Discussion Papers in Economics

AGENT-BASED MACROECONOMICS AND DYNAMIC STOCHASTIC GENERAL EQUILIBRIUM MODELS: WHERE DO WE GO FROM HERE?

By

Özge Dilaver
(University of Surrey)
Robert Jump
(Kingston University London)
&
Paul Levine
(University of Surrey)

DP 01/16

School of Economics
University of Surrey
Guildford
Surrey GU2 7XH, UK
Telephone +44 (0)1483 689380
Facsimile +44 (0)1483 689548
Web www.econ.surrey.ac.uk
ISSN: 1749-5075

Agent-based Macroeconomics and Dynamic Stochastic General Equilibrium Models: Where do we go from here?*

Özge Dilaver
School of Social Sciences
University of Surrey

Robert Jump
School of Economics, Politics and History
Kingston University London

Paul Levine
School of Economics
University of Surrey

January 7, 2016

Abstract

Agent-based computational economics (ACE) has been used for tackling major research questions in macroeconomics for at least two decades. This growing field positions itself as an alternative to dynamic stochastic general equilibrium (DSGE) models. In this paper we first review the arguments raised against DSGE in the ACE literature. We then review existing ACE models, and their empirical performance. We then turn to a literature on behavioural New Keynesian models that attempts to synthesise these two approaches to macroeconomic modelling by incorporating some of the insights of ACE into DSGE modelling. We highlight the individually rational New Keynesian model following Deak et al. (2015) and discuss how this line of research can progress.

JEL Classification: E03, E12, E32.
Keywords: agent-based computational economics, agent-based macroeconomics, dynamic stochastic general equilibrium models, new Keynesian behavioural models

*We acknowledge financial support from the ESRC, grant reference ES/K005154/1. We would like to thank participants at a seminar on DSGE and ACE at the University of Surrey, held on 20/11/15, and Domenico Delli Gatti for encouraging comments and feedback. All remaining errors are the responsibility of the authors.

1 Introduction

Agent based (AB) modelling is a computational research method that is frequently used in studies of complex social phenomena. As the name suggests, simple representations of decision-makers in social, economic, or political contexts are at the core of this method. By generating a high number of heterogeneous agents that can respond to individual and local as well as aggregate variables, researchers can simulate adaptive behaviour, interdependent decision making, spatial patterns and social networks in a broad range of contexts. The AB literature in economics has been given a number of names, but it is most commonly referred to as agent-based computational economics (ACE).

The present paper reviews both the macroeconomic ACE literature and the literature that attempts to incorporate some of the insights of ACE into DSGE modelling. The purpose of the present study is therefore synthesis - we review and justify a literature that has responded to dissatisfaction with conventional DSGE modelling over the past 5-10 years. As well as presenting and placing very recent research within this literature, we discuss how the program as a whole can develop. We begin with a brief overview of some relevant criticisms of DSGE modelling from the ACE perspective, followed by an examination of some key models in the macroeconomic ACE literature and their empirical performance. We then discuss the extent to which the limitations highlighted in ACE studies have been addressed in recent DSGE models, particularly the behavioural New Keynesian model. Finally, we present very recent work following Deak et al. (2015) on the individually rational New Keynesian model, which provides the behavioural New Keynesian model with more robust microfoundations. We hope that this will provide a framework for future work towards a more realistic macroeconomics.

The paper is organised in five sections. Section 2 reviews the criticisms that the macro ACE literature directs towards DSGE models. Section 3 provides an overview of the macro ACE literature, and some key models. Section 4 reviews the empirical performance of these key models. Section 5 considers how the insights of the macro ACE literature can be incorporated into the DSGE framework, and discusses the behavioural and individually rational New Keynesian models. Finally, section 6 concludes.

2 ACE Critiques of DSGE

In this section we review some of the criticisms of DSGE modelling from an ACE perspective. We do not present a comprehensive review of the criticisms directed at DSGE from ACE scholars. Rather, we present those criticisms that seem most important in achieving a constructive re-evaluation of the DSGE and ACE approaches. Specifically, we consider three arguments: that DSGE models ignore heterogeneity amongst agents, complex dynamics, and bounded rationality.

2.1 Representative Agents versus Heterogeneous Interacting Agents

The representative agent (RA) assumption is the idea that a single agent can stand for an entire sector of the economy. This is, arguably, at the centre of both DSGE analysis and criticism towards it. Alongside a number of epistemological concerns (see Delli Gatti et al. (2005), Delli Gatti et al. (2010)), ACE critics argue that the RA assumption ignores heterogeneity, non-normal distributions, interactions between agents, and is incompatible with observations of scaling effects (Delli Gatti et al. (2005)). By ignoring interactions and interdependencies between agents, RA overlooks the occurrence of large aggregate fluctuations as a consequence of small idiosyncratic shocks and does not allow any room for emergent macroscopic patterns (Delli Gatti et al. (2005), Gaffeo et al. (2007), see also Gabaix (2011) in this respect). Finally, the RA assumption is also criticised in terms of the purpose it serves. In situations where preferences are influenced by policy regimes (Bowles (1998), Delli Gatti et al. (2010)), the RA approach might not be able to deliver policy-independent microfoundations.

2.2 Complexity and Endogenous Cycles

In light of the arguments that DSGE modelling ignores heterogeneity and local interaction by specifying representative agents, ACE scholars criticise mainstream macroeconomics for ignoring the fundamental complexity of economic dynamics. Whilst complexity is often only approximately defined, the basic determinants are a high dimensional state space and a degree of non-linearity such that superposition is not present. Superposition is a property of linear systems, whereby the net response of a system to two or more simultaneous impulses is given by the sum of the responses of the system to the same impulses separately. In particular, two identical impulses, differing only in sign, will cancel each other out. With non-linear models this property fails to hold, and as such the response of a non-linear system to an impulse is not necessarily proportional to the size of the shock, and the state of the system will matter in determining the response to any given shock. Thus small shocks can give rise to large business cycles, and phenomena such as financial fragility can be studied.

The macro ACE community argue that this type of complexity is pervasive in macroeconomics, with particular importance being given to endogenous business cycles, or business cycles that are not driven by aggregate shocks. Dosi et al. (2008), for example, argue that real business cycle models and New Keynesian DSGE models are both inadequate because a large part of the dynamics are driven by aggregate technology shocks, and "both streams of literature dramatically underestimate the role of endogenous technological shocks occurring at the microeconomic level" (*ibid.*). Gaffeo et al. (2007), similarly, argue that "complexity arises because of the dispersed, localized, non-linear interactions of a large number of heterogeneous components", and that the economy should be modelled as such.[1]

2.3 Rational Expectations versus Bounded Rationality

The final ACE criticism of the DSGE paradigm that we wish to highlight is concerned with rational expectations. This criticism is similar to existing critiques of the rational expectations hypothesis, which extend back to the early days of New Classical economic theory, but arises as an almost necessary conclusion from the focus on heterogeneity and complexity. An interesting extension of the rational expectations critique from the ACE community is linked to the concept of "emergence" - since real-life versions of economic agents are clearly not equipped with perfect information and foresight, rational expectations are not a property of individual agents but of the system as a whole. In other words, rational expectations supposes that expectations are correct *on average*, and economic theory refers to an instance of "emergence" in the RE hypothesis. This, of course, is contested. For example, Howitt argues that, "even blind faith in individual rationality does not guarantee that the system as a whole will find this fixed point [of rational expectations]" (Howitt (2012)).

2.4 Summary

Given the above, ACE models move away from DSGE models along the lines of heterogeneity, complexity, and rationality. It is worth noting at the outset that a number of dynamic general equilibrium models have already addressed these issues, to a certain extent - the highly nonlinear models of Branch and McGough (2010), or the internal rationality models of Eusepi and Preston (2011), for example and there is a large and growing DSGE literature with heterogeneous agent

[1] It should be noted that endogenous business cycles have a history that long pre-dates ACE - Hicks, Kalecki, and Kaldor were early advocates, and Goodwin (1967) is a classic example. This literature continues in contemporary Post Keynesian and Marxian economics.

models in the spirit of Krusell and Smith (1998). Before these can be considered in greater detail in section 5, some key macroeconomic ACE models are examined in section 3, and their empirical performance is reviewed in section 4.

3 Key Macro ACE Models

Although it is often difficult to classify and compare ACE models, it is possible to identify three major families of models within the macro ACE literature. These are the Keynes meets Schumpeter (K&S) model (Dosi et al. (2006), Dosi et al. (2008), Dosi et al. (2010), Dosi et al. (2013)), the CATS model[2] (Delli Gatti et al. (2005), Delli Gatti et al. (2007), Delli Gatti et al. (2010), Gaffeo et al. (2007), Russo et al. (2007), Ricetti et al. (2013)), and the Eurace model (Deissenberg et al. (2008), Dawid et al. (2009), Cincotti et al. (2010), Dawid and Neugart (2011), Raberto et al. (2011)). The following subsections briefly examine each of these model families in turn, focusing on the areas of disagreement with DSGE described above: agent heterogeneity, complexity, and rationality.

3.1 The K&S Model

The K&S model was first developed in Dosi et al. (2006) and Dosi et al. (2008) for exploring industry dynamics and for simulating endogenous business cycles with Keynesian features. Later versions of the model emphasise the synthesis that the model creates between Keynesian approaches to demand dynamics and Schumpeterian approaches to innovation, and add Minskyan credit dynamics to the analysis (Dosi et al. (2010), Dosi et al. (2013)). Given this, early versions of the model incorporated a great deal of agent heterogeneity, with relatively little direct interaction. In Dosi et al. (2006) and Dosi et al. (2008), firms are divided into consumption and capital goods firms, with the latter producing heterogeneous capital goods driven by idiosyncratic shocks to firm level technology. In Dosi et al. (2008), all direct interaction takes place between consumption and capital goods firms, with all other interaction at an aggregate level - the real wage, for example, evolves according to an aggregate wage equation.

