniques. These developments are meant to fill the gap in the literature mentioned above and lead to accuracy improvements when applying these techniques in finite sample. Moreover, the ability to bootstrap tests for jumps opens the possibility to further improvements to the way these tests are applied based on combinations of tests and or sampling frequencies. In fact, the second main contribution of this paper is to propose a test averaging technique that requires bootstrap and substantially improves the performance of the jump tests.

As showed above, the literature on tests for jumps is very rich. However, when applied to real or simulated price data, the jump testing procedures tend to lead to different results. Dumitru and Urga (2012); Theodosiou and Žikeš (2010) perform thorough comparisons between various jump testing procedures and attempt to rank them considering both the power and size criteria. This inconsistency of results is partly due to the way tests are built, leading to different levels of size and power. Moreover, the inconsistency is deepened at very high sampling frequencies, where securities prices are contaminated with microstructure noise. The noise generates both size and power distortions for all jump tests. So far, the majority of the existing literature unequivocally proposed sub-sampling as a unique solution to overcome the problems generated by the presence of noise. As an exception, Dumitru and Urga (2012) point out that sub-sampling leads to loss of power and inefficiency, by "throwing away" a lot of data. They propose combining various tests and/ or sampling frequencies through both reunions and intersections in order to extract more information on jump occurrence. However, this multiple testing procedure, despite over-performing the existing single-testing procedures, lacks rigor, by not providing a proper inference.

This paper proposes a new jump detection procedure, that manages to rigorously combine results obtained at different sampling frequencies and from different jump tests.This new procedure is more efficient, by extracting information from multiple sampling frequencies and/or multiple procedures. The use of different time scales in the high frequency literature is not entirely new. Zhang et al. (2005) and Zhang (2006) use two or more time scales to estimate volatility in the presence of microstructure noise. While in their case, the main purpose was to extract noise from the final volatility estimate, here we attempt to extract more information on the occurrence of jumps. We use Fisher's method to average p-values obtained from applying the same testing procedure at different frequencies. We use bootstrap to obtain the empirical distribution of the Fisher statistic. In order to control the size of the test, a double bootstrap procedure is required when combining different frequencies. We apply this new jump detection procedure to the Barndorff-Nielsen and Shephard (2006) and Andersen et al. (2012) tests.

The rest of the paper is organized as follows. Section 2 explains the theoretical background and past relevant contributions in the field of high frequency testing for jumps. Section 3 contains the new bootstrap methods for the tests for jumps. Section 4 covers all methodological issues concerning jump identification based on averaging p-values. Section 5 contains simulation results for the new procedure. Section 6 concludes the paper.

2 Setup, notation and existing theory

Following Barndorff-Nielsen and Shephard (2006), the logarithmic price process, P_t, is a Brownian semimartingale plus jump process:

$$\mathrm{d}P_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t + \mathrm{d}J_t \tag{1}$$

where the drift, μ_t, and the volatility, σ_t, are assumed càdlàg, and W_t a Brownian motion at time t. J_t is the jump process at time t, defined as $J_t = \sum_{j=1}^{N_t} c_{t_j}$ where c_{t_j} represents the size of the jump at time t_j and N_t is a counting process, representing the number of jumps up to time t and assumed to be finite for all t.

The quadratic variation of the price process up to a certain point in time, t, (QV_t), usually a trading day, can be defined as follow:

$$QV_t = \int_0^t \sigma_s^2 \mathrm{d}s + \sum_{j=1}^{N_t} c_{t_j}^2, \tag{2}$$

where $\int_0^t \sigma_s^2 \mathrm{d}s = IV_t$ is the integrated variance or volatility.

There are several estimators in the field of high frequency econometrics for both the quadratic variance and the integrated volatility of the price. Most of these estimators are based on equally spaced data. Thus, the interval $[0, t]$ is split into n equal subintervals of length δ. The j-th intraday return r_j on day t is defined as $r_j = p_{t-1+j\delta} - p_{t-1+(j-1)\delta}$.

Andersen and Bollerslev (1998) proposed RV_t to estimate the quadratic variance:

$$RV_t = \sum_{j=1}^{n} r_j^2 \xrightarrow{p} QV_t, \qquad \text{for} \quad \delta \to 0 \tag{3}$$

where $\xrightarrow{p}$ stands for convergence in probability.

2.1 Tests for jumps

The Barndorff-Nielsen and Shephard (2006) test (BNS test hereafter) Barndorff-Nielsen and Shephard (2006) propose the first robust to jumps estimator of the integrated variance, the realized bipower variation (BV_t), constructed to reduce the impact of jump returns on the volatility estimate by multiplying them with adjacent jump-free returns:

$$BV_t = \frac{\pi}{2} \sum_{j=2}^{n} |r_j||r_{j-1}| \tag{4}$$

In the absence of jumps ($N_t = 0$), both RV_t and BV_t consistently estimate the integrated variance. Barndorff-Nielsen and Shephard (2006) establish a CLT for RV_t and BV_t, when $N_t = 0$:

$$\delta^{-1/2}\left[\begin{pmatrix} RV_t \\ \frac{\pi}{2}BV_t \end{pmatrix} - \begin{pmatrix} \int_0^t \sigma_s^2 \mathrm{d}s \\ \int_0^t \sigma_s^2 \mathrm{d}s \end{pmatrix}\right] \xrightarrow{L} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \int_0^t \sigma_s^4 \mathrm{d}s \begin{pmatrix} 2 & 2 \\ 2 & \frac{\pi^2}{4}+\pi-3 \end{pmatrix}\right) \tag{5}$$

where $\xrightarrow{L}$ stands for convergence in law.

We can test for jumps by comparing RV_t with BV_t under the null of no jumps. This leads to a Hausman-type test, where RV_t is more efficient, but consistent only under the null, whereas BV_t is consistent under both hypotheses. Barndorff-Nielsen and Shephard (2006) propose the difference statistics, whereas Huang and Tauchen (2005) point out that the use of the ratio statistic leads to a less oversized test. $\int_0^t \sigma_s^4 \mathrm{d}s$ is usually estimated by using the tripower quarticity, TQ_t:

$$TQ_t = n\,1.74\left(\frac{n}{n-2}\right)\sum_{j=3}^{n} |r_{j-2}|^{4/3}|r_{j-1}|^{4/3}|r_j|^{4/3}. \tag{6}$$

In this paper, we will consider the following feasible difference and ratio statistics:

$$\frac{\delta^{-1/2}(RV_t - BPV_t)}{\sqrt{0.61\,TQ_t}} \tag{7}$$

$$\frac{1-\frac{BV_t}{RV_t}}{\sqrt{0.61\,\delta\,\max\left(1, \frac{TQ_t}{BV_t^2}\right)}} \tag{8}$$

The Andersen et al. (2012) test (Med test hereafter) Andersen et al. (2012) propose to estimate integrated volatility in the presence of jumps based on the nearest neighbour truncation. The minimum realized variance ($MinRV_t$) and median realized variance ($MedRV_t$) eliminate jumps by taking respectively the minimum and the median over adjacent returns:

$$\begin{aligned} MinRV_t &= 2.75\,\tfrac{n}{n-1}\textstyle\sum_{j=2}^{n} \min(|r_j|, |r_{j-1}|)^2 \\ MedRV_t &= 1.42\,\tfrac{n}{n-2}\textstyle\sum_{j=3}^{n} \mathrm{med}(|r_j|, |r_{j-1}|, |r_{j-2}|)^2. \end{aligned} \tag{9}$$

Here we consider the test for jumps based on the $MedRV_t$ estimator. The test statistic is based on the same argument as the BNS procedure, i.e. the comparison between a robust to jumps estimator and RV_t. For simplicity, we will only refer to the ratio test:

$$\frac{1-\frac{MedRV_t}{RV_t}}{\sqrt{0.96\ \delta\ \max\left(1, \frac{MedRQ_t}{MedRV_t^2}\right)}} \xrightarrow{L} \mathcal{N}(0,1), \tag{10}$$

where $MedRQ_t = 0.92\frac{n^2}{n-2}\sum_{j=3}^{n} \text{med}(|r_j|, |r_{j-1}|, |r_{j-2}|)^4$ the median realized quarticity which estimate the integrated quarticity.

3 The bootstrap for the BNS and Med tests for jumps

In this section, we propose i.i.d. and WILD bootstrap methods for the BNS and Med tests. As Gonçalves and Meddahi (2009) explain, intraday returns are independent, but usually heteroskedastic, making the wild bootstrap (WB hereafter) the appropriate bootstrap method. However, at the same time, i.i.d. bootstrap is valuable as a benchmark and if we assume that intraday volatility does not vary substantially.

We denote the bootstrap intraday return r_j^*. In the case of the i.i.d. bootstrap, r_j^* is i.i.d. from $\{r_j : j = 1, \ldots, n\}$. For the WB, $r_j^* = r_j \cdot \eta_j$, where η_j are i.i.d. with moments given by $\mu_q^* = E^*|\eta_j|^q$. P^* denotes the probability measure under bootstrap re-sampling, conditional on the original sample. Let E^* and Var^* denote the expected value and the variance under the P^* probability measure.

3.1 Consistency of the bootstrap for the BNS test

I.i.d. bootsrap In order to be able to bootstrap one of the test statistics in equations 7 and 8, we prove a CLT-type result similar to the one in equation 5. The bootstrap realized variance and realized bipower variation are defined as follows:

$$RV_t^* = \sum_{j=1}^{n} r_j^{*2} \tag{11}$$

$$BV_t^* = \frac{\pi}{2}\sum_{j=2}^{n} |r_j^*||r_{j-1}^*| \tag{12}$$

Gonçalves and Meddahi (2009) derive a CLT result for the bootstrapped realized variance. Following their steps, we first compute the expected values and variances of RV_t^* and BV_t^*, as well as the covariances between the two, under the bootstrap measure, P^*.

$$\begin{aligned} E^*(RV_t^*) &= RV_t \\ E^*(BV_t^*) &= \tfrac{\pi}{2}\tfrac{n-1}{n^2}\left(\sum_{i=1}^n |r_i|\right)^2 \\ Var^*(RV_t^*) &= \sum_{i=1}^n r_i^4 - \tfrac{1}{n}RV_t^2 \\ Var^*(BV_t^*) &= \tfrac{\pi^2}{4}\left[\tfrac{n-1}{n^2}RV_t^2 + \tfrac{2(n-2)}{n^3}RV_t\left(\sum_{i=1}^n |r_i|\right)^2 - \tfrac{3n-5}{n^4}\left(\sum_{i=1}^n |r_i|\right)^4\right] \\ cov^*(RV_t^*, BV_t^*) &= \tfrac{\pi}{2}\left[2\tfrac{n-1}{n^2}\sum_{i=1}^n |r_i^3|\sum_{i=1}^n |r_i| - \tfrac{n-1}{n^3}RV_t^2\left(\sum_{i=1}^n |r_i|\right)^2\right] \end{aligned} \tag{13}$$

As seen above, in the case of i.i.d. bootstrap, the BV_t^* statistic is not centered in BV_t. Moreover, the variances of both bootstrapped estimators are different from the variances of the original estimators. These discrepancies occur because for the original estimators, the asymptotics are based on a local Gaussian assumption, that does not longer hold in the case of i.i.d. bootstrap (Gonçalves and Meddahi, 2009).