Agent behaviour in this context is boundedly rational. Whilst households consume all of their income (an extreme form of the Keynesian consumption heuristic), firms price and produce in an approximately Post Keynesian manner. For example, production levels are determined by naive

[2]Also referred to as the MBU model in Delli Gatti et al. (2011). MBU stands for “Macroeconomics from the Bottom Up”, whereas CATS stands for “Complex AdapTive System” (*ibid.*).

or adaptive expectations over demand levels, and pricing is given by a mark-up over unit costs, where the mark-up evolves according to the following heuristic:

$$\mu_{jt} = \mu_{jt-1}\left(1 + \frac{f_{jt-1} - f_{jt-2}}{f_{jt-2}}\right). \tag{1}$$

In (1), μ_j denotes firm j's mark-up, whilst f_j denotes firm j's market share. Whilst this heuristic could be interpreted as pursuing profit maximisation, there is no attempt in the Dosi et al papers to formally justify this. Given the above, the K&S model produces macroeconomic complexity in the sense of endogenous business cycles, which appear to be driven by pervasive non-linearity and idiosyncratic shocks.

3.2 The CATS Model

Like the K&S model, the CATS model was initially built for studying business cycles, and incorporates heterogeneity and bounded rationality, but the structures of the two families of models are quite different. In general, the CATS models are simpler than the K&S models in terms of agent types, and more complex in terms of direct interaction. For example, the Russo et al. (2007) model is a one sector model, with idiosyncratic R&D shocks at the firm level. Again, firms' pricing strategies are boundedly rational, and evolve according to the following heuristic:

$$P^s_{jt} = \begin{cases} P_{jt-1}(1+\eta_{jt}) & \text{if } S_{jt-1} = 0 \\ P_{jt-1}(1-\eta_{jt}) & \text{if } S_{jt-1} > 0 \end{cases} \tag{2}$$

In (2), S_j denotes the firm's stock of unsold goods, and η_j is a firm specific idiosyncratic shock. Hence the firm raises its price if it sells all its produced output in the previous period, and lowers it otherwise. This is not associated with profit maximisation, but is associated with Simon's "satisficing" approach to firm behaviour (P^s_j denotes the firm's "satisfying" price; this is equal to the selling price if it covers unit cost). As with the K&S models, the CATS models produce macroeconomic complexity in the sense of endogenous business cycles driven by idiosyncratic shocks and pervasive non-linearity.

3.3 The Eurace Model

The Eurace model was produced in an attempt to construct an agent-based model of the European economy. The aim is to simulate interactions of a very large number of heterogenous agents within a complex environment that represents NUTS-2 regions of the EU-27 countries (Deissenberg et al. (2008)). As with the K&S and CATS model, the Eurace model successfully produces endogenous business cycles driven by non-linearity at the microeconomic level. In this larger model the heuristic behaviour of at least some of the agents is brought closer to individual rationality - that is, explicit utility and/or profit maximisation in the context of bounded rationality. In particular, the pricing decision of consumption goods firms is predicated on the belief in a CES demand function. Denoting the expected price elasticity as ε^e, firms set prices in the Eurace model as follows:

$$p_{jt} = \frac{\bar{c}_{jt-1}}{1 + 1/\varepsilon^e_{jt}} \tag{3}$$

In (3), $\bar{c}_{jt-1}$ is a measure of unit costs that takes into account past costs and inventory levels. Household behaviour is also based on individual rationality, where the decision rule is justified by appeal to prospect theory, and in particular the theory of loss aversion.

3.4 Summary

The overview of some of the key elements of the K&S, CATS, and Eurace models provided above can only scratch the surface of what are vibrant and continuing research programs. Later versions of the K&S and CATS models, for example, incorporate credit and banking networks, and examine the role of government policy in controlling fluctuations and growth. Nevertheless, we hope to have given an indication of the manner in which existing macro ACE models incorporate heterogeneity, complexity, and bounded rationality.

4 Empirical Performance of Macro ACE Models

This section reviews the empirical performance of the models examined in section 3. As above, we consider the three most prominent families of macro ACE models sequentially: the Dosi et al K&S model, the Delli Gatti et al CATS model, and the Eurace model. In addition, we examine two more recent models: the Lengnick (2013) model, and the Assenza et al. (2015) model. First, we provide a brief overview of the empirical method behind existing ACE studies, and examine the empirical performance of the existing models in sections 4.2, 4.3, 4.4, and 4.5. Finally, we briefly consider the prospects for future empirical work in section 4.6.

4.1 Estimation in ACE

For the purpose of estimation, one can write the general form of an agent based model as follows:

$$\mathbf{X_t} = F(\mathbf{X_{t-1}}, \epsilon_{\mathbf{t-1}}, \theta) \tag{4}$$

Here, $\mathbf{X}$ is the matrix of individual states, ϵ is the vector of exogenous shocks, and θ is a vector of parameters. Assuming that we can derive a vector or matrix of observable variables $\mathbf{Y_t}$ from $\mathbf{X_t}$, and that the former correspond to a set of observed variables $\mathbf{Q}$, the question faced is how to estimate θ such that the "distance" between $\mathbf{Y}$ and $\mathbf{Q}$ is minimised. Interpreting ϵ as a vector of stochastic shocks with a known distribution, and therefore the output of the model described by (4) as a joint probability distribution for $\mathbf{X}$, *and* assuming that the model described by (4) corresponds to the data generating process behind $\mathbf{Q}$ up to the unknown values of θ, is usually referred to as the probability approach to economics. This can be traced back to Haavelmo (1944), and allows formal statistical inference to be employed in the model estimation procedure.

Given the above, the estimation and validation procedures employed in the macroeconomic ACE literature are predominantly informal, in a similar spirit to the approach taken by the early real business cycle literature (see Dejong & Dave 2007, ch.6). The chief approach is the classic calibration exercise, which involves the assignment of numerical values to model parameters based on a priori belief, external information, or steady state requirements. A well known example is the unique determination of the steady state real interest rate by the rate of time preference in real business cycle models; if one observes an average annual real interest rate of 5% over time, for example, a natural parameterisation for the rate of time preference is 5%.

Model calibration, as recorded in DeJong and Dave (2007), can be traced back to Kyldland and Prescott (1982). In turn, Kydland and Prescott trace their approach to the econometric method of Ragnar Frisch:

> In this review (Frisch) discusses what he considers to be 'econometric analysis of the genuine kind' . . . and gives four examples of such analysis. None of these examples involve the estimation and statistical testing of some model. None involve an attempt

to discover some true relationship. All use a model, which is an abstraction of a complex reality, to address some clear-cut question. (Kydland and Prescott 1991a, in Dejong and Dave *ibid.*: 123)

This approach, in which the choice of numerical values for model parameters is independent of formal statistical considerations, is completely at odds with the Haavelmo (1944) approach described above. Whether or not one agrees with this approach or not, the key observation is that if one does in fact compare summary statistics of the model generated joint probability distribution with the equivalent statistics observed in economic data, one is implicitly taking the probability approach to economics. Whilst the calibration approach to parameter choice can certainly be seen in the earlier published output of Dosi et al and Delli Gatti et al, later papers uniformly provide second moment comparisons alongside the calibration exercise. In this regards, ACE is moving beyond the simple calibration exercise approach to quantitative macroeconomics. In what follows, we consider the validation and recorded performance of the models considered in section 3, as well as the Lengnick (2013) model and the Assenza (2015) model.

4.2 The K&S Model

Out of the papers studying the K&S model, Dosi et al. (2008) provides the most in depth empirical validation. Dosi et al. (2008) initially identifies a number of stylised facts to facilitate basic qualitative validation of the model. These include the standard US business cycle facts, the lumpiness and finance dependent nature of individual firm investment expenditure, pronounced and persistent productivity dispersion across firms, and the distinctive distributions of firm size and firm growth rates. The basic calibration follows the antecedent models in Dosi et al. (2005) and Dosi et al. (2006), but is otherwise unexplained. Nevertheless, the model reproduces the basic stylised facts that the authors target, and this result appears to be robust to the exact parameterisation used (see Dosi et al. (2006)).

Of greater interest are the cross-correlograms presented. These compare the correlations at plus and minus four lags of band pass filtered consumption and GDP, investment and GDP, stock accumulation and GDP, employment and GDP, and unemployment and GDP. Interestingly, given the detailed modelling of firm level investment in the model, and the ability of the model to match cross-sectional stylised facts, aggregate investment still performs poorly in comparison to the other time series. This is in line with the failure of standard New Keynesian DSGE models to match aggregate investment data satisfactorily.

4.3 The CATS Model

Out of the papers studying the CATS model family, Gaffeo et al. (2008) provides the most in depth empirical validation. This paper is particularly interesting, from our perspective, in that it explicitly attempts to "rival the explanatory power of DSGE models" (*ibid.*: 443). As with Dosi et al. (2008), the calibration is unexplained. However, instead of a list of stylised facts, the authors regard endogenous business cycles as a basic explandum, and compare the model's time series co-movements with US data. Unfortunately the correlograms presented are mostly not comparable with those of Dosi et al. (2008), describing the correlations at plus and minus four lags of Hodrick-Prescott filtered employment and GDP, productivity and GDP, price index and GDP, interest rate and GDP, and the real wage and GDP. Neither is there an attempt to compare these correlations with a standard New Keynesian DSGE model - although on balance, it seems fair to say that the model performs relatively poorly compared to Dosi et al. (2008).

4.4 The Eurace Model

Unsurprisingly, the Eurace model, given its size, is also not subject to formal estimation. Given this, the Eurace model builders approach the calibration problem in the same manner as early versions of the K&S and CATS models - a set of stylised facts is identified, and the region of the parameter space that can reproduce those facts is identified. In general, as before, this seems to be a relatively informal method, but Dawid et al. (2009) cite a number of varied empirical studies to justify the choice of calibration. It is difficult, from the available literature, to judge the empirical performance of the calibrated Eurace model.

4.5 Recent Models

There exist two prominent macro ACE models that attempt to combine the insights of the K&S and CATS frameworks into simplified, more manageable models. The aim of Lengnick (2013), the first of these, is to "take the most prominent ACE macro models and reduce them in complexity" (pp.104). Again, the calibration is unexplained, but the model succeeds in generating artificial Phillips curves, Beveridge curves, and the long run neutrality and short run non-neutrality of money. The only cross-correlogram presented is between the price level and GDP, and the model appears to perform as least as well as Gaffeo et al. (2008) along this dimension.

The second attempt to combine the K&S and CATS frameworks is presented in Assenza et al. (2015) - in this case, by including capital goods in a zero growth CATS framework. Again, the calibration is relatively arbitrary, although a small number of the parameter choices are explained (e.g. the desired level of capacity utilisation is chosen to match average capacity utilisation in the USA). However, this paper presents by far the most in depth moment comparison exercise, comparing autocorrelation functions and cross-correlograms of HP filtered GDP, consumption, investment, and unemployment, against equivalent US time series, as well as absolute standard deviations. The model performs strikingly well along these dimensions. Again, however, it is interesting to note that investment still performs poorly, in a similar manner to DSGE models - it is considerably more volatile than in the data. Interestingly, the model correlogram between unemployment and total debt is qualitatively similar to that in US data, although there appears to be a small phase shift, and the model correlations are lower at each lag than in the data.

Finally, the authors conclude with the observation that: "Bringing the model to the data will be key to assess the effects of policy moves in a quantitative framework. The econometric practice in agent based modelling is still in its infancy so that there is a long way to go before the model is fully operative . . . however, the results obtained so far are promising and bode well for the future." This, more than anything, illustrates the apparent shift to viewing macroeconomic agent based models as complete probability models in the sense of Haavelmo (1944), a view that does not appear to be shared in earlier presentations of the CATS model.

4.6 GMM and Indirect Inference

Viewing agent based models as complete probability models allows a considerably greater arsenal of econometric techniques to be employed in model validation, at least in principle. The obvious possibilities are already regularly used in the DSGE literature - namely, the generalised method of moments (analytical or simulated), indirect inference, and full information maximum likelihood (with or without Bayesian priors). Interestingly, a version of the CATS model was estimated in 2007 using indirect inference, which is itself a special case of the generalised method of moments. However, the moments used to estimate the model describe the firm size distribution in Italian data - specifically, Pareto exponents estimated on those data. Given that the aggregate dynamics of macroeconomic agent based models often appear to be driven by idiosyncratic shocks which are not averaged out, given highly skewed size distributions, the distributions of firm size considered in Bianchi et al. (2007) are of macroeconomic interest (also see Gabaix (2011) in this regard). Nevertheless, these statistics are arguably of second-order importance if the basic explananda are still macroeconomic aggregates in the traditional sense. In principle, however, there is no reason why indirect inference or simulated GMM cannot be used to estimate macroeconomic agent based models against aggregate time series data.

Before we consider the use of limited information methods to estimate macroeconomic agent based models, consider the theoretically ideal estimation procedure - maximum likelihood estimation. The maximum likelihood estimator uses the following objective function:

$$Q(\theta) = \frac{1}{t}\sum_{t=1}^{T} \ln L(x_t|\theta) \tag{5}$$

Maximum likelihood chooses the parameter vector θ that maximise the log-likelihood of observing a sample x given the model in question. In practice, in a highly non-linear model, even if we know the distribution of the exogenous shock processes, we will not be able to derive closed form solutions for L and thus computing $Q(\theta)$ is a challenging task. This is compounded in agent based models, where computing the relevant time series for one parameterisation alone can be computationally costly. As a result, and as argued by Grazzini and Richiardi (2015), the only feasible way forward with complex agent based models appears to be via some type of simulated minimum distance estimation. In this case, (5) is approximated by the following objective function (DeJong and Dave, 2007, pp. 154):

$$\Gamma(\theta) = -g(X,\theta)'\Omega g(X,\theta) \tag{6}$$

Here, $g(x_t,\theta) = 1/t\sum f(x_t,\theta)$, where $f(x_t,\theta)$ is a vector valued function relating the parameter vector to the observed variables, summarising the "moment conditions" used in estimation. One then wishes to minimise the norm of $g(X,\theta)$ - that is, choose θ to minimise the distance between the model generated moments and the observed sample moments. The choice of weights Ω used in (6) will then determine the norm used, and the optimal weights turn out to be those that minimise the asymptotic variance of $\hat{\theta}$ (see Hansen (1982) or DeJong and Dave (2007) for further discussion).

Equation (6) is therefore an approximation to (5), disregarding a certain amount of the available sample information in return for computational feasibility. This type of estimation procedure, as argued in Grazzini and Richiardi (2015), would appear to be the logical step forward for the empirical validation of macroeconomic agent based models - particularly because a version of the procedure appears to have been informally performed in the calibration exercises reviewed above. Unfortunately, as indicated in Bianchi et al. (2007), agent based models are a priori no less likely to suffer from flat objective function problems than DSGE models - a difficult problem for any extremum estimator approach, whether GMM, indirect inference, or maximum likelihood. In the state of the art in structural macroeconometrics, this problem is usually solved by the incorporation of prior assumptions and Bayesian techniques - but an application of this method to the K&S model or CATS model is hard to imagine in the near future. Moreover, unlike in DSGE models, where the order of integration and ergodicity are usually known a priori, it may not be immediately obvious whether or not agent based models are stationary or ergodic. As defined in Grazzini and Richiardi (2015), stationarity implies the existence of a statistical equilibrium, whilst ergodicity implies that, regardless of initial conditions, the model will always settle at the same statistical equilibrium. Whilst ergodicity in the strict sense is not necessary for the use of simulated minimum distance estimation, stationarity is necessary.[3] As well as the difficulty of applying Bayesian techniques to ACE, therefore, one also faces the difficulty of establishing ergodicity and stationarity when this is not known a priori.

Given the above, there appear to be two possibilities. The first is to apply simulated minimum distance methods to macroeconomic agent based models as they stand, and deal with the associated flat objective function problems pragmatically, taking into account the antecedent problems of stationarity and ergodicity. The second is to incorporate some of the insights of agent based modelling into more traditional macroeconomic models, such that stationarity and ergodicity can be established a priori, and Bayesian techniques can be used.

4.7 Summary

Two major conclusions stand out from sections 3 and 4: first, that macro ACE models are very rich, if rather varied, and second, that their complexity and analytical intractability make them very difficult to validate empirically. Related to this is the absence of formal estimation procedures in the vast majority of macroeconomics ACE papers. However, the available evidence - particularly that presented in Assenza et al (2015), suggests that ACE is a fruitful modelling approach in macroeconomics. Given the relative ease of estimating and validating DSGE models, and given the significant amount of expertise in DSGE modelling and estimation in the profession as a whole, this suggests that a fruitful way forward might be the incorporation of ACE insights into DSGE modelling.

[3] Grazzini and Richiardi are concerned with *weak* stationarity; their definition of statistical equilibrium is then a stationary aggregate outcome over a time window $[T_0, T_\tau]$ given initial conditions. Given this, stationarity is a necessary condition for estimation by simulated minimum distance, as without this, the estimator will not be consistent. Grazzini and Richiardi argue that ergodicity is not necessary, as given initial conditions the model can still produce stationary output (*ibid.*: 154).

5 Bridging the Gap

Section 2 of this paper isolated what we believe to be the three most important differences between DSGE and ACE models. To recap, these are:

- The representative agent assumption versus heterogeneous interacting agents.
- Dynamics driven by exogenous shocks versus complexity and endogenous cycles.
- Rationality, in particular rational expectations, versus bounded rationality.

This section reviews studies within the DSGE framework that incorporate bounded rationality, heterogeneous agents, and endogenous business cycles. In particular, we review those studies that incorporate the Brock-Hommes complexity framework into DSGE models, and in so doing respond to the three critiques of DSGE identified here. These models are known as behavioural New Keynesian models. In the sense that these papers have features that characterize ACE, they bridge the gap between the two modelling approaches. In section 5.1 we review the Brock and Hommes complexity approach, and in section 5.2 the behavioural New Keynesian model. In section 5.3 we review Adam and Marcet's approach to individual rationality, and explain how this can be incorporated into the behavioural New Keynesian model to improve the microfoundations of the latter. Section 5.4 discusses the theoretical and numerical properties of the individually rational New Keynesian model, following Deak et al. (2015).

5.1 Brock-Hommes Complexity

The complexity framework comprehensively described in Hommes (2013), and first introduced in Brock and Hommes (1997), provides a minimal way of generating complex dynamics via heterogeneous agents with varying degrees of rationality. As such it provides a simple method of answering the major critiques of DSGE outlined above, but until recently was only explored in the context of partial equilibrium models. A simple cob-web model demonstrates the main features. The model is of a partial equilibrium with two types of producers. A proportion $n_{1,t}$ form rational expectations of the price level p_t at time t, denoted by $\mathbb{E}_t(p_t)$. This amounts to perfect foresight so $\mathbb{E}_t(p_t) = p_t$. The remaining proportion of producers, $1 - n_{1,t}$, are boundedly rational in a manner to be defined. Their expectations, formed at time $t-1$, are denoted by $\mathbb{E}^*_{t-1}(p_t)$. We assume linear demand and supply curves subject to random shocks $\epsilon_{d,t}$ and $\epsilon_{s,t}$ respectively. Given $n_{1,t}$ and our definition of

$\mathbb{E}^*_{t-1}(\cdot)$, the market-clearing price is given by:

$$
\begin{aligned}
D(p_t) &= a - dp_t + \epsilon_{d,t} &(7)\\
S(p_t, \mathbb{E}^*_{t-1}(p_t), n_{1,t}) &= s(n_{1,t}p_t + (1 - n_{1,t})\mathbb{E}^*_{t-1}(p_t)) + \epsilon_{s,t} &(8)\\
D(p_t) &= S(p_t, \mathbb{E}^*_{t-1}(p_t), n_{1,t}) &(9)
\end{aligned}
$$

where a, d and s are fixed parameters. These pin down the deterministic steady state of the price level denoted by p. We assume that boundedly rational agents eventually forecast correctly, so that in the steady state $p_t = \mathbb{E}^*_{t-1}(p_t) = p$ which is given by $p = \frac{a}{d+s}$.