To estimate the variances and covariances of RV_t^* and BV_t^*, we define the following consistent estimators:

$$\begin{aligned} \widehat{Var^*(RV_t^*)} &= \sum_{i=1}^n r_i^{*4} - \tfrac{1}{n}\sum_{i=1}^n r_i^{*2} \\ \widehat{Var^*(BV_t^*)} &= \tfrac{\pi^2}{4}\left[\tfrac{n-1}{n^2}\left(\sum_{i=1}^n r_i^{*2}\right)^2 + \tfrac{2(n-2)}{n^3}\sum_{i=1}^n r_i^{*2}\left(\sum_{i=1}^n |r_i^*|\right)^2 - \tfrac{3n-5}{n^4}\left(\sum_{i=1}^n |r_i^*|\right)^4\right] \\ cov^*(\widehat{RV_t^*, BV_t^*}) &= \tfrac{\pi}{2}\left[2\tfrac{n-1}{n^2}\sum_{i=1}^n |r_i^{*3}|\sum_{i=1}^n |r_i| - \tfrac{n-1}{n^3}\sum_{i=1}^n r_i^{*2}\left(\sum_{i=1}^n |r_i^*|\right)^2\right] \end{aligned} \tag{14}$$

In the above equations, each power variation -type of sum, $\sum_{i=1}^n |r_i{}^m|$, $m \in \{1, 2, 3, 4\}$, is estimated using the bootstrapped counterpart, $\sum_{i=1}^n |r_i^{*m}|$. This estimation procedure will be jump robust, as bootstrapped returns are sampled under the null.

Theorem 1. CLT of the bootstrapped vector $(RV_t^*; BV_t^*)'$ (Consistency of the i.i.d. bootstrap) Suppose the price process can be described as in 1. Let $(RV_t^*; BV_t^*)'$ be the vector of bootstrapped statistics. As $n \to \infty$ ($\delta \to 0$),

$$\sqrt{n}\left[\begin{pmatrix} RV_t^* \\ BV_t^* \end{pmatrix} - \begin{pmatrix} E^*(RV_t^*) \\ E^*(BV_t^*) \end{pmatrix}\right] \xrightarrow{L} \mathcal{N}\left(O, \widehat{\Omega^*}\right), \tag{15}$$

where

$$O = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \widehat{\Omega^*} = n\begin{pmatrix} \widehat{Var^*(RV_t^*)} & cov^*(\widehat{RV_t^*, BV_t^*}) \\ cov^*(\widehat{RV_t^*, BV_t^*}) & \widehat{Var^*(BV_t^*)} \end{pmatrix} \tag{16}$$

Given that the original centered vector $(RV_t; BV_t)'$ is also multivariate normal (see 5), the consistency of the bootstrap follows.

Proof: To prove Theorem 1, two steps must be followed:

1. Show that:

$$\sqrt{n}\left[\begin{pmatrix} RV_t^* \\ BV_t^* \end{pmatrix} - \begin{pmatrix} E^*(RV_t^*) \\ E^*(BV_t^*) \end{pmatrix}\right] \xrightarrow{L} \mathscr{N}\left(O, \Omega^*\right),$$

where

$$\Omega^* = n\begin{pmatrix} Var^*(RV_t^*) & cov^*(RV_t^*, BV_t^*) \\ cov^*(RV_t^*, BV_t^*) & Var^*(BV_t^*) \end{pmatrix}$$

2. Show that $\widehat{\Omega^*} \xrightarrow{p} \Omega^*$.

We show here the proof for 1. The proves for 2 are included in Appendix 1. We need to prove that

$$\sqrt{n}\left[\begin{pmatrix} \sum_{j=2}^n r_j^{*2} \\ \sum_{j=2}^n |r_j^*||r_{j-1}^*| \end{pmatrix} - \begin{pmatrix} E^*(RV_t^*) \\ E^*(BV_t^*) \end{pmatrix}\right] \xrightarrow{L} \mathscr{N}\left(O, \Omega^*\right)$$

This is equivalent to proving that:

$$\sqrt{n}\,(c_1\, c_2)\left[\begin{pmatrix} \sum_{j=2}^n r_j^{*2} \\ \sum_{j=2}^n |r_j^*||r_{j-1}^*| \end{pmatrix} - \begin{pmatrix} E^*(RV_t^*) \\ E^*(BV_t^*) \end{pmatrix}\right] \xrightarrow{L} \mathscr{N}\left(0, c'\Omega^* c\right),$$

where $c = (c_1 c_2)'$ is a vector of constants.

Thus, we have:

$$\begin{aligned}
&(c_1\, c_2)\left[\begin{pmatrix} \sum_{j=2}^n r_j^{*2} \\ \sum_{j=2}^n |r_j^*||r_{j-1}^*| \end{pmatrix} - \begin{pmatrix} E^*(RV_t^*) \\ E^*(BV_t^*) \end{pmatrix}\right] = \\
&\sum_{j=2}^n \left[c_1 r_j^{*2} + c_2\frac{\pi}{2}|r_j^*||r_{j-1}^*| - c_1 {r_j}^2 - c_2\frac{\pi}{2}\frac{1}{n}({r_j}^2 + |r_j^*|\sum_{i\neq j}|r_i|)\right] = \\
&\sum_{j=2}^n z_j^*
\end{aligned}$$

Given that r_j^* is i.i.d., z_j^* is a 1-dependent process, with dependence disappearing after the first period, a general CLT result applies.

Wild Bootstrap In the case of the WB, we follow the same steps as for the i.i.d. bootstrap. We start by computing all moments under the bootstrap probability measure. For the realized variance, the moments are those derived by Gonçalves and Meddahi (2009): $E^*(RV_t^*) = \mu_2^* RV_t$, $Var^*(RV_t^*) = (\mu_4^* - \mu_2^{*2})\sum r_i^4$. The moments of the bootstrapped realized bipower variation are:

$$E^*(BV_t^*) = \frac{\pi}{2}\mu_1^{*2} BV_t \tag{17}$$

$$Var^*(BV_t^*) = \frac{\pi^2}{4}\left[(\mu_2^{*2} - \mu_1^{*4})\sum r_i^2 r_{i+1}^2 + 2(\mu_1^{*2}\mu_2^* - \mu_1^{*4})\sum |r_{i-1} r_i^2 r_{i+1}|\right] \tag{18}$$

The covariance between RV_t^* and BV_t^* is given by:

$$\begin{aligned} cov^*(RV_t^*, BV_t^*) =& \frac{\pi}{2}\left[E^*\left(\sum r_i^2 u_i^2 \sum |r_i u_i||r_{i+1} u_{i+1}|\right) - \mu_2^* \mu_1^{*2} \sum r_i^2 \sum |r_i r_{i+1}|\right] \qquad (19) \\ =& \frac{\pi}{2}\Bigg[\mu_1^*\mu_3^* \sum |r_i^3 r_{i+1}| + \mu_1^*\mu_3^* \sum |r_i r_{i+1}^3| + \mu_2^*\mu_1^{*2} \sum \sum_{i \neq j \neq j+1} |r_i^2 r_j r_{j+1} - \\ & \mu_2^*\mu_1^{*2}\left(\sum |r_i^3 r_{i+1}| + \sum |r_i r_{i+1}^3| + \sum \sum_{i \neq j \neq j+1} |r_i^2 r_j r_{j+1}|\right)\Bigg] \\ =& \frac{\pi}{2}\mu_1^*(\mu_3^* - \mu_2^*\mu_1^*)\left(\sum |r_i^3 r_{i+1}| + \sum |r_i r_{i+1}^3|\right) \end{aligned}$$

Gonçalves and Meddahi (2009) propose estimating $Var^*(RV_t^*)$ with $\widehat{Var^*(RV_t^*)} = \frac{\mu_4^* - \mu_2^{*2}}{\mu_4^*}\sum r_i^{*4}$, which is a centered estimator. Following their lead, we propose the following centered estimators for $Var^*(BV_t^*)$ and $cov^*(RV_t^*, BV_t^*)$:

$$\widehat{Var^*(BV_t^*)} = \frac{\pi^2}{4}\left(\frac{\mu_2^{*2} - \mu_1^{*4}}{\mu_2^{*2}}\sum r_i^{*2} {r_{i+1}^*}^2 + 2\frac{\mu_1^{*2}\mu_2^* - \mu_1^{*4}}{\mu_1^{*2}\mu_2^*}\sum |r_{i-1}^* r_i^{*2} r_{i+1}^*|\right) \tag{20}$$

$$\widehat{cov^*(RV_t^*, BV_t^*)} = \frac{\pi}{2}\frac{\mu_1^*(\mu_3^* - \mu_2^*\mu_1^*)}{\mu_1^*\mu_3^*}\left(\sum |r_i^{*3} r_{i+1}^*| + \sum |r_i^* {r_{i+1}^*}^3|\right) \tag{21}$$

Theorem 2. CLT of the bootstrapped vector $(RV_t^*; BV_t^*)'$ **(Consistency of the wild bootstrap)** Suppose the price process can be described as in 1. Let $(RV_t^*; BV_t^*)'$ be the vector of bootstrapped statistics. As $n \to \infty$ $(\delta \to 0)$,

$$\sqrt{n}\left[\begin{pmatrix} RV_t^* \\ BV_t^* \end{pmatrix} - \begin{pmatrix} E^*(RV_t^*) \\ E^*(BV_t^*) \end{pmatrix}\right] \xrightarrow{L} \mathcal{N}\left(O, \widehat{\Omega_{WB}^*}\right), \tag{22}$$

where

$$O = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \widehat{\Omega_{WB}^*} = n\begin{pmatrix} \widehat{Var^*(RV_t^*)} & \widehat{cov^*(RV_t^*, BV_t^*)} \\ \widehat{cov^*(RV_t^*, BV_t^*)} & \widehat{Var^*(BV_t^*)} \end{pmatrix} \tag{23}$$

Proof: The proof follows the same two steps as in Theorem 1. Step 1 follows from the same arguments as in the case of Theorem 1.