The learning literature adopts two basic approaches to modelling boundedly rational expectations. The first is usually referred to as statistical learning, where agents are competent econometricians who make observations of the price (in this example), have some idea of the data generating process and estimate it using standard techniques. We leave a discussion of this approach to our later application to a macroeconomic model below. Here we adopt the second approach, which assumes that agents use simple heuristic forecasting rules. A general formulation that nests particular examples found in the literature is an adaptive expectations rule of the form

$$
\mathbb{E}^*_{t-1}(p_t) = \mathbb{E}^*_{t-2}(p_{t-1}) + \lambda(p_{t-1} - \mathbb{E}^*_{t-2}(p_{t-1})\,;\ \lambda \in [0, 1] \qquad (10)
$$

The key component of the Brock-Hommes framework giving rise to complex dynamics is the method by which the proportions of rational and non-rational producers are updated over time. Here the literature adopts a basic general framework set out in Young (2004). To limit the departure from rationality, the approach of reinforcement learning proposes that, although adaptation can be slow and there can be a random component of choice, the higher the 'payoff' (defined appropriately) from taking an action in the past, the more likely it will be taken in the future. Here the payoff is defined as last period's squared forecasting error plus the costs of obtaining that forecasting rule. Then the updated fractions of rational producers is given by a discrete logit model:

$$
\begin{aligned}
n_{1,t} &= \frac{\exp(-\gamma[(p_t - \mathbb{E}_t(p_t))^2 + C])}{\exp(-\gamma([p_t - \mathbb{E}_t(p_t))^2 + C]) + \exp(-\gamma([p_t - \mathbb{E}^*_{t-1}(p_t)]^2)}\\
&= \frac{\exp(-\gamma C)}{\exp(-\gamma C) + \exp(-\gamma([p_t - \mathbb{E}^*_{t-1}(p_t)]^2)} &(11)
\end{aligned}
$$

The key features of (11) is that the best-performing rule will attract the most followers, and that there is a fixed per period cost, C, of making rational predictions. The parameter γ is referred to in the literature as the intensity of choice and dictates how quickly agents will switch to the best-performing rule. The steady state proportion of rational producers is given by $n_1 = \frac{\exp(-\gamma C)}{\exp(-\gamma C)+1}$.

The stability properties of this model depend on the parameter values s, d, a, C that determine the steady state and λ, γ that determine the speed of learning. For a high proportion of rational producers the model exhibits local stability: in response to an exogenous shock price and output return to their steady state values. As C increases above zero the proportion of rational producers falls and we enter regions of local instability. However depending on λ, γ, the trajectories are locally unstable, but do not explode. Rather they show chaotic patterns: random-like complex behaviour. As forecast errors under adaptive expectations become large, non-rational but intelligent producers switch behaviour by investing the amount C needed to make rational forecasts. Then forecast errors fall and they switch back to non-rational forecasting.

5.2 The Behavioural New Keynesian Model

The Brock-Hommes framework has been used by a number of authors to propose a behavioural version of the standard New Keynesian (NK) model (see e.g. Woodford (2003), Gali (2008)). These include Branch and McGough (2010), Branch et al. (2012), Branch and Evans (2011), De Grauwe and Katwasser (2012), De Grauwe (2011), De Grauwe (2012a), De Grauwe (2012b), Jang and Sacht (2012), Massaro (2013) and Jang and Sacht (2014).

The basic three-equation linearized work-horse NK model used in this literature in its rational expectations form is as follows:

$$y_t = \mathbb{E}_t y_{t+1} - (r_t - \mathbb{E}_t \pi_{t+1}) + u_{1,t} \tag{12}$$

$$\pi_t = \beta \mathbb{E}_t[\pi_{t+1}] + \lambda(y_t + u_{2,t}) \tag{13}$$

$$r_t = \rho_r r_{t-1} + (1 - \rho_r)(\theta_\pi \pi_t + \theta_y y_t) + u_{3,t} \tag{14}$$

where y_t, π_t and r_t are the output gap, the inflation rate and the nominal interest rate respectively. The shock processes $u_{i,t}$, $i = 1, 3$ should be interpreted as shocks to preferences, marginal costs and monetary policy, respectively, and are usually AR(1) processes. (12) is the linearized Euler equation for consumption which is equated with output in equilibrium (there is no government expenditure). (13) is the NK Phillips curve and (14) is the nominal interest rate rule with persistence responding to current inflation and the output gap. Expectations up to now are formed assuming rational expectations and perfect information of the state vector (which includes the shock processes).[4]

As in the cob-web model example, the model becomes behavioural by a departure from RE and the introduction of two groups of agents forming expectations through different learning rules.

[4] Habit in consumption and price indexing result in additional lags in y_t in (12) and in π_t in (13) providing additional persistence mechanisms that help to fit the model to data.

In De Grauwe (2012b) there are two groups using fundamentalist (f) and extrapolative (e) rules with (possibly) non-RE market expectations denoted by $\mathbb{E}^*$. The market forecasts are assumed to be simple weighted averages:

$$\begin{aligned} \mathbb{E}_t^* y_{t+1} &= \alpha_{f,t}\mathbb{E}_t^* y_{t+1}^f + (1-\alpha_{f,t})\mathbb{E}_t^* y_{t+1}^e & (15) \\ \mathbb{E}_t^* \pi_{t+1} &= \beta_{f,t}\mathbb{E}_t^* \pi_{t+1}^f + (1-\beta_{f,t})\mathbb{E}_t^* \pi_{t+1}^e & (16) \end{aligned}$$

We refer to this approach to non-rational expectations as the Euler Learning (EL) approach. The model is completed by the expressions for the weights $\alpha_{f,t}, \beta_{f,t}$, and the learning rules for the output gap and inflation. The former follows the Brock-Hommes framework as follows:

$$\begin{aligned} \alpha_{f,t} &= \frac{\exp(\gamma U_{f,t}(\{y_t\})}{\exp(\gamma U_{f,t}(\{y_t\})) + \exp(\gamma U_{e,t}(\{y_t\}))} & (17) \\ \beta_{f,t} &= \frac{\exp(\gamma U_{f,t}(\{\pi_t\}))}{\exp(\gamma U_{f,t}(\{\pi\}_t)) + \exp(\gamma U_{e,t}(\{\pi\}_t))} & (18) \end{aligned}$$

where $U_{f,t}(\{x_t)\})$ is the payoff measure of the fundamentalist rule for outcome $\{x_t\} = \{y_t\}, \{\pi_t\}$, given by a MSE predictor:

$$U_{f,t}(\{x_t\})) = \rho U_{f,t-1}(\{x_t\})) - (1-\rho)[x_{t-1} - \mathbb{E}_{f,t-2}\, x_{t-1}]^2 \qquad (19)$$

Equations (12)-(14), with $\mathbb{E}$ replaced with $\mathbb{E}^*$, and equations (15)-(19) completes the behavioural New Keynesian model without specifying the forecasting rules. Whilst De Grauwe, for example, uses a selection of boundedly rational predictors, rules in the spirit of Brock and Hommes (1997), Hommes (2013), and Branch and McGough (2010) are

$$\begin{aligned} \mathbb{E}_t^* y_{t+1}^f &= \mathbb{E}_t y_{t+1}^f & (20) \\ \mathbb{E}_t^* y_{t+1}^e &= \mathbb{E}_{t-1}^* y_t^e + \lambda_y (y_t - \mathbb{E}_{t-1}^* y_t^e)\,; \quad \lambda_y \in [0,1] & (21) \\ \mathbb{E}_t^* \pi_{t+1}^f &= \mathbb{E}_t \pi_{t+1}^f & (22) \\ \mathbb{E}_t^* \pi_{t+1}^e &= \mathbb{E}_{t-1}^* \pi_t^e + \lambda_\pi (\pi_t - \mathbb{E}_{t-1}^* \pi_t^e)\,; \quad \lambda_\pi \in [0,1] & (23) \end{aligned}$$

This assumes fundamentalists are rational and the extrapolative learners use a general adaptive expectations rule. As before, we have:

$$\begin{aligned} \alpha_{f,t} &= \frac{\exp(\gamma(U_{f,t}(\{y_t\}) - C))}{\exp(\gamma(U_{f,t}(\{y_t\}) - C)) + \exp(\gamma U_{e,t}(\{y_t\}))} & (24) \\ \beta_{f,t} &= \frac{\exp((\gamma U_{f,t}(\{\pi_t\}) - C))}{\exp((\gamma U_{f,t}(\{\pi\}_t) - C)) + \exp(\gamma U_{e,t}(\{\pi\}_t))} & (25) \end{aligned}$$

where C represents the relative costs of being rational. Thus, by incorporating the Brock-Hommes complexity framework into the workhorse New Keynesian DSGE model, the behavioural New Keynesian model incorporates heterogeneity and bounded rationality into a DSGE framework. In addition, and as with the simple Cobweb model presented in section 5.1, the behavioural NK model can generate persistent and asymmetric fluctuations in response to small shocks, and generate endogenous business cycles characterised by bounded instability and chaos (see e.g. Branch et al. (2012)). Hence the behavioural NK model answers - at least to some extent - the three ACE critiques of DSGE modelling outlined above.

5.3 Individual Rationality

As Graham (2011) has pointed out, the form of learning implied by the NK behavioural model above follows the Euler equation approach and in effect assumes that agents forecast their own decisions - for the household their consumption decision, and for the firm their price decision. In the statistical learning approach pioneered in Evans and Honkapohja (2001), agents know the minimum state variable (MSV) form of the equilibrium (equivalent to the saddle-path under rational expectations) and use direct observations of these states to update their parameter estimates each period using a discounted least-squares estimator. Then a statistical learning equilibrium is one where this perceived law of motion and the actual law of motion coincide.