Step 2 concerns the consistency of the estimators of $\widehat{\Omega_{WB}^*}$ and is detailed in Appendix 2.

3.2 Consistency of the bootstrap for the Med test[2]

I.i.d. bootstrap In proposing a bootstrapped Med test, we follow the same steps as in the case of the BNS test. We first compute the expected value and the variance of MedRV under the bootstrap probability measure, P^*:

$$E^*(MedRV_t^*) = \frac{\pi}{6-4\sqrt{3}+\pi} \cdot \frac{1}{n^2} \sum_{j=1}^{n} -(2+3n-6jn+6j^2-6j) r_{(j)}^2, \tag{24}$$

where $r_{(j)}$ is the j-th order statistic in the returns sample.

$$\begin{aligned} Var^*(MedRV_t^*) = & \left(\frac{\pi}{6-4\sqrt{3}+\pi} \frac{n}{n-2} \right)^2 \left(\sum_{j=1}^{n} C(j) \cdot r_{(j)}^4 + \sum_{j=2}^{n} \sum_{k=1}^{j-1} C_1(j,k) \cdot r_{(j)}^2 r_{(k)}^2 + \right. \\ & \left. \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} C_2(j,k) \cdot r_{(j)}^2 r_{(k)}^2 \right), \end{aligned} \tag{25}$$

where $r_{(j)}$ and $r_{(k)}$ are the j-th and k-th order statistics, while $C(j)$, $C_1(j,k)$ and $C_2(j,k)$ are defined as:

$$\begin{aligned} C(j) = & \frac{1}{n^6} \big(36 + 96n - 216j - 468jn + 540j^2 + 35n^3 + 10n^5 j - 2n^4 j^2 + 8n^2 j^4 - \\ & 16j^3 n^3 - 10n^4 + 324j^4 - 648j^3 + 81n^2 - 5n^5 + 10n^4 j - 124n^3 j^2 + 28n^3 j - 140nj^4 + \\ & 264j^3 n^2 + 24j^2 n^2 - 368j^3 n + 760j^2 n - 224n^2 j \big) \\ C_1(j,k) = & -\frac{1}{n^6} \big(-36 - 136n + 108j + 298jn + 108k - 108j^2 + n^3 - 84jkn^2 + 120nkj^2 + \\ & 312nk^2 j + 330nk + 312n^2 k - 324jk + 324kj^2 + 324k^2 j - 174nk^2 - 108k^2 - \\ & 700jkn + 16n^4 - 151n^2 - 58n^3 j + 42j^2 n^2 - 78j^2 n + 64n^2 j - 32n^4 k + 24k^2 n^3 - \\ & 10n^3 k - 30k^2 n^2 - 324j^2 k^2 + 116n^3 jk + 108j^2 nk^2 - 84j^2 n^2 k - 132n^2 jk^2 \big) \\ C_2(j,k) = & -\frac{1}{n^6} \big(-36 - 136n + 108j - 700jkn + 116n^3 jk + 108j^2 nk^2 - 132j^2 n^2 k - \\ & 84n^2 jk^2 - 32n^4 j + 24n^3 j^2 - 10n^3 j - 30j^2 n^2 - 174j^2 n + 312n^2 j + 330jn + \\ & 108k + 16n^4 + n^3 - 108j^2 - 151n^2 + 312nkj^2 - 84jkn^2 + 120nk^2 j - 58n^3 k + \\ & 42k^2 n^2 - 324j^2 k^2 - 108k^2 + 324kj^2 + 324k^2 j + 298nk - 78nk^2 + 64n^2 k - 324jk \big) \end{aligned}$$

$C(j)$, $C_1(j,k)$ and $C_2(j,k)$ are obtained when computing the expectations over the product $\sum_{j=3}^{n} \text{med}(|r_j^*|, |r_{j-1}^*|, |r_{j-2}^*|)^2 \sum_{j=3}^{n} \text{med}(|r_j^*|, |r_{j-1}^*|, |r_{j-2}^*|)^2$. The computations were performed

[2] This draft only includes the i.i.d. bootstrap method for the Med test. We will include the Wild bootstrap method in a further version of this paper.

via counting the number of cases when 1, 2 or 3 returns overlap in the expectation of the product[3]:

$$\text{med}(|r_i^*|,|r_{i-1}^*|,|r_{i-2}^*|)^2\text{med}(|r_j^*|,|r_{j-1}^*|,|r_{j-2}^*|)^2, \quad i,j=3\dots n$$

The covariance between RV^* and $MedRV^*$ is:

$$\begin{aligned} cov^*(RV_t^*, MedRV_t^*) =& \frac{\pi}{6-4\sqrt{3}+\pi}\frac{n}{n-2}\left\{3\frac{n-2}{n^4}\left[\sum_{j=1}^{n}(-1+2jn-2j^2+2j)r_{(j)}^4 + \right.\right. \\ & \sum_{j=2}^{n}\sum_{k=1}^{j-1} r_{(j)}^2 r_{(k)}^2 (2jn-2j^2-2n+3j-1)+ \\ & \left.\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} r_{(j)}^2 r_{(k)}^2 (2jn-2j^2-n+j)\right]+ \\ & \left.[n^2-3(n-2)]\frac{1}{n^4}\sum_{j=1}^{n} -(2+3n-6jn+6j^2-6j)r_{(j)}^2 RV\right\} - \\ & E^*(MedRV_t^*)RV \end{aligned} \tag{26}$$

Theorem 3. CLT of the bootstrapped vector $(RV_t^*; MedRV_t^*)'$ (Consistency of the i.i.d. bootstrap) Suppose the price process can be described as in 1. Let $(RV_t^*; MedRV_t^*)'$ be the vector of bootstrapped statistics. As $n \to \infty$ ($\delta \to 0$),

$$\sqrt{n}\left[\begin{pmatrix} RV_t^* \\ MedRV_t^* \end{pmatrix} - \begin{pmatrix} E^*(RV_t^*) \\ E^*(MedRV_t^*) \end{pmatrix}\right] \xrightarrow{L} \mathcal{N}\left(O, \widehat{\Omega_{Med}^*}\right), \tag{27}$$

where

$$O = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \widehat{\Omega_{Med}^*} = n\begin{pmatrix} \widehat{Var^*(RV_t^*)} & \widehat{cov^*(RV_t^*, MedRV_t^*)} \\ \widehat{cov^*(RV_t^*, MedRV_t^*)} & \widehat{Var^*(MedRV_t^*)} \end{pmatrix} \tag{28}$$

Proof: The proof follows the same two steps as in Theorem 1. Step 1 follows from the same arguments as in the case of Theorem 1.

Step 2 concerns the consistency of the estimators of $\widehat{\Omega_{Med}^*}$ and is detailed in Appendix 3.

4 Averaging tests for jumps

In this section, we propose combining p-values obtained at different sampling frequencies or from different testing procedures to gather information on the presence of jumps in the price

[3] I am very grateful to Tom Holden for the useful suggestions in computing these coefficients.

process. Our procedure consists of applying one or more testing procedures at different sampling frequencies, obtain the corresponding p-values and then average them by using Fisher's method. Simulation studies looking at the size and power properties of the BNS and Med tests (see, for instance Dumitru and Urga, 2012) show that their size tends to grow as the sampling frequency becomes lower (i.e. as δ becomes higher). However, to be able to combine the results from different frequencies, the tests must have the same size for all frequencies. Thus, instead of using the asymptotic distribution to obtain p-values, we propose using the empirical distribution of the bootstrapped statistics.

The combination of test statistics with simple null hypotheses has been long used in the statistics and econometrics literature, with the oldest and most famous including Tippett (1931)'s $\min(p_i)$ statistics, Fisher (1932)'s X^2 statistic and Liptak (1958)'s $\sum_{j=1}^{p} \Phi^{-1}(1-p_i)$ statistic, where p_i represents the i-th p-value, $i = 1 \dots p$. Here, as we attempt to extract information from more frequencies, we focus on Fisher (1932)'s X^2, defined as:

$$X^2 = -2 \sum_{j=1}^{p} \log p_i \tag{29}$$

When the combined test statistics are independent, $X^2 \xrightarrow{L} \chi^2(2p)$. However, this is not the case here, as the test is applied to the same data, but sampled at different frequencies. Brown (1975) and Kost and McDermott (2002) proposed approximating the distribution of the X^2 with that of a scaled χ^2 variable, but assuming certain approximations for the covariances between the p-values. The advances in both computational technology and power allow us to easily simulate the distribution of the X^2 statistic.

There are several important contributions in the field of econometrics that use meta-analysis methods like Tippett (1931), Fisher (1932) or Liptak (1958) to combine test results and simulate the distributions of the "combined" statistic. Maddala and Wu (1999) apply Fisher (1932)'s method for unit root tests for panel data; Smeekes and Taylor (2012) use unions of rejections of unit root tests; Dufour et al. (2004) use combined procedures to test for heteroskedasticity when there are unknown breakpoints in the variance. A very relevant contribution is the one by Godfrey (2005), who suggests applying double bootstrap methods to control the overall significance level of several diagnostic tests applied for an ordinary least squares regression model.

In this paper, even if the test statistic is asymptotically pivotal, the asymptotic size depends on the sampling frequency. Thus, following Godfrey (2005), we apply a double-bootstrap procedure to control the overall significance level of our test averaging procedure.

4.1 The double bootstrap procedure

Let us assume we are interested in finding out whether jumps occurred during the interval $[0, t]$, which could be, for instance, a trading day. Moreover, let the data be sampled at sampling intervals of the form $k_i \cdot \delta$, where $k_i > 0$ mutiplies δ, $i = 1, \ldots, p$, where p is the number of frequencies we choose to combine. We denote with z the test statistic of either the BNS test or the Med test. In applying the double bootstrap, we take the following steps:

1. compute the test statistic on the original data for different sampling frequencies (k_i), z_{t,k_i}
2. for each day and each frequency, re-sample the returns under the null and obtain B replicates for the statistics, z^*_{t,k_i}
3. compute the p-value corresponding to the original z_{t,k_i} statistics, based on the bootstrapped distribution:
$$p^*_{t,k_i} = \frac{\sum_{j=1}^{B} z_{t,k_i} > z^*_{t,k_i}(j)}{B} \tag{30}$$
4. for each new sample from 1 to B, generate B_1 sub-samples and compute the statistic $z^{**}_{t,k_i,b}$, where $b = 1, \ldots, B$
5. for each new sample from 1 to B, compute corresponding p-value:
$$p^{**}_{t,k_i,b} = \frac{\sum_{j=1}^{B_1} z^*_{t,k_i} > z^{**}_{t,k_i}(j)}{B_1} \tag{31}$$

Thus, this procedure generates for each k_i, one p-value from the first round of bootstrap and B p-values from the second round of bootstrap. The Fisher X^2 statistic is computed as:

$$X^2_t = -2 \sum_{i=1}^{p} \log p^*_{t,k_i} \tag{32}$$

The distribution of X^2_t can be obtained as:

$$X^{2**}_t = -2 \sum_{i=1}^{p} \log p^{**}_{t,k_i,b}, \tag{33}$$

where X^{2**}_t is a vector with B elements.

To re-sample under the null, if a jump is identified on that day, we remove the corresponding return from the data and sample from the remaining returns. To identify the jump return on a certain day, we take the maximum standardized return of that day, where the standardization is performed as in Andersen et al. (2007) and Lee and Mykland (2008).

5 Simulation study

To assess the effectiveness of our bootstrap methods and test averaging procedure, we simulate a stochastic volatility process with finite jumps. We use a similar simulation setup as in Huang and Tauchen (2005) and Dumitru and Urga (2012). The stochastic volatility model for the log of the price process follows the subsequent dynamics:

$$\begin{aligned} \mathrm{d}p_t &= 0.03\mathrm{d}t + \exp[0.125\,\upsilon_t]\mathrm{d}W_{p_t}, \\ \mathrm{d}\upsilon_t &= -0.1\upsilon_t\mathrm{d}t + \mathrm{d}W_{\upsilon_t}, \qquad \mathrm{corr}(\mathrm{d}W_p, \mathrm{d}W_\upsilon) = -0.62 \end{aligned} \tag{34}$$

where p_t is the log-price process, the W's are standard Brownian motions, υ_t the volatility factor. This is the process that we simulate under the null hypothesis of no jumps.

Under the alternative, we add rare compound Poisson jumps, arriving with intensity $\lambda = 0.5$ and having normally distributed sizes with mean 0 and standard deviation $\sigma_{jump} = 1.5$.

To the simulated stochastic volatility plus jump model, we add i.i.d. microstructure noise normally distributed with mean 0 and $\sigma_{noise} = 0.04$.

In this version of the paper, we only report results for i.i.d. bootstrap. Results for WB will be included in further versions.

Table 1 reports the size and power of the i.i.d. bootstrapped and asymptotic BNS and Med ratio tests. The nominal significance level is 5%.

Table 1: **Size and power for the i.i.d. bootstrapped and asymptotic BNS and Med statistics**

	1 min		2 min		5 min		10 min		15 min	
	Size	Power	Size	Power	Size	Power	Size	Power	Size	Power
BNS test										
Bootstrap	0.0131	0.7615	0.0440	0.7758	0.0946	0.7544	0.1293	0.7216	0.1353	0.6745
Asymptotic	0.0131	0.7503	0.0368	0.7544	0.0729	0.7390	0.0991	0.6786	0.1011	0.6438
Med test										
Bootstrap	0.0160	0.7206	0.0470	0.7422	0.0748	0.7020	0.0819	0.6543	0.0814	0.5980
Asymptotic	0.0090	0.7530	0.0264	0.7627	0.0460	0.7233	0.0527	0.6613	0.0533	0.5933

In the case of the BNS test, we observe very similar results for the asymptotic and bootstrap tests at high sampling frequencies. Thus, at 1 minute, size is 0.13 in both cases, while power is slightly higher in the case of the bootstrap (0.76 versus 0.75 for the asymptotic test). At higher frequencies, the bootstrap test shows slightly higher size and power than the asymptotic one. Moreover, this discrepancy tends to increase slowly with the decrease in the sampling frequencies. When prices are contaminated with microstructure noise, this behaviour can be beneficial at high frequencies, when the tests tend to be undersized. In this case, the bootstrap test can help recover some power.

In the case of the Med test, the bootstrap version also tends to be oversized in comparison to the asymptotic one. The power is very close to the asymptotic one, but is slightly lower for most frequencies (up to 10 minutes).

We assess the performance of our test averaging procedure by combining 2 or 3 sampling frequencies for both BNS and Med tests. Table 2 reports the results for the BNS test. In the table, we also include the asymptotic benchmark for comparison purposes. The nominal significance level is 5%.

We note that the test combinations manage to outperform the asymptotic results in almost all cases. This is because in the presence of i.i.d. microstructure noise, the BNS asymptotic test becomes severely undersized at high frequencies. However, at lower frequencies, where size is getting close to the nominal one, power gets considerably smaller. Our procedure, by combining higher with lower frequencies, manages to maintain a high power, combined with a manageable size. Usually, for every 2 or 3 combined sampling frequencies, our procedure leads to a power higher than the asymptotic levels for the individual frequencies. The size is always comprised in the range of the asymptotic size levels for those frequencies. For instance, combining tests applied on data sampled every 1 and 5 minutes renders a size of 0.028, coupled with a very high power of 0.768. This power is higher than the one obtained for the asymptotic test at either 1 or 5 minutes. Size is in the interval of the asymptotic levels for 1 and 5 minutes (0.009 and 0.0791). The behaviour exemplified here is verified for all combinations of sampling frequencies.

Table 3 shows the results obtained from combining sapling frequencies for the Med test. The performance is similar to the one described above for the BNS test. However, the effect in this case is slightly weaker. When combining 2 or 3 sampling frequencies, size either lies in the range of the corresponding asymptotic levels, or becomes larger. To keep the size at lower levels, it is beneficial to combine very high frequencies with lower ones. Power lies in the range of the asymptotic levels, but towards the upper bound. For instance, when combining p-values from 1 and 10 minutes, the size of 0.031 is in the range of the asymptotic size levels (0.008, 0.055). The power (0.725) is in the range of the asymptotic levels, but still very high. Moreover, this effect is also very pronounced when combining 3 sampling frequencies (see, for instance, '1-2-5' or '1-2-10').

Further developments of this paper include applying our procedure to infrequent trading data, in order to assess the impact of infrequent trading on jump detection.

Table 2: **Size and power from averaging p-values over frequencies. The BNS statistic**

Frequencies (min)	'1-2'	'1-5'	'1-10'	'1-15'	'2-5'	'2-10'	'2-15'	'5-10'	'5-15'	'10-15'
'Size'	0.0181	0.0276	0.0284	0.0266	0.0632	0.0665	0.0700	0.1044	0.1037	0.1373
'Power'	0.7866	0.7680	0.7354	0.7050	0.7798	0.7489	0.7221	0.7442	0.7105	0.7015
Frequencies (min)	'1-2-5'	'1-2-10'	'1-2-15'	'1-5-10'	'1-5-15'	'1-10-15'	2-5-10'	'2-5-15'	'2-10-15'	'5-10-15'
'Size'	0.0216	0.0213	0.0189	0.0326	0.0268	0.0352	0.0543	0.0583	0.0665	0.1037
'Power'	0.7803	0.7535	0.7409	0.7469	0.7283	0.7143	0.7565	0.7439	0.7243	0.7231
'BNS asymp'	'1 min'	'2 min'	'5 min'	'10 min'	'15 min'					
'Size'	0.0090	0.0356	0.0791	0.0987	0.1147					
'Power'	0.7442	0.7577	0.7248	0.6766	0.6249					

Table 3: **Size and power from averaging p-values over frequencies. The Med statistic**

Frequencies (min)	'1-2'	'1-5'	'1-10'	'1-15'	'2-5'	'2-10'	'2-15'	'5-10'	'5-15'	'10-15'
'Size'	0.0279	0.0312	0.0305	0.0282	0.0669	0.0617	0.0646	0.1069	0.0905	0.1243
'Power'	0.7779	0.7569	0.7249	0.6938	0.7684	0.7314	0.7059	0.7259	0.6888	0.6813
Frequencies (min)	'1-2-5'	'1-2-10'	'1-2-15'	'1-5-10'	'1-5-15'	'1-10-15'	'2-5-10'	'2-5-15'	'2-10-15'	'5-10-15'
'Size'	0.0276	0.0249	0.0230	0.0292	0.0295	0.0361	0.0564	0.0525	0.0656	0.0981
'Power'	0.7699	0.7479	0.7324	0.7379	0.7164	0.7009	0.7474	0.7284	0.7159	0.7129
'med asymp'	'1 min'	'2 min'	'5 min'	'10 min'	'15 min'					
'Size'	0.0083	0.0273	0.0445	0.0552	0.0555					
'Power'	0.7540	0.7645	0.7268	0.6553	0.5908					

6 Conclusion

This paper brings two important contributions to the current literature on high frequency econometrics.

First, we propose bootstrap methods for the Barndorff-Nielsen and Shephard (2006) and Andersen et al. (2012) tests for jumps. We consider both i.i.d. and Wild bootstrap and provide consistency results for these methods. These contributions fill an important gap in the literature on bootstrapping realized-type estimators based on high frequency data, which lacks developments on robust to jumps estimators. Further improvements to the current paper will also provide consistency results for bootstrapping the realized bipower variation and the median realized variation.