An alternative approach was first introduced by Eusepi and Preston (2011) into an RBC model. This assumes that agents are individually rational (IR) given their beliefs over aggregate states and prices. As with the Euler equation approach, agents cannot form model-consistent expectations and instead learn about these variables using their knowledge of the MSV form of the equilibrium. The two approaches then differ with respect to what agents learn about - their own decisions in the first approach, and variables exogenous to the agents in the second approach.

The construction of an IR equilibrium for an NK model goes through the following steps.[5]:

- Solve the household budget constraint forward in time and impose the transversality condition.
- Use the first-order conditions and either linearize or assume point expectations to obtain consumption as a function of expected nominal interest rates, inflation, wages and profits.
- For monopolistically competitive retail firms express the Calvo contract for the price optimizing firm as a function of expected aggregate demand, aggregate inflation, real marginal cost and mark-up shocks (again either linearizing or assuming point expectations).

[5]See Deak et al. (2015) for details.

- Finally, choose the expectations formation mechanisms.

The result is an individually rational NK model. As above, in this instance we will use adaptive expectations to illustrate the basic model. One-step ahead forecasts are then given by:

$$\mathbb{E}_t^* x_{t+1} = \mathbb{E}_t^* x_t + \lambda_x (x_{t-j} - \mathbb{E}_t^* x_t)\,; \quad j = 0, 1 \tag{26}$$

Individually rational households make intertemporal decisions for their consumption demand and hours supply given adaptive expectations of the wage rate, the nominal interest rate, inflation and profits. Individually rational price-setting retail firms form adaptive expectations of aggregate demand, aggregate inflation and real marginal costs. These macro-variables may be observed with or without a one-period lag ($j = 1, 0$).

For example, for households the forecast of the future nominal interest is given by (26) with $x = r_n$ so that

$$\mathbb{E}_t^* r_{n,t+1} = \mathbb{E}_t^* r_{n,t} + \lambda_{r_n}(r_{n,t-j} - \mathbb{E}_t^* r_{n,t}) = \sum_{i=1}^{\infty} \lambda_{r_n}^i r_{n,t-j-i}\,; \quad j = 0, 1 \tag{27}$$

Finally, the model is closed in the same way as the behavioural NK model considered above, with two groups of households and firms, one adopting rational expectations and one individually rational. The proportions of rational households and firms are given by

$$\begin{aligned} n_{h,t} &= \frac{\exp(\gamma\Phi_{h,t}^{RE})}{\exp(\gamma\Phi_{h,t})^{RE} + \exp(\gamma\Phi_{h,t}^{IR})} \\ n_{f,t} &= \frac{\exp(\gamma\Phi_{f,t}^{RE})}{\exp(\gamma\Phi_{f,t}^{RE}) + \exp(\gamma\Phi_{f,t}^{IR})} \end{aligned}$$

where fitness for households given by

$$\begin{aligned} \Phi_{h,t}^{RE} &= \mu_h^{RE}\Phi_{h,t-1}^{RE} - \Big(\text{weighted sum of forecast errors} + C_h\Big) \\ \Phi_{h,t}^{IR} &= \mu_h^{IR}\Phi_{h,t-1}^{IR} - \Big(\text{weighted sum of forecast errors}\Big) \end{aligned}$$

As before, C_h is a fixed cost of the rational expectations operator for households and firms.

5.4 The Individually Rational New Keynesian Model

The individually rational NK model described above incorporates heterogeneity and bounded rationality into DSGE modelling in the same manner as the behavioural NK models described in

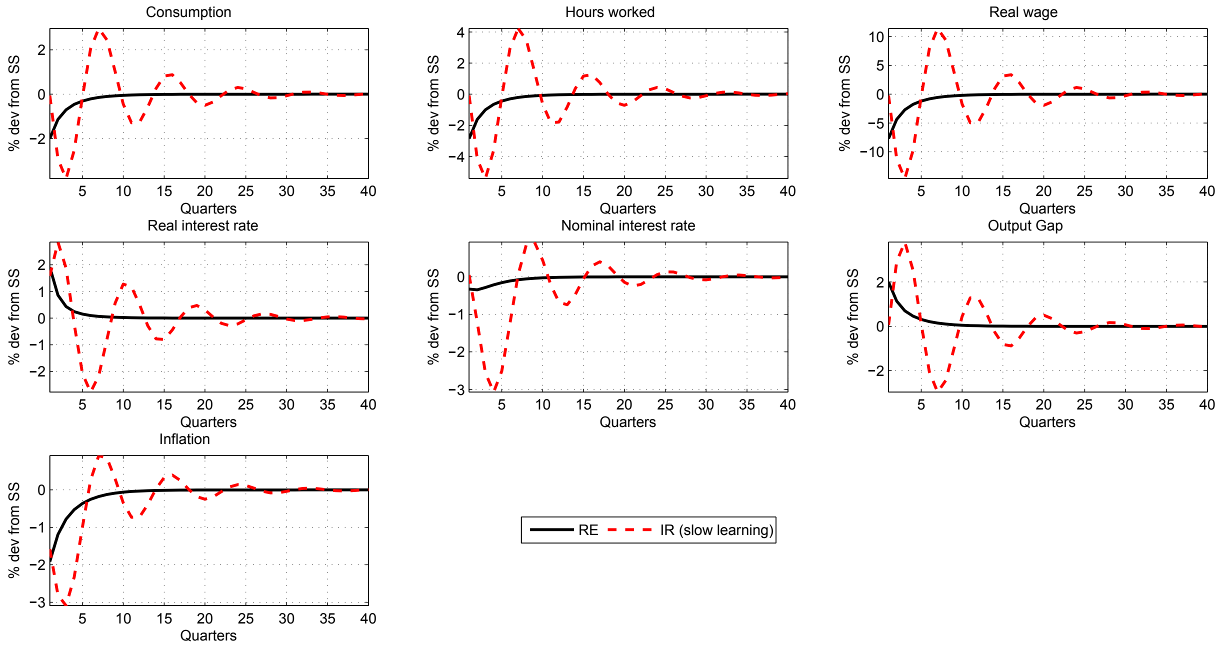

Figure 1: RE versus IR-RE Composite Expectations with $n_h = n_f = 0.5$, $\lambda_x = 0.04$; lagged observations: Monetary Policy Shock

section 5.2, and may be considered an advance on this literature. The importance of the advance lies in the more rigorous microfoundations of the individually rational model, where households are boundedly rational over the exogenous variables of interest to them, rather than their own future decisions.

As well as incorporating heterogeneity and bounded rationality, the individually rational NK model also produces complex dynamics. This follows from the highly nonlinear nature of the model, and results in a highly asymmetric joint distribution in the stochastic steady state. Table 1 describes the steady state distribution for a third order solution of the individually rational NK model described in section 5.3 with the following parameterisation: $\mu_h^{RE} = \mu_h^{IR} = \mu_f^{RE} = \mu_f^{IR} = 0.7$, $C_h = C_f = 0$, $\sigma_A = \sigma_{MS} = 0.01$; $\sigma_{MPS} = 0.001$, where the last three parameters are the standard deviations of the technology, mark-up, and monetary policy shocks, respectively. Aggregate consumption, hours, inflation, and interest rates exhibit extremely high kurtosis, or fat tails, and high skewness. Although the stochastic means of most endogenous variables are close to their deterministic steady state values, the stochastic means of the proportions of rational and boundedly rational agents are quite different from their deterministic steady state values.

Figure 1 provides more information about the model's dynamic behaviour by plotting the impulse response functions for the monetary policy shock with $n_h = n_f = 0.5$, $\lambda_x = 0.04$ and standard values for the other parameters in (12)–(14).[6] Despite the fundamental parameterisation

[6] See Deak et al. (2015)

	Deterministic Mean	Stochastic Mean	SD	Skewness	Kurtosis
$\frac{C_t}{C}$	1.0000	0.9912	0.0260	-2.037	15.56
$\frac{H_t}{H}$	1.0000	1.0036	0.0166	2.595	19.09
$\frac{W_t}{W}$	1.0000	0.9981	0.0204	0.560	1.951
$\frac{\Pi_t}{\Pi}$	1.0000	1.0054	0.0169	1.644	9.904
$\frac{R_{n,t}}{R_n}$	1.0000	1.0054	0.0145	1.708	10.25
$n_{h,t}$	0.5000	0.5795	0.1872	1.964	12.48
$n_{f,t}$	0.5000	0.5195	0.0290	2.169	12.09

Table 1: Third Order Solution of NK IR-RE Model

being exactly the same for the RE model and the IR model, the latter produces highly persistent and highly cyclical impulse response functions. Thus as well as a highly non-normal stochastic steady state, the individually rational NK model also exhibits large responses to small shocks. In this sense, the stable parameterisation here produces business cycles that are *more* endogenous than those generated by rational expectations NK models. However, as with the behavioural NK models examined in section 5.2, the individually rational NK model can also produce bounded chaotic dynamics and therefore endogenous business cycles in the proper sense of the term.[7]

5.5 Summary

This section reviewed the behavioural New Keynesian model that incorporates Brock-Hommes complexity into the conventional New Keynesian DSGE model. Existing behavioural NK models use a form of Euler equation learning, which essentially means that households and firms forecast their own future decisions. Thus an individually rational NK model was discussed, following recent work by Deak et al. (2015), which can be considered an advance on this literature. Both the behavioural and individually rational NK models incorporate heterogeneity and bounded rationality into the standard New Keynesian framework, and in so doing generate complex dynamics. Therefore, this literature begins to answer the criticisms levelled at DSGE from the ACE community.

6 Concluding Remarks

This paper has considered the major points of contention between macro agent-based computational economics and dynamic stochastic general equilibrium modelling, and has reviewed a literature which attempts to synthesise, and thus bridge the gap between, these two approaches.

[7] See the appendix for a full description of the individually rational New Keynesian model.

First, we considered the major criticisms directed at DSGE modelling by the ACE community, and then reviewed the main families of mature macro ACE models. We then reviewed the empirical performance of these models. We then reviewed the literature surrounding the behavioural New Keynesian model, and discussed the individually rational New Keynesian model following recent work by Deak et al. (2015). This model improves the microfoundations of the behavioural NK model, whilst answering the criticisms directed at DSGE modelling in the ACE literature.

We hope that the research program outlined in this review will constitute a profitable direction for the economics profession following the perceived failure of DSGE modelling in the light of the 2008 financial crisis. Two avenues immediately suggest themselves. First, local interaction, in the shape of matching functions or replicator dynamics, could be incorporated into the individually rational NK model. A more challenging line of research would be to construct a similar model using the heterogeneous agent methodology of Krusell and Smith (1998), which would bring the methodology closer to the microsimulation structure of a lot of agent-based models.

The DSGE literature reviewed in the present paper is not, of course, the only way in which ACE and DSGE models can be brought closer together. Sinitskaya and Tesfatsion (2015) work from the opposite direction by introducing forward-looking optimizing agents into an ACE framework. They use an equivalent to individual rationality which they refer to as "constructive rational decision making". This is a novel macro ACE model in having individually rational optimizers: households maximize expected intertemporal utility over an infinite time-horizon and firms do the same with their utility being taken as profit. But there are important differences of an ACE nature: the time interval is divided into 6 sub-intervals, and agents adopt optimized parameterized decision rules proportional to expected market-clearing prices and then update these parameters through reinforcement learning. A third line of research, therefore, would be to explore the similarities between the individually rational NK model presented here and the Sinitskaya and Tesfatsion (2015) model, and potentially compare their empirical performance.

Finally, a limitation that affects any macroeconomics that seeks to incorporate bounded rationality is the gap in our empirical knowledge with respect to the microfoundations of economic behaviour. For example, Lengnick (2013) argues that identity should be considered as part of individual decisions, and concepts such as reciprocity, fairness, and loss aversion should be incorporated into macroeconomic models. Yet, he notes that simple behavioural rules in ACE models are usually either derived from survey studies or "common-sensical reasoning". A profitable way forward here may be a sustained effort to incorporate the results of experimental economics into macroeconomic analysis - extending work already done with robust maxmin decision rules (Hansen and Sargent (2008)), smooth ambiguity utility (Ilut and Schneider (2014), Ju and Miao (2012)), prospect theory (Kahneman and Tversky (1979), Kahneman et al. (1990), Barberis (2013) Dhami and Al-Nowaihi (2010)), and hyperbolic discounting (Harris and Laibson (2001), Krusell and Smith (2002)). This ambitious project is more in keeping with the inter-disciplinary nature of agent-based modelling and awaits future research.

References

Assenza, T., D. Delli Gatti, and J. Grazzini (2015). Emergent dynamics of a macroeconomic agent based model with capital and credit. *Journal of Economic Dynamics and Control 50*, 5–28.

Barberis, N. C. (2013). Thirty Years of Prospect Theory in Economics: A Review and Assessment. *Journal of Economic Perspectives 27*(1), 173–196.

Bianchi, C., P. Cirillo, M. Gallegati, and P. Vagliasindi (2007). Validating and calibrating agent-based models: A case study. *Computational Economics 30*(3), 245–264.

Bowles, S. (1998). Endogenous preferences: the cultural consequences of markets and other economic institutions. *Journal of Economic Literature 36*(1), 75 – 111.

Branch, W. A. and G. W. Evans (2011). Monetary Policy and Heterogeneous Agents. *Economic Theory 47*(2), 365–393.

Branch, W. A., G. W. Evans, and B. McGough (2012). Finite Horizon Learning. CDMA Working Paper, 12/04.

Branch, W. A. and B. McGough (2010). Dynamic predictor election in a new keynesian model with heterogeneous agents. *Journal of Economic Dynamics and Control 34*(8), 1492–1508.

Brock, W. A. and C. H. Hommes (1997). A Rational Route to Randomness. *Econometrica 65*, 1059–1095.

Calvo, G. (1983). Staggered Prices in a Utility-Maximizing Framework. *Journal of Monetary Economics 12*(3), 383–398.

Cincotti, S., M. Raberto, and A. Teglio (2010). Credit money and macroeconomic instability in the agent-based model and simulator eurace. *Economics 3*, 2010 – 2026.

Dawid, H., S. Gemkow, P. Harting, and M. Neugart (2009). On the effects of skill upgrading in the presence of spatial labor market frictions: An agent-based analysis of spatial policy design. *Journal of Artificial Societies and Social Simulation 12*(4).

Dawid, H. and M. Neugart (2011). Agent-based models for economic policy design. *Eastern Economic Journal 37*(1), 44 – 50.

De Grauwe, P. (2011). Animal Spirits and Monetary Policy. *Economic Theory 47*(2–3), 423–457.

De Grauwe, P. (2012a). Booms and Busts in Economic Activity: A Behavioral Explanation. *Journal of Economic Behavior and Organization 83*(3), 484–501.

De Grauwe, P. (2012b). *Lectures on Behavioral Macroeconomics*. Princeton University Press.

De Grauwe, P. and P. R. Katwasser (2012). Animal Spirits in the Foreign Exchange Market. *Journal of Economic Dynamics and Control 36*(8), 1176–1192.

Deak, S., P. Levine, and B. Yang (2015). A New Keynesian Behavioural Model with Individual Rationality and Heterogeneous Agents. Presented at the CEF2015 Conference, June 2015, Taiwan and School of Economics, University of Surrey, Forthcoming Working Paper.

Deissenberg, C., S. van der Hoog, and H. Dawid (2008). Eurace: A massively parallel agent-based model of the european economy. *Applied Mathematics and Computation 204*, 541 – 552.

DeJong, D. and C. Dave (2007). *Structural Macroeconometrics*. Princeton University Press.

Delli Gatti, D., S. Desiderio, E. Gaffeo, P. Cirilo, and M. Gallegati (2011). *Macroeconomics from the Bottom-up*. Springer-Verlag Italia.

Delli Gatti, D., C. Di Guilmi, E. Gaffeo, G. Giulioni, M. Gallegati, and A. Palestrini (2005). A new approach to business fluctuations: heterogeneous interacting agents, scaling laws and financial fragility. *Journal of Economic Behavior and Organization 56*(4), 489 – 512.

Delli Gatti, D., E. Gaffeo, and M. Gallegati (2010). Complex agent-based macroeconomics: a manifesto for a new paradigm. *Journal of Economic Interaction and Coordination 5*(2), 111 – 135.

Delli Gatti, D., E. Gaffeo, M. Gallegati, G. Giulioni, A. Kirman, A. Palestrini, and A. Russo (2007). Complex dynamics and empirical evidence. *Information Sciences 177*(5), 1204 – 1221.

Dhami, S. and A. Al-Nowaihi (2010). Optimal Taxation in the Presence of Tax Evasion: Expected Utility versus Prospect Theory. *Journal of Economic Behavior and Organization 75*(2), 313 – 337.

Dixit, A. K. and J. E. Stiglitz (1977). Monopolistic competition and optimal product diversity. *American Economic Review 67*(3), 297–308.

Dosi, G., G. Fagiolo, M. Napoletano, and A. Roventini (2013). Income distribution, credit and fiscal policies in an agent-based keynesian model. *Journal of Economic Dynamics and Control 37*(8), 1598 –1625.

Dosi, G., G. Fagiolo, and A. Roventini (2005). Animal spirits, lumpy investment and endogenous business cycles. *LEM Working Paper Series 2005/04*.

Dosi, G., G. Fagiolo, and A. Roventini (2006). An evolutionary model of endogenous business cycles. *Computational Economics 27*(1), 3–34.

Dosi, G., G. Fagiolo, and A. Roventini (2008). The microfoundations of business cycles: An evolutionary, multi-agent model. *Journal of Evolutionary Economics 18*, 413–432.

Dosi, G., G. Fagiolo, and A. Roventini (2010). Schumpeter meeting keynes: A policy-friendly model of endogenous growth and business cycle. *Journal of Economic Dynamics and Control 34*(9), 1748 – 1767.

Eusepi, S. and B. Preston (2011). Expectations, learning, and business cycle fluctuations. *American Economic Review 101*(6), 2844–2872.

Evans, G. W. and S. Honkapohja (2001). *Learning and Expectations in Macroeconomics*. Princeton University Press.

Gabaix, X. (2011). The granular origins of aggregate fluctuations. *Econometrica 79*(3), 733–772.

Gaffeo, E., M. Catalano, F. Clementi, D. Delli Gatti, M. Gallegati, and A. Russo (2007). Reflections on modern macroeconomics: Can we travel along a safer road. *Physica A 382*(1), 89–97.

Gaffeo, E., M. Catalano, D. Delli Gatti, S. Desiderio, and M. Gallegati (2008). Adaptive microfoundations for emergent macroeconomics. *Eastern Economic Journal 34*, 441–463.

Gali, J. (2008). *Monetary Policy, Inflation and the Business Cycle*. Princeton University Press.

Goodwin, R. (1967). A growth cycle. In C. Feinstein (Ed.), *Social, Capitalism and Economic Growth*. Cambridge: Cambridge University Press.

Graham, L. (2011). Individual Rationality, Model-Consistent Expectations and Learning. Mimeo, University College London.

Grazzini, J. and M. Richiardi (2015). Estimation of Ergodic Agent-Based Models by Simulated Minimum Distance. *Journal of Economic Dynamics and Control 51*, 148–165.

Haavelmo, T. (1944). The probability approach in econometrics. *Econometrica 12*(supp.), iii–iv+1–115.

Hansen, L. (1982). Large sample properties of generalized method of moments estimators. *Econometrica 50*(4), 1029–1054.

Hansen, L. and T. J. Sargent (2008). *Robustness*. Princeton University Press.

Harris, C. and D. Laibson (2001). Dynamic Choices of Hyperbolic Consumers. *Econometrica 69*(4), 935–957.

Hommes, C. (2013). *Behavioral Rationality and Heterogeneous Expectations in Complex Economic Systems*. CUP.