Second, we propose a new procedure to detect jumps based on high frequency data that uses Fisher (1932)'s method to average p-values from one/ different tests applied at different sampling frequencies. The procedure is proven to be more efficient than applying individual tests on only one sampling frequency. This is because we discard less data and extract information from multiple frequencies and/ or procedures. When combining p-values obtained at different sampling frequencies, we apply a double bootstrap procedure to control the overall size of the test. In the case of the Barndorff-Nielsen and Shephard (2006) test, we show that averaging p-values of tests applied at different frequencies outperforms the simple asymptotic test, by delivering a higher power, combined with a manageable size. Similar, but weaker results are found for the Andersen et al. (2012) test. In this case, the performance of test combinations is higher when combining results from higher frequencies with results from lower frequencies.

We plan to extend the results in this paper in various ways. First, we will complete the current paper with results on bootstrapping the realized bipower variation and the median realized variation, as well as the Wild bootstrap for the median RV test. Second, we will extend the simulation setup to various types of microstructure noise, including infrequent trading. Third, we will add theoretical results on the accuracy of the bootstrap.

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A Appendix 1

We need to show that the estimators of all the terms in Ω^* are consistent. For this, we prove that both the estimation biases and the variances of $\widehat{Var^*(RV_t^*)}$, $\widehat{Var^*(BV_t^*)}$ and $\widehat{cov^*(RV_t^*, BV_t^*)}$ converge in probability to 0. Gonçalves and Meddahi (2009) show that $E^*(\widehat{Var^*(RV_t^*)}) = Var^*(RV_t^*)$. They also show that $\widehat{Var^*(RV_t^*)}$ is consistent for $Var^*(RV_t^*)$. First, we compute the expected value for the estimator of the variance of BV_t:

$$
\begin{aligned}
E^*(\widehat{Var^*(BV_t^*)}) =& E^*\Bigg\{\frac{\pi^2}{4}\Bigg[\frac{n-1}{n^2}\left(\sum r_i^{*4} + \sum\sum_{j\neq i} r_i^{*2} r_j^{*2}\right) + 2\frac{n-2}{n^3}\sum r_i^{*2}\Big(\sum r_i^{*2} + \\
& \sum\sum_{j\neq i} |r_i^*||r_j^*|\Big) - \frac{3n-5}{n^4}\Bigg(\sum r_i^{*4} + 3\sum\sum_{j\neq i} r_i^{*2} r_j^{*2} + 4\sum\sum_{j\neq i}|r_i^*|^3|r_j^*| + \\
& 6\sum\sum\sum_{l\neq j\neq i}|r_i^*||r_j^*||r_l^*|^2 + \sum\sum\sum\sum_{k\neq l\neq j\neq i}|r_i^*||r_j^*||r_l^*||r_k^*|\Bigg)\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
E^*(\widehat{Var^*(BV_t^*)}) =& \frac{\pi^2}{4}E^*\Bigg[\frac{n^3+n^2-7n+5}{n^4}\sum r_i^{*4} + \frac{n^3+n^2-13n+15}{n^4}\sum\sum_{j\neq i} r_i^{*2} r_j^{*2} + \\
& \sum\sum_{j\neq i}|r_i^*|^3|r_j^*|\left(\frac{4n-8}{n^3} - 4\frac{3n-5}{n^4}\right) + \sum\sum\sum_{l\neq j\neq i}|r_i^*||r_j^*||r_l^*|^2\Big(\frac{2n-4}{n^3} - \\
& 6\frac{3n-5}{n^4}\Big) - \frac{3n-5}{n^4}\sum\sum\sum\sum_{k\neq l\neq j\neq i}|r_i^*||r_j^*||r_l^*||r_k^*|\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
E^*(\widehat{Var^*(BV_t^*)}) =& \frac{\pi^2}{4}\Bigg[\frac{n^3+n^2-7n+5}{n^4}\sum {r_i}^4 + \frac{n^2-n}{n^2}\frac{n^3+n^2-13n+15}{n^4}\left(\sum {r_i}^2\right)^2 + \\
& \frac{n^2-n}{n^2}\frac{4n^2-20n+20}{n^4}\sum|r_i|^3\sum|r_i| + \frac{n(n-1)(n-2)}{n^3}\frac{2n^2-22n+30}{n^4} \\
& \sum|r_i|^2\left(\sum|r_i|\right)^2 - \frac{n(n-1)(n-2)(n-3)}{n^4}\frac{3n-5}{n^4}\left(\sum|r_i|\right)^4\Bigg] \\
\approx & \frac{\pi^2}{4}\left[\frac{S_4}{n} + \frac{S_2^2}{n} + \frac{4S_3S_1}{n^2} + \frac{2S_2S_1^2}{n^2} - \frac{3S_1^4}{n^3}\right],
\end{aligned}
$$

where $S_k = \sum |r_i|^k$, $k = 1\ldots 4$. At the same time,

$$
Var^*(BV_t^*) \approx \frac{\pi^2}{4}\left[\frac{S_2^2}{n} + \frac{2S_2S_1^2}{n^2} - \frac{3S_1^4}{n^3}\right] \tag{35}
$$

Consequently, the bias will be:

$$
\begin{aligned}
Bias(V\widehat{ar^*(BV}_t^*)) &\approx \left[\frac{S_4}{n} + \frac{4S_3S_1}{n^2}\right] \\
&\xrightarrow{p} \frac{1}{n^2}\mu_4^{-1}\int_0^t \sigma_s^4 \mathrm{d}s + \frac{4}{n^2}\mu_3^{-1}\mu_1^{-1}\int_0^t \sigma_s^3 \mathrm{d}s \int_0^t \sigma_s \mathrm{d}s \\
&= O_p(\delta^2)
\end{aligned}
$$

The variance of $V\widehat{ar^*(BV}_t^*)$ is:

$$
Var^*(V\widehat{ar^*(BV}_t^*)) = E^*(V\widehat{ar^*(BV}_t^*))^2 - \left(E^*(V\widehat{ar^*(BV}_t^*))\right)^2 \tag{36}
$$

We will compute first the second order moment of the estimator:

$$
\begin{aligned}
E^*(V\widehat{ar^*(BV}_t^*))^2 &= \frac{\pi^4}{16}E^*\Bigg[\frac{(n-1)^2}{n^4}\left(\sum r_i^{*2}\right)^4 + 4\frac{(n-2)^2}{n^6}\left(\sum r_i^{*2}\right)^2\left(\sum |r_i^*|\right)^4 - \\
&\quad \frac{(3n-5)^2}{n^8}\left(\sum |r_i^*|\right)^8\Bigg] \\
&= \frac{\pi^4}{16}E^*\left[\frac{(n-1)^2}{n^4}A + 4\frac{(n-2)^2}{n^6}B - \frac{(3n-5)^2}{n^8}C\right]
\end{aligned}
$$

$$
\begin{aligned}
A &= \sum r_i^{*8} + 3\sum\sum_{j\neq i} r_i^{*4}r_j^{*4} + 4\sum\sum_{j\neq i} r_i^{*6}r_j^{*2} + 6\sum\sum\sum_{l\neq j\neq i} |r_i^{*2}r_j^{*2}r_l^*|^4 + \\
&\quad \sum\sum\sum\sum_{k\neq l\neq j\neq i} r_i^{*2}r_j^{*2}r_l^{*2}r_k^{*2}
\end{aligned}
$$

$$
E^*(A) = S_8 + 3\frac{n(n-1)}{n^2}S_4^2 + 4\frac{n(n-1)}{n^2}S_6S_2 + 6\frac{n(n-1)(n-2)}{n^3}S_4S_2^2 + \frac{n(n-1)(n-2)(n-3)}{n^4}(S_2)^4
$$

$$
\begin{aligned}
B = &\left(\sum r_i^{*4} + \sum\sum_{j\neq i} r_i^{*2} r_j^{*2}\right)\left(\sum r_i^{*4} + 3\sum\sum_{j\neq i} r_i^{*2} r_j^{*2} + \right.\\
&\left. 4\sum\sum_{j\neq i} |r_i^*|^3 |r_j^*| + 6\sum\sum\sum_{\neq} |r_i^* r_j^* r_l^*|^2 + \sum\sum\sum\sum_{\neq} |r_i^* r_j^* r_l^* r_k^*|\right)\\
&= \sum r_i^{*8} + \sum\sum_{\neq} r_i^{*4} r_j^{*4} + 4\left(2\sum\sum_{\neq} r_i^{*6} r_j^{*2} + \sum\sum\sum_{\neq} r_i^{*2} r_j^{*2} r_l^{*4}\right) + \\
&4\left(\sum\sum_{\neq} |r_i^*|^7 |r_j^*| + \sum\sum_{\neq} |r_i^*|^5 |r_j^*|^3 + \sum\sum\sum_{\neq} |r_i^{*4} r_j^{*3} r_l^*|\right) + \\
&6\left(\sum\sum\sum_{\neq} |r_i^{*6} r_j^* r_l^*| + 2\sum\sum\sum_{\neq} |r_i^{*5} r_j^{*2} r_l^*| + \sum\sum\sum\sum_{\neq} |r_i^{*4} r_j^{*2} r_l^* r_k^*|\right) + \\
&3\left(\sum\sum\sum\sum_{\neq} |r_i^{*2} r_j^{*2} r_l^{*2} r_k^{*2}| + 4\sum\sum\sum_{\neq} |r_i^{*4} r_j^{*2} r_l^{*2}| + 2\sum\sum_{\neq} r_i^{*4} r_j^{*4}\right) + \\
&4\left(\sum\sum\sum\sum_{\neq} |r_i^{*3} r_j^{*2} r_l^{*2} r_k^*| + 2\sum\sum\sum_{\neq} |r_i^{*3} r_j^{*3} r_l^{*2}| + 2\sum\sum\sum_{\neq} |r_i^{*5} r_j^{*2} r_l^*| + 2\sum\sum_{\neq} |r_i^{*5} r_j^{*3}|\right) + \\
&6\left(\sum\sum\sum\sum\sum_{\neq} |r_i^{*2} r_j^{*2} r_k^{*2} r_l^* r_m^*| + 4\sum\sum\sum\sum_{\neq} |r_i^{*3} r_j^{*2} r_l^{*2} r_k^*| + 2\sum\sum\sum_{\neq} |r_i^{*3} r_j^{*3} r_l^{*2}| + \right.\\
&\left. 2\sum\sum\sum\sum_{\neq} |r_i^{*4} r_j^{*2} r_l^* r_k^*| + 4\sum\sum\sum_{\neq} |r_i^{*4} r_j^{*3} r_l^*|\right)
\end{aligned}
$$

where for simplicity we used $\neq$ to specify that $i \neq j \neq k \neq l \neq m \ldots$.

The expectation of B is:

$$
\begin{aligned}
E^*(B) \approx &S_8 + 7S_4^2 + 3S_2^4 + 16S_2^2 S_4 + 6S_1^2 S_2^3 + 20S_2 S_3^2 + 12S_3 S_5 + 6S_1^2 S_6 + 8S_2 S_6 + \\
&4S_1 S_7 + 28S_1 S_3 S_4 + 18S_1^2 S_2 S_4 + 20S_1 S_2 S_5 + 28S_1 S_2^2 S_3
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
C = &\Bigg(\sum r_i^{*4} + 3\sum\sum_{\neq} r_i^{*2}r_j^{*2} + 4\sum\sum_{\neq} |r_i^*|^3|r_j^*| + 6\sum\sum\sum_{\neq} |r_i^*r_j^*r_k^*|^2 + \sum\sum\sum\sum_{\neq} |r_i^*r_j^*r_k^*r_l^*|\Bigg)^2 \\
= &\Bigg(\sum r_i^{*8} + \sum\sum_{\neq} r_i^{*4}r_j^{*4}\Bigg) + 9\Bigg(\sum\sum\sum\sum_{\neq} r_i^{*2}r_j^{*2}r_k^{*2}r_k^{*2} + 4\sum\sum\sum_{\neq} r_i^{*2}r_j^{*2}r_k^{*4} + \\
&2\sum\sum_{\neq} r_i^{*4}r_j^{*4}\Bigg) + 16\Bigg(\sum\sum\sum\sum_{\neq} |r_i^{*3}r_j^{*3}r_k^*r_l^*| + \sum\sum\sum_{\neq} |r_i^{*6}r_j^*r_k^*| + \\
&2\sum\sum\sum_{\neq} |r_i^{*4}r_j^{*3}r_k^*| + \sum\sum\sum_{\neq} |r_i^{*3}r_j^{*3}r_k^{*2}| + \sum\sum_{\neq} r_i^{*6}r_j^{*2} + \sum\sum_{\neq} r_i^{*4}r_j^{*4} + \Bigg) \\
&36\Bigg(\sum\sum\sum\sum\sum\sum_{\neq} |r_i^{*2}r_j^{*2}r_k^*r_l^*r_m^*r_p^*| + 4\sum\sum\sum\sum\sum_{\neq} |r_i^{*3}r_j^{*2}r_k^*r_l^*r_m^*| + \\
&\sum\sum\sum\sum\sum_{\neq} |r_i^{*4}r_j^*r_k^*r_l^*r_m^*| + 4\sum\sum\sum\sum\sum_{\neq} |r_i^{*2}r_j^{*2}r_k^{*2}r_l^*r_m^*| + \\
&4\sum\sum\sum\sum_{\neq} |r_i^{*4}r_j^{*2}r_k^*r_l^*| + 4\sum\sum\sum\sum_{\neq} |r_i^{*3}r_j^{*3}r_k^*r_l^*| + 4\sum\sum\sum\sum_{\neq} |r_i^{*3}r_j^{*2}r_k^{*2}r_l^*| + \\
&2\sum\sum\sum_{\neq} r_i^{*2}r_j^{*2}r_l^{*4} + 4\sum\sum\sum_{\neq} r_i^{*3}r_j^{*3}r_h^{*2}\Bigg) + \Bigg(\sum\sum\sum\sum\sum\sum\sum\sum_{\neq} |r_i^*r_j^*r_k^*r_l^*r_m^*r_o^*r_p^*r_q^*| \\
&+ 16\sum\sum\sum\sum\sum\sum\sum_{\neq} |r_i^{*2}r_j^*r_k^*r_l^*r_m^*r_o^*r_p^*| + 72\sum\sum\sum\sum\sum\sum_{\neq} |r_i^{*2}r_j^{*2}r_k^*r_l^*r_m^*r_o^*| + \\
&96\sum\sum\sum\sum\sum_{\neq} |r_i^{*2}r_j^{*2}r_k^{*2}r_l^*r_m^*| + 24\sum\sum\sum\sum_{\neq} |r_i^{*2}r_j^{*2}r_l^{*2}r_k^{*2}|\Bigg) + 8\Bigg(\sum\sum\sum_{\neq} |r_i^{*4}r_j^{*3}r_k^*| + \\
&\sum\sum_{\neq} |r_i^{*5}r_j^{*3}| + \sum\sum_{\neq} |r_i^{*7}r_j^*|\Bigg) + 6\Bigg(\sum\sum\sum_{\neq} |r_i^{*4}r_j^{*2}r_k^{*2}| + 2\sum\sum_{\neq} |r_i^{*6}r_j^{*2}|\Bigg) + \\
&12\Bigg(\sum\sum\sum\sum_{\neq} |r_i^{*4}r_j^{*2}r_k^*r_l^*| + 2\sum\sum\sum_{\neq} |r_i^{*5}r_j^{*2}r_k^*| + \sum\sum\sum_{\neq} |r_i^{*6}r_j^*r_k^*|\Bigg) + \\
&2\Bigg(\sum\sum\sum\sum\sum_{\neq} |r_i^{*4}r_j^*r_k^*r_l^*r_m^*| + 4\sum\sum\sum\sum_{\neq} |r_i^{*5}r_j^*r_k^*r_l^*|\Bigg) + 24\Bigg(\sum\sum\sum\sum_{\neq} |r_i^{*3}r_j^{*2}r_k^{*2}r_l^*| + \\
&2\sum\sum\sum_{\neq} |r_i^{*3}r_j^{*3}r_k^{*2}| + 2\sum\sum\sum_{\neq} |r_i^{*5}r_j^{*2}r_k^*| + 2\sum\sum_{\neq} |r_i^{*5}r_j^{*3}|\Bigg) +
\end{aligned}
$$

$$
\begin{aligned}
&36\left(\sum\sum\sum\sum\sum_{\neq}|r_i^{*2}r_j^{*2}r_k^{*2}r_l^{*}r_m^{*}| + 4\sum\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*2}r_k^{*2}r_l^{*}| + 2\sum\sum\sum\sum_{\neq}|r_i^{*4}r_j^{*2}r_k^{*}r_l^{*}| +\right.\\
&\left.2\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*3}r_k^{*2}| + 4\sum\sum\sum_{\neq}|r_i^{*4}r_j^{*3}r_k^{*}|\right) + 48\left(\sum\sum\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*2}r_k^{*}r_l^{*}r_m^{*}| +\right.\\
&2\sum\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*2}r_k^{*2}r_l^{*}| + 2\sum\sum\sum\sum_{\neq}|r_i^{*4}r_j^{*2}r_k^{*}r_l^{*}| + \sum\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*3}r_k^{*}r_l^{*}| +\\
&\sum\sum\sum\sum_{\neq}|r_i^{*5}r_j^{*}r_k^{*}r_l^{*}| + 2\sum\sum\sum_{\neq}|r_i^{*4}r_j^{*2}r_k^{*2}| + 2\sum\sum\sum_{\neq}|r_i^{*4}r_j^{*3}r_k^{*}| +\\
&\left.2\sum\sum\sum_{\neq}|r_i^{*5}r_j^{*2}r_k^{*}|\right) + 6\left(\sum\sum\sum\sum\sum\sum_{\neq}|r_i^{*2}r_j^{*2}r_k^{*}r_l^{*}r_m^{*}r_o^{*}| +\right.\\
&\left.8\sum\sum\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*2}r_k^{*}r_l^{*}r_m^{*}| + 12\sum\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*3}r_k^{*}r_l^{*}|\right) +\\
&8\left(\sum\sum\sum\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*}r_k^{*}r_l^{*}r_m^{*}r_o^{*}| + 4\sum\sum\sum\sum\sum_{\neq}|r_i^{*4}r_j^{*}r_k^{*}r_l^{*}r_m^{*}| +\right.\\
&\left.4\sum\sum\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*2}r_k^{*}r_l^{*}r_m^{*}| + 12\sum\sum\sum\sum_{\neq}|r_i^{*4}r_j^{*2}r_k^{*}r_l^{*}|\right) +\\
&12\left(\sum\sum\sum\sum\sum\sum\sum_{\neq}|r_i^{*2}r_j^{*}r_k^{*}r_l^{*}r_m^{*}r_o^{*}r_p^{*}| + 4\sum\sum\sum\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*}r_k^{*}r_l^{*}r_m^{*}r_o^{*}| +\right.\\
&8\sum\sum\sum\sum\sum\sum_{\neq}|r_i^{*2}r_j^{*2}r_k^{*}r_l^{*}r_m^{*}r_o^{*}| + 24\sum\sum\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*2}r_k^{*}r_l^{*}r_m^{*}| +\\
&\left.12\sum\sum\sum\sum\sum_{\neq}|r_i^{*2}r_j^{*2}r_k^{*2}r_l^{*}r_m^{*}| + 15\sum\sum\sum\sum_{\neq}|r_i^{*3}r_j^{*2}r_k^{*2}r_l^{*}|\right)
\end{aligned}
$$

$$
\begin{aligned}
E^*(C) =& S_8 + 35S_4^2 + 32S_2^4 + S_1^8 + 210S_2^2S_4 + 280S_1^2S_3^2 + 28S_1^2S_6 + 280S_2S_3^2 + 210S_1^4S_2^2 + 70S_1^4S_4 +\\
& 420S_1^2S_2^3 + 28S_1^6S_2 + 28S_2S_6 + 56S_3S_5 + 8S_1S_7 + 56S_1^3S_5 + 56S_1^5S_3 + 280S_1S_3S_4 +\\
& 560S_1^3S_2S_3 + 420S_1^2S_2S_4 + 588S_1S_2^2S_3 + 168S_1S_2S_5
\end{aligned}
$$

$$
\begin{aligned}
\left[E^*(Var^*(\hat{B}V_t^*))\right]^2 \approx& \frac{\pi^4}{16}\frac{1}{n^2}\left(S_4^2 + S_2^4 + \frac{16}{n^2}S_1^2S_3^2 + \frac{4}{n^2}S_1^4S_2^2 +\right.\\
& \frac{9}{n^4}S_1^8 + 2S_2^2S_4 + \frac{8}{n}S_1S_3S_4 + \frac{4}{n}S_1^2S_2S_4 - \frac{6}{n^2}S_1^4S_4 + \frac{8}{n}S_1S_2^2S_3 + \frac{4}{n}S_1^2S_2^3 -\\
& \left.\frac{6}{n^2}S_1^4S_2^2 + \frac{16}{n^2}S_1^3S_2S_3 - \frac{24}{n^3}S_1^5S_3 - \frac{12}{n^3}S_1^6S_2\right)
\end{aligned}
$$