Howitt, P. (2012). What have central bankers learned from modern macroeconomic theory? *Journal of Macroeconomics 34*(1), 11–22.

Ilut, C. L. and M. Schneider (2014). Ambiguous Business Cycles. *American Economic Review 104*(8), 2368–99.

Jang, T.-S. and S. Sacht (2012). Identification of Animal Spirits in a Bounded Rationality Model: An Application to the Euro Area. Economics Working Papers 2012-02, Christian-Albrechts-University of Kiel, Department of Economics.

Jang, T.-S. and S. Sacht (2014). Animal spirits and the business cycle: Empirical evidence from moment matching. Economics Working Papers 2014-06, Christian-Albrechts-University of Kiel, Department of Economics.

Ju, M. and J. Miao (2012). Ambiguity, Learning and Asset Returns. *Econometrica 80*(2), 559–591.

Kahneman, D., J. Knetsch, and R. H. Thaler (1990). Experimental Tests of the Endowment Effect and the Coase Theorem. *Journal of Political Economy 98*(2), 1325 – 1348.

Kahneman, D. and A. Tversky (1979). Prospect Theory: An Analysis of Decision under Risk. *Econometrica 47*(2), 263 – 91.

Krusell, P. and A. Smith (2002). Consumption-Savings Decisions with Quasi-Geometric Discounting. *Econometrica 71*, 365 – 375.

Krusell, P. and A. A. Smith, Jr. (1998, October). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy 106*(5), 867–896.

Kyldland, F. and E. Prescott (1982). Time to build and aggregate fluctuations. *Econometrica 50*(1), 345–370.

Lengnick, M. (2013). Agent-based macroeconomics: A baseline model. *Journal of Economic Behavior and Organization 86*, 102–120.

Massaro, D. (2013). Heterogeneous Expectations in Monetary DSGE Models. *Journal of Economic Dynamics and Control 37*(3), 680–692.

Raberto, M., A. Teglio, and S. Cincotti (2011). Debt deleveraging and business cycles: An agent-based perspective. *Economics Discussion Paper 2011-31*.

Ricetti, L., A. Russo, and M. Gallegati (2013). Unemployment benefits and financial leverage in an agent based macroeconomic model. *Economics 2013-42*.

Russo, A., M. Catalano, E. Gaffeo, M. Gallegati, and M. Napoletano (2007). Industrial dynamics, fiscal policy and r&d: Evidence from a computational experiment. *Journal of Economic Behavior & Organization 64*, 426–447.

Sinitskaya, E. and L. Tesfatsion (2015). Macroeconomics as Constructively Rational Games. *Journal of Economic Dynamics and Control 61*, 152–182.

Woodford, M. (2003). *Interest and Prices. Foundations of a Theory of Monetary Policy*. Princeton University Press.

Young, H. P. (2004). *Strategic Learning and its Limits*. Oxfor University Press, New York.

A Appendix

A.1 The Rational Expectations Model

A.1.1 Households

Household j choose between work and leisure and therefore how much labour they supply. Let $C_t(j)$ be consumption and $H_t(j)$ be the proportion of this available for work or leisure spent at the former. The single-period utility is

$$U_t(j) = U(C_t(j), H_t(j)) = \log(C_t(j)) - \kappa \frac{H_t(j)^{1+\phi}}{1+\phi} \tag{A.1}$$

In a stochastic environment, the value function of the representative household at time t is given by

$$V_t(j) = V_t(B_{t-1}(j)) = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s U(C_{t+s}(j), H_{t+s}(j)) \right] \tag{A.2}$$

For the household's problem at time t is to choose paths for consumption $\{C_t(j)\}$, labour supply $\{H_t(j)\}$ and holdings of financial savings to maximize $V_t(j)$ given by (A.2) given its budget constraint in period t

$$B_t(j) = R_t B_{t-1}(j) + W_t H_t(j) + \Gamma_t - C_t(j) \tag{A.3}$$

where $B_t(j)$ is the given net stock of financial assets at the end of period t, W_t is the wage rate and R_t is the ex post real interest rate paid on assets held at the beginning of period t given by

$$R_t = \frac{R_{n,t-1}}{\Pi_t} \tag{A.4}$$

where $R_{n,t}$ and Π_t are the nominal interest and inflation rates respectively and Γ_t are profits from wholesale and retail firms owned by households. W_t, R_t and Γ_t are all exogenous to household j. As usual all variables are expressed in real terms relative to the price of final output.

The first order conditions are

$$U_{C,t}(j) = \beta \mathbb{E}_t \left[R_{t+1} U_{C,t+1}(j) \right] \tag{A.5}$$

$$\frac{U_{L,t}(j)}{U_{C,t}(j)} = W_t \tag{A.6}$$

An equivalent representation of the Euler consumption equation (A.5) is

$$1 = \mathbb{E}_t \left[\Lambda_{t,t+1}(j) R_{t+1} \right] \tag{A.7}$$

where $\Lambda_{t,t+1}(j) \equiv \beta\frac{U_{C,t+1}(j)}{U_{C,t}(j)}$ is the *real stochastic discount factor* for household j, over the interval $[t, t+1]$.

For our choice of utility function $U_{C,t} = \frac{1}{C_t}$ and $U_{H,t} = -\kappa H_t^{\phi}$ so these become

$$\frac{1}{C_t(j)} = \beta\mathbb{E}_t\left[\frac{R_{t+1}}{C_{t+1}(j)}\right] \tag{A.8}$$

$$\kappa C_t(j)H_t(j)^{\phi} = W_t \Rightarrow H_t = \left(\frac{W_t}{\kappa C_t(j)}\right)^{\frac{1}{\phi}} \tag{A.9}$$

In a symmetric equilibrium of identical households $C_t(j) = C_t$, aggregate per household consumption and $H_t(j) = H_t$, average hours worked,

A.1.2 Firms in the Wholesale

Wholesale firms employ a Cobb-Douglas production function to produce a homogeneous output

$$Y_t^W = F(A_t, H_t) = A_t H_t^{\alpha} \tag{A.10}$$

where A_t is total factor productivity. Profit-maximizing demand for labour results in the first order condition

$$W_t = \frac{P_t^W}{P_t}F_{H,t} = \alpha\frac{P_t^W}{P_t}\frac{Y_t^W}{H_t} \tag{A.11}$$

Hence from (A.9) and (A.11) we have

$$H_t = \left[\frac{\alpha\frac{P_t^W}{P_t}}{\kappa}\right]^{\frac{1}{1+\phi}} \tag{A.12}$$

A.1.3 Firms in the Retail Sector

The retail sector uses a homogeneous wholesale good to produce a basket of differentiated goods for aggregate consumption

$$C_t = \left(\int_0^1 C_t(m)^{(\zeta-1)/\zeta}dm\right)^{\zeta/(\zeta-1)} \tag{A.13}$$

where ζ is the elasticity of substitution. For each m, the consumer chooses $C_t(m)$ at a price $P_t(m)$ to maximize (A.13) given total expenditure $\int_0^1 P_t(m)C_t(m)dm$. This results in a set of consumption demand equations for each differentiated good m with price $P_t(m)$ of the form

$$C_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta}C_t \Rightarrow Y_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta}Y_t \tag{A.14}$$

where $P_t = \left[\int_0^1 P_t(m)^{1-\zeta} dm\right]^{\frac{1}{1-\zeta}}$. P_t is the aggregate price index. C_t and P_t are Dixit-Stigliz aggregates – see Dixit and Stiglitz (1977).

For each variety m the retail good is produced from wholesale production according to an iceberg technology

$$Y_t(m) = Y_t^W = A_t H_t(m)^\alpha \tag{A.15}$$

Following Calvo (1983), we now assume that there is a probability of $1 - \xi$ at each period that the price of each retail good m is set optimally to $P_t^0(m)$. If the price is not re-optimized, then it is held fixed.[8] For each retail producer m, given its real marginal cost MC_t, the objective is at time t to choose $\{P_t^0(m)\}$ to maximize discounted profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} Y_{t+k}(m) \left[P_t^0(m) - P_{t+k} MC_{t+k}\right] \tag{A.16}$$

subject to (A.14), where $\Lambda_{t,t+k} \equiv \beta^k \frac{U_{C,t+k}/P_{t+k}}{U_{C,t}/P_t}$ is now the *nominal* stochastic discount factor over the interval $[t, t+k]$. The solution to this is

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} Y_{t+k}(m) \left[P_t^0(m) - \frac{1}{(1 - 1/\zeta)} P_{t+k} MC_{t+k}\right] = 0 \tag{A.17}$$

and by the law of large numbers the evolution of the price index is given by

$$P_{t+1}^{1-\zeta} = \xi P_t^{1-\zeta} + (1 - \xi)(P_{t+1}^0)^{1-\zeta} \tag{A.18}$$

In order to set up the model in non-linear form as a set of difference equations, required for software packages such a Dynare, we need to represent the price dynamics as difference equations. First define k periods ahead inflation as

$$\Pi_{t,t+k} \equiv \frac{P_{t+k}}{P_t} = \frac{P_{t+1}}{P_t} \frac{P_{t+2}}{P_{t+1}} \cdot \cdot \frac{P_{t+k-1}}{P_{t+k}} = \Pi_{t+1} \Pi_{t+2} \cdot \cdot \Pi_{t+k}$$

Note that $\Pi_{t,t+1} = \Pi_{t+1}$ and $\Pi_{t,t} = 1$.

Next using (A.14) with $P_{t+k}(m) = P_0(m)$, the price set at time t which survives with probability ξ, we have that

$$\Lambda_{t,t+k} Y_{t+k}(m) = \beta^k \frac{U_{C,t+k}}{U_{C,t}} \frac{P_t}{P_{t+k}} \left(\frac{P_0(m)}{P_{t+k}}\right)^{-\zeta} Y_{t+k} = \beta^k \frac{U_{C,t+k}}{U_{C,t}} \Pi_{t,t+k}^{\zeta-1} \left(\frac{P_0(m)}{P_t}\right)^{-\zeta} Y_{t+k}$$

[8]Thus we can interpret $\frac{1}{1-\xi}$ as the average duration for which prices are left unchanged.