Putting together the expectations of A, B and C, we obtain the final formula for $Var^*(Var^*\hat{(}BV_t^*))$ is given bellow:

$$
\begin{aligned}
Var^*(\widehat{Var^*(BV_t^*)}) =& \frac{\pi^4}{16}\Bigg[S_8\frac{n^4+4n^2-3}{n^6} + S_4^2\frac{2n^4+28n^2-105}{n^6} + S_2^4\frac{12n^2-99}{n^6} - S_1^8\frac{12}{n^6} + \\
& S_2^2S_4\frac{4n^4+64n^2-630}{n^6} + S_1^2S_2^3\frac{24n^2-4n^3-1260}{n^6} + S_2S_3^2\frac{80n^2-840}{n^6} + \\
& S_3S_5\frac{48n^2-168}{n^6} + S_1^2S_6\frac{24n^2-84}{n^6} + S_2S_6\frac{4n^4+32n^2-84}{n^6} + S_1S_7\frac{16n^2-24}{n^6} - \\
& S_1^2S_3^2\frac{16n^2+840}{n^6} + S_1^4S_2^2\frac{2n^2-630}{n^6} + S_1^4S_4\frac{6n^2-210}{n^6} + S_1^6S_2\frac{12n-84}{n^6} - \\
& S_1^3S_5\frac{168}{n^6} + S_1^5S_3\frac{24n-168}{n^6} + S_1S_3S_4\frac{-8n^3+112n^2-840}{n^6} + \\
& S_1^2S_2S_4\frac{-4n^3+72n^2-1260}{n^6} + S_1S_2S_5\frac{80n^2-504}{n^6} + S_1S_2^2S_3\frac{-8n^3+112n^2-1764}{n^6} - \\
& S_1^3S_2S_3\frac{16n^2+1680}{n^6}\Bigg] = O_p(\delta^2)
\end{aligned}
$$

The last step in showing that $\widehat{\Omega^*}$ is consistent for Ω^* is to look at the bias and variance of the estimator of $cov^*(BV_t^*, RV_t^*)$. The expected value of the latter estimator is given by:

$$
\begin{aligned}
E^*(cov^*(\widehat{BV_t^*, RV_t^*})) =& \pi\frac{n-1}{n^2}\Bigg[S_4 + \frac{n^2-n}{n^2}S_1S_3 - \frac{1}{n}\Bigg(S_4 + \frac{n^2-n}{n^2}S_2^2 + \\
& \frac{2(n^2-n)}{n^3}S_1S_3 + \frac{n(n-1)(n-2)}{n^3}S_1^2S_2\Bigg)\Bigg]
\end{aligned}
$$

The bias of $cov^*(\widehat{BV_t^*, RV_t^*})$ is:

$$
\begin{aligned}
Bias(cov^*(\widehat{BV_t^*, RV_t^*})) =& \pi\frac{n-1}{n^2}\Bigg(S_4\frac{n-1}{n} - S_1S_3\frac{3n-2}{n^2} - S_2^2\frac{n-1}{n^2} + \\
& S_1^2S_2\frac{3n-2}{n^3}\Bigg) = O_p(\delta^2)
\end{aligned}
$$

The variance of the covariance estimator is:

$$
\begin{aligned}
Var^*(cov^*(\widehat{BV_t^*, RV_t^*})) =& \frac{\pi^2}{4}E^*\left\{\left[2\frac{n-1}{n^2}\sum|r_i^{*3}|\sum|r_i^*| - 2\frac{n-1}{n^3}\sum|r_i^{*2}|\left(\sum|r_i^*|\right)^2\right]^2\right\} - \\
& \left(E^*(cov^*(\widehat{BV_t^*, RV_t^*}))\right)^2 \\
=& \pi^2\left(\frac{n-1}{n}\right)^2 E^*\left\{\left[\sum|r_i^{*3}|\sum|r_i^*| - \frac{1}{n}\sum|r_i^{*2}|\left(\sum|r_i^*3|\right)^2\right]^2\right\} - \\
& \left(E^*(cov^*(\widehat{BV_t^*, RV_t^*}))\right)^2 \\
=& \pi^2\left(\frac{n-1}{n}\right)^2 E^*\left\{\mathbf{cov1}\right\} - \mathbf{cov2}
\end{aligned}
$$

$$\begin{aligned}\mathbf{cov1} &= \left(\sum |r_i^{*3}| \sum |r_i^*|\right)^2 - \frac{2}{n} \sum |r_i^{*3}| \sum |r_i^*| \sum |r_i^{*2}| \left(\sum |r_i^*|\right)^2 + \frac{1}{n^2} \left(\sum |r_i^{*2}|\right)^2 \left(\sum |r_i^*|\right)^4 \\ &= \mathbf{cov11} - \frac{2}{n}\mathbf{cov12} + \frac{1}{n^2}B\end{aligned}$$

B in the above equation is the same term appearing in the formula for $Var^*(\widehat{Var^*(BV_t^*)})$. Its expectation is given in equation 37.

$$\begin{aligned}\mathbf{cov11} = &\left(\sum r_i^{*4} + \sum\sum_{\neq} |r_i^{*3} r_j^*|\right)^2 \\ &\sum r_i^{*8} + \sum\sum_{\neq} r_i^{*4} r_j^{*4} + 2\sum\sum_{\neq} |r_i^* r_j^{*7}| + 2\sum\sum_{\neq} |r_i^{*3} r_j^{*5}| + 2\sum\sum\sum_{\neq} |r_i^* r_j^{*3} r_k^{*4}| + \\ &\sum\sum\sum\sum_{\neq} |r_i^{*3} r_j^{*3} r_k^* r_l^*| + 2\sum\sum\sum_{\neq} |r_i^{*6} r_j^* r_k^*| + 2\sum\sum\sum_{\neq} |r_i^* r_j^{*3} r_k^{*4}| + \\ &\sum\sum_{\neq} |r_i^{*2} r_j^{*6}| + \sum\sum_{\neq} r_i^{*4} r_j^{*4}\end{aligned}$$

$$E^*(\mathbf{cov11}) = S_8 + 2S_4^2 + 2S_1S_7 + 2S_3S_5 + S_2S_6 + S_1^2S_3^2 + 2S_1^2S_6 + 4S_1S_3S_4$$

$$\begin{aligned}\mathbf{cov12} = &\left(\sum r_i^{*4} + \sum\sum_{\neq} |r_i^{*3} r_j^*|\right) \Bigg(\sum r_i^{*4} + \sum\sum_{\neq} r_i^{*2} r_j^{*2} + 2\sum\sum_{\neq} |r_i^{*3} r_j^*| + \\ &\sum\sum\sum_{\neq} |r_i^{*2} r_j^* r_k^*|\Bigg) \\ = &\sum r_i^{*8} + 3\sum\sum_{\neq} r_i^{*4} r_j^{*4} + 3\sum\sum\sum_{\neq} |r_i^{*2} r_j^{*2} r_k^{*4}| + 4\sum\sum_{\neq} |r_i^{*2} r_j^{*6}| + \\ &5\sum\sum_{\neq} |r_i^{*3} r_j^{*5}| + 3\sum\sum_{\neq} |r_i^* r_j^{*7}| + 3\sum\sum\sum_{\neq} |r_i^{*6} r_j^* r_k^*| + 4\sum\sum\sum_{\neq} |r_i^{*2} r_j^{*3} r_k^{*3}| + \\ &3\sum\sum\sum\sum_{\neq} |r_i^{*3} r_j^{*3} r_k^* r_l^*| + \sum\sum\sum\sum_{\neq} |r_i^{*5} r_j^* r_k^* r_l^*| + 9\sum\sum\sum_{\neq} |r_i^* r_j^{*3} r_k^{*4}| + \\ &3\sum\sum\sum\sum_{\neq} |r_i^{*4} r_j^{*2} r_k^* r_l^*| + 6\sum\sum\sum_{\neq} |r_i^* r_j^{*2} r_k^{*5}| + 3\sum\sum\sum\sum_{\neq} |r_i^{*3} r_j^{*2} r_k^{*2} r_l^*| + \\ &\sum\sum\sum\sum\sum_{\neq} |r_i^{*3} r_j^{*2} r_k^* r_l^* r_m^*|\end{aligned}$$

$$\begin{aligned}E^*(\mathbf{cov12}) = &S_8 + 3S_4^2 + 3S_2^2S_4 + 4S_2S_6 + +5S_3S_5 + 3S_1S_7 + 3S_1^2S_6 + 4S_2S_3^2 + 3S_1^2S_3^2 + S_1^3S_5 + 9S_1S_3S_4 + \\ &3S_1^2S_2S_4 + 6S_1S_2S_5 + 3S_1S_2^2S_3 + S_1^3S_2S_3\end{aligned}$$

$$
\begin{aligned}
Var^*(cov^*(\widehat{BV_t^*, RV_t^*})) =& \frac{\pi^2}{n^2}\Bigg(S_8\frac{n^2-2n+1}{n^2}+S_4^2\frac{n^2-6n+7}{n^2}+S_2^4\frac{2}{n^2}+2S_1S_7\frac{n^2-3n+2}{n^2}+\\
&2S_3S_5\frac{n^2-5n+6}{n^2}-S_1^2S_3^2\frac{6}{n}+2S_1^2S_6\frac{n^2-3n+3}{n^2}+S_2S_6\frac{n^2-8n+8}{n^2}-\\
&4S_2^2S_4\frac{n-4}{n^2}-4S_2S_3^2\frac{2n-5}{n^2}-S_1^3S_5\frac{2}{n}+S_1^2S_2^3\frac{4}{n^2}-S_1^4S_2^2\frac{1}{n^2}+\\
&S_2^2S_4\frac{2}{n}+2S_1S_3S_4\frac{n^2-9n+14}{n^2}-2S_1^2S_2S_4\frac{2n-9}{n^2}-4S_1S_2S_5\frac{3n-5}{n^2}-\\
&4S_1S_2^2S_3\frac{n-7}{n^2}-S_1^2S_2S_3\frac{2}{n}\Bigg)\\
=&O_p(\delta^2)
\end{aligned}
$$

B Appendix 2

Just as in the i.i.d. case, we show that the estimators of all the terms in Ω^*_{WB} are consistent. Gonçalves and Meddahi (2009) show the consistency of $\widehat{Var^*(RV_t^*)}$ for $Var^*(RV_t^*)$ for the case of the wild bootstrap. We are left with proving the consistency for the estimators of the variance of BV_t^* and for the covariance between BV_t^* and RV_t^*. The estimation biases for all these estimators equal 0. Next, we derive their variances and show they converge to 0 in probability.

To compute the variance of $\widehat{Var^*(BV_t^*)}$, we define the following notations: $a = \frac{\mu_2^{*2}-\mu_1^{*4}}{\mu_2^{*2}}$, $b = \frac{\mu_1^{*2}\mu_2^*-\mu_1^{*4}}{\mu_1^{*2}\mu_2^*}$, $a' = \mu_2^{*2}-\mu_1^{*4}$ and $b' = \mu_1^{*2}\mu_2^*-\mu_1^{*4}$.

$$
\begin{aligned}
Var^*(\widehat{Var^*(BV_t^*)}) =& \frac{\pi^4}{16}\left[E^*\left(a\sum {r_i^*}^2 {r_{i+1}^*}^2+2b\sum |r_{i-1}^* {r_i^*}^2 r_{i+1}^*|\right)^2-\right.\\
&\left.\left(a'\sum r_i^2 r_{i+1}^2+2b'\sum |r_{i-1}r_i^2 r_{i+1}|\right)^2\right]\\
=& \frac{\pi^4}{16}\left[\mathbf{A}-\mathbf{B}\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A} =& E^*\left[a^2\left(\sum r_i^2 u_i^2 r_{i+1}^2 u_{i+1}^2\right)^2 + 4b^2\left(\sum |r_{i-1}{r_i}^2 r_{i+1} u_{i-1}{u_i}^2 u_{i+1}|\right)^2 + \right.\\
&\left. 4ab\sum r_i^2 u_i^2 r_{i+1}^2 u_{i+1}^2 \sum |r_{i-1}{r_i}^2 r_{i+1} u_{i-1}{u_i}^2 u_{i+1}|\right]\\
=& a^2\left(\mu_4^{*2}\sum r_i^4 r_{i+1}^4 + 2\mu_2^{*2}\mu_4^*\sum r_i^2 r_{i+1}^4 r_{i+2}^2 + \mu_2^{*4}\sum\sum_{i\neq j\neq j\pm 1} r_i^2 r_{i+1}^2 r_j^2 r_{j+1}^2\right) +\\
& 4b^2\left(\mu_2^{*2}\mu_4^*\sum r_i^2 r_{i+1}^4 r_{i+2}^2 + \mu_1^{*4}\mu_2^{*2}\sum\sum_{i\neq j\neq j\pm 1\neq j\pm 2} |r_{i-1}{r_i}^2 r_{i+1}||r_{j-1}{r_j}^2 r_{j+1}| + \right.\\
& \left. 2\mu_1^{*2}\mu_3^{*2}\sum |r_{i-2}r_{i-1}^3 r_i^3 r_{i+1}| + 2\mu_1^{*2}\mu_2^{*3}\sum |r_{i-2}r_{i-1}^2{r_i}^2 r_{i+1}^2 r_{i+2}|\right) +\\
& 4ab\left(\mu_1^*\mu_3^*\mu_4^*\sum |r_{i-1}^3{r_i}^4 r_{i+1}| + \mu_1^*\mu_3^*\mu_4^*\sum |r_{i-1}{r_i}^4 r_{i+1}^3| + \mu_1^*\mu_2^{*2}\mu_3^*\sum |r_{i-2}r_{i-1}^2{r_i}^3 r_{i+1}^2| + \right.\\
& \left. \mu_1^*\mu_2^{*2}\mu_3^*\sum |r_{i-2}^2 r_{i-1}^3 r_i^2 r_{i+1}|\right)
\end{aligned}
$$

In the above equation, we denote the sums of the type $\sum |r_{i-m}^{k_m} r_{i-m+1}^{k_{m-1}} \ldots r_{i+p-1}^{k_{p-1}} r_{i+p}^{k_p}|$ with $S_{k_m k_{m-1}\ldots k_{p-1}k_p}$.

$$
\begin{aligned}
Var^*(\widehat{Var^*(BV_t^*)}) =& \frac{\pi^4}{16}\left[a'^2 S_{44}\left(\frac{\mu_4^{*2}}{\mu_2^{*4}} - 1\right) + 2a'^2 S_{242}\left(\frac{\mu_4^*}{\mu_2^{*2}} - 1\right) + 4b'^2 S_{242}\left(\frac{\mu_4^*}{\mu_2^{*2}} - 1\right) + \right.\\
& 8b'^2 S_{1331}\left(\frac{\mu_3^{*2}}{\mu_1^{*2}\mu_2^{*2}} - 1\right) + 8b'^2 S_{12221}\left(\frac{\mu_2^*}{\mu_1^{*2}} - 1\right) +\\
& 4a'b' S_{341}\left(\frac{\mu_3^*\mu_4^*}{\mu_1^*\mu_2^{*3}} - 1\right) + 4a'b' S_{143}\left(\frac{\mu_3^*\mu_4^*}{\mu_1^*\mu_2^{*3}} - 1\right) +\\
& \left. 4a'b' S_{1232}\left(\frac{\mu_3^*}{\mu_1^*\mu_2^*} - 1\right) + 4a'b' S_{2321}\left(\frac{\mu_3^*}{\mu_1^*\mu_2^*} - 1\right)\right]
\end{aligned}
$$

All the above sums, if multiplied by the right scale, i.e. $n^{\frac{k_m+\ldots+k_p}{2}-1}$, are realized multipower variations which converge in probability to $\mu_{k_m}\ldots\mu_{k_p}\int_0^t \sigma_s^{k_m+\ldots+k_p}\mathrm{d}s$. In our case, all sums converge to multipliers of $\int_0^t \sigma_s^8 \mathrm{d}s$. Thus, $Var^*(\widehat{Var^*(BV_t^*)}) = O_p(\delta^3)$.

The variance of the estimator for the covariance between RV_t^* and BV_t^* is defined as:

$$
Var^*(\widehat{cov^*(RV_t^*, BV_t^*)}) = E^*\left[(\widehat{cov^*(RV_t^*, BV_t^*)})^2\right] - \left[E^*(\widehat{cov^*(RV_t^*, BV_t^*)})^2\right] \tag{38}
$$

We use the following notation: $c = \frac{\pi}{2}\mu_1^*(\mu_3^* - \mu_2^*\mu_1^*)$.

$$
\begin{aligned}
E^*\left[(cov^*(\widehat{RV_t^*, BV_t^*}))^2\right]^2 =& \frac{c^2}{\mu_1^{*2}\mu_3^{*2}}\Big[\mu_2^*\mu_6^*\sum r_i^6 r_{i+1}^2 + 2\mu_1^*\mu_3^*\mu_4^*\sum |r_{i-1}^3 {r_i}^4 r_{i+1}| + \\
& \mu_1^{*2}\mu_3^{*2}\sum\sum_{i\neq j\neq j\pm 1}|r_i^3 r_{i+1} r_j^3 r_{j+1}| + \mu_2^*\mu_6^*\sum r_i^2 r_{i+1}^6 + 2\mu_1^*\mu_3^*\mu_4^*\sum |r_{i-1}{r_i}^4 {r_{i+1}}^3| + \\
& \mu_1^{*2}\mu_3^{*2}\sum\sum_{i\neq j\neq j\pm 1}|r_i r_{i+1}^3 r_j r_{j+1}^3| + 2\Big(\mu_4^{*2}\sum r_i^4 r_{i+1}^4 + \mu_1^{*2}\mu_6^*\sum |r_i r_{i+1}^6 r_{i+2}| + \\
& \mu_3^{*2}\mu_2^*\sum |r_i^3 r_{i+1}^2 r_{i+2}^3| + \mu_1^{*2}\mu_3^{*2}\sum\sum_{i\neq j\neq j\pm 1}|r_i^3 r_{i+1} r_j r_{j+1}^3|\Big)\Big]
\end{aligned}
$$

To write the final expression for $Var^*(cov^*(\widehat{RV_t^*, BV_t^*}))$, we use the same sum notation as in the case of $Var^*(V\widehat{ar^*(BV}_t^*))$.

$$
\begin{aligned}
Var^*(cov^*(\widehat{RV_t^*, BV_t^*})) =& c^2\Bigg[S_{62}\left(\frac{\mu_2^*\mu_6^*}{\mu_1^{*2}\mu_3^{*2}} - 1\right) + 2S_{341}\left(\frac{\mu_4^*}{\mu_1^*\mu_3^*} - 1\right) + \\
& S_{26}\left(\frac{\mu_2^*\mu_6^*}{\mu_1^{*2}\mu_3^{*2}} - 1\right) + 2S_{143}\left(\frac{\mu_4^*}{\mu_1^*\mu_3^*} - 1\right) + 2S_{44}\left(\frac{\mu_4^{*2}}{\mu_1^{*2}\mu_3^{*2}} - 1\right) + \\
& 2S_{161}\left(\frac{\mu_6^*}{\mu_3^{*2}} - 1\right) + 2S_{323}\left(\frac{\mu_2^*}{\mu_1^{*2}} - 1\right)\Bigg] \\
=& O_p(\delta^3)
\end{aligned}
$$