Hence cancelling out $\left(\frac{P_0(m)}{P_t}\right)^{-\zeta}$ and multiplying by $\frac{U_{C,t}}{P_t}$ we can write (A.17) as

$$E_t \sum_{k=0}^{\infty} (\xi\beta)^k U_{C,t+k} \Pi_{t,t+k}^{\zeta-1} Y_{t+k} \left[\frac{P_t^0(m)}{P_t} - \Pi_{t+k} MC_{t+k} MS_{t+k} \right] = 0 \tag{A.19}$$

We seek a symmetric equilibrium where firms who are either re-setting their prices or are locked into a contract are identical. In such an equilibrium, the price dynamics can be written as difference equations as follows (see Appendix A):

$$\frac{P_t^0}{P_t} = \frac{J_t}{JJ_t} \tag{A.20}$$

$$JJ_t - \xi \mathbb{E}_t \left[\Pi_{t+1}^{\zeta-1} JJ_{t+1} \Lambda_{t,t+1} \right] = Y_t \tag{A.21}$$

$$J_t - \xi \mathbb{E}_t \left[\Pi_{t+1}^{\zeta} J_{t+1} \Lambda_{t,t+1} \right] = \left(\frac{1}{1 - \frac{1}{\zeta}} \right) Y_t MC_t MS_t \tag{A.22}$$

$$1 = \xi \Pi_t^{\zeta-1} + (1-\xi) \left(\frac{J_t}{JJ_t} \right)^{1-\zeta} \tag{A.23}$$

$$\Delta_t = \xi \Pi_t^{\frac{\zeta}{\alpha}} \Delta_{t-1} + (1-\xi) \left(\frac{J_t}{JJ_t} \right)^{\frac{\zeta}{\alpha}} \tag{A.24}$$

(A.25)

$$MC_t = \frac{P_t^W}{P_t} = \frac{W_t}{F_{H,t}} \tag{A.26}$$

where (A.41) uses (A.11). We have introduced a mark-up shock MS_t. Notice that the real marginal cost, MC_t, is no longer fixed as it in the flexi-price case model.

Price dispersion lowers aggregate output as follows. Market clearing in the labour market gives

$$H_t = \sum_{m=1}^{n} H_t(m) = \sum_{m=1}^{n} \left(\frac{Y_t(m)}{A_t} \right)^{\frac{1}{\alpha}} = \left(\frac{Y_t}{A_t} \right)^{\frac{1}{\alpha}} \sum_{m=1}^{n} \left(\frac{P_t(m)}{P_t} \right)^{-\frac{\zeta}{\alpha}} \tag{A.27}$$

using (A.14).

Hence equilibrium for good m gives

$$Y_t = \frac{Y_t^W}{\Delta_t^{\alpha}} \tag{A.28}$$

where price dispersion is defined by

$$\Delta_t \equiv \left(\sum_{m=1}^{n} \left(\frac{P_t(m)}{P_t} \right)^{-\frac{\zeta}{\alpha}} \right) \tag{A.29}$$

Price dispersion is linked to inflation as follows. Assuming as before that the number of firms is large we obtain the following dynamic relationship:

$$\Delta_t = \xi \Pi_t^{\frac{\zeta}{\alpha}} \Delta_{t-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{-\frac{\zeta}{\alpha}} \tag{A.30}$$

Proof

See Appendix B.

A.1.4 Profits

To close the model with internal rationality we require total profits from retail and wholesale firms. Γ_t, remitted to households. This is given in real terms by

$$\Gamma_t = \underbrace{Y_t - \frac{P_t^W}{P_t} Y_t^W}_{\text{retail}} + \underbrace{\frac{P_t^W}{P_t} Y_t^W - W_t H_t}_{\text{Wholesale}} = Y_t - \alpha \frac{P_t^W}{P_t} Y_t^W \tag{A.31}$$

using the first-order condition (A.11).

A.1.5 Flexi-Price Output

In order to evaluate the output gap we need the flexi-price limit as $\xi \to 0$ (but retaining monopolistic competition with $c > 0$ and $\zeta < \infty$ and the mark-up shock MS_t). As for the sticky-price model labour supply and demand conditions are

$$W_t = C_t \kappa H_t^{\phi} = Y_t \kappa H_t^{\phi} = A_t \kappa H_t^{\alpha+\phi} \tag{A.32}$$

$$W_t = \frac{P_t^M}{P_t} \frac{\alpha Y_t^W}{H_t} = \alpha M C_t A_t H_t^{\alpha-1} \tag{A.33}$$

$$\text{Hence } H_t = \left(\frac{\alpha M C_t}{\kappa}\right)^{\frac{1}{1+\phi}} \tag{A.34}$$

As $\xi \Rightarrow 0$ we have

$$\frac{P_t^0}{P_t} = 1 = \frac{MC_t MS_t}{1 - \frac{1}{\zeta}} \tag{A.35}$$

It follow that

$$H_t^F = \left(\frac{\alpha\left(1-\frac{1}{\zeta}\right)}{MS_t \kappa}\right)^{\frac{1}{1+\phi}} = \left(\frac{\alpha}{\kappa MS_t}\right)^{\frac{1}{1+\phi}} \tag{A.36}$$

$$Y_t^F = A_t (H_t^F)^{\alpha} \tag{A.37}$$

A.1.6 Closing the Model

The model is closed with a resource constraint

$$Y_t = C_t + G_t \tag{A.38}$$

where G_t is an exogenous demand process and a monetary policy rule for the nominal interest rate is given by the following Taylor-type rule

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\left(\theta_\theta \log\left(\frac{\Pi_t}{\Pi}\right) + \theta_y \log\left(\frac{Y_t}{Y}\right)\right) + \epsilon_{MPS,t}, \tag{A.39}$$

or

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\theta_\pi \log\left(\frac{\Pi_t}{\Pi}\right) + (1-\rho_r)\theta_y \log\left(\frac{Y_t}{Y_t^F}\right) + \epsilon_{MPS,t}, \tag{A.40}$$

where Y_t^F is the flexi-price level of output and $\epsilon_{MPS,t}$ is a monetary policy i.i.d shock [9] and finally there are two exogenous AR1 shock processes to technology and marginal cost (the latter being interpreted as a mark-up shock):

$$\begin{aligned}
\log A_t - \log A &= \rho_A(\log A_{t-1} - \log A) + \epsilon_{A,t} \\
\log G_t - \log G &= \rho_G(\log G_{t-1} - \log G) + \epsilon_{G,t} \\
\log MS_t - \log MS &= \rho_{MS}(\log MS_{t-1} - \log MS) + \epsilon_{MS,t}
\end{aligned}$$

and $\epsilon_{M,t}$ is an i.i.d. shock to monetary policy.

A.1.7 Summary of Model

Households:

$$\begin{aligned}
U_t &= U(C_t, H_t) = \log C_t - \kappa \frac{H_t^{1+\phi}}{1+\phi} \\
U_{C,t} &= \beta \mathbb{E}_t \left[R_{t+1} U_{C,t+1}\right] \\
\textbf{or } \mathbb{E}_t[\Lambda_{t,t+1} R_{t+1}] &= 1 \text{ where } \Lambda_{t-1,t} = \frac{\beta U_{C,t}}{U_{C,t-1}} \\
R_t &= \frac{R_{n,t-1}}{\Pi_t} \\
U_{C,t} &= \frac{1}{C_t} \\
U_{H,t} &= -\kappa H_t^{\phi}
\end{aligned}$$

[9](A.41) is an 'implementable' form of the Taylor rule which stabilizes output about its steady state.

$$\frac{U_{L,t}}{U_{C,t}} = W_t$$

Firms:

$$\begin{aligned}
Y_t^W &= F(A_t, H_t) = A_t H_t^{\alpha} \\
Y_t &= \frac{Y_t^W}{\Delta_t^{\alpha}} \\
\frac{P_t^W}{P_t} F_{H,t} &= \frac{P_t^W}{P_t} \frac{\alpha Y_t^W}{H_t} = W_t \\
\frac{P_t^0}{P_t} &= \frac{J_t}{JJ_t} \\
JJ_t &= \xi \mathbb{E}_t \left[\Pi_{t+1}^{\zeta-1} JJ_{t+1} \Lambda_{t,t+1} \right] + Y_t \\
J_t &= \xi \mathbb{E}_t \left[\Pi_{t+1}^{\zeta} J_{t+1} \Lambda_{t,t+1} \right] + \left(\frac{1}{1 - \frac{1}{\zeta}} \right) Y_t MC_t MS_t \\
1 &= \xi \Pi_t^{\zeta-1} + (1 - \xi) \left(\frac{J_t}{JJ_t} \right)^{1-\zeta} \\
\Delta_t &= \xi \Pi_t^{\frac{\zeta}{\alpha}} \Delta_{t-1} + (1 - \xi) \left(\frac{J_t}{JJ_t} \right)^{-\frac{\zeta}{\alpha}}
\end{aligned}$$

$$\begin{aligned}
MC_t &= \frac{P_t^W}{P_t} = \frac{W_t}{F_{H,t}} \\
\Gamma_t &= Y_t - \alpha MC_t Y_t^W
\end{aligned}$$

Closure:

$$\begin{aligned}
Y_t &= C_t + G_t \\
\log \left(\frac{R_{n,t}}{R_n} \right) &= \rho_r \log \left(\frac{R_{n,t-1}}{R_n} \right) + (1 - \rho_r) \left(\theta_\theta \log \left(\frac{\Pi_t}{\Pi} \right) + \theta_y \log \left(\frac{Y_t}{Y} \right) \right) + \epsilon_{M,t} \\
\text{or } \log \left(\frac{R_{n,t}}{R_n} \right) &= \rho_r \log \left(\frac{R_{n,t-1}}{R_n} \right) + (1 - \rho_r) \theta_\pi \log \left(\frac{\Pi_t}{\Pi} \right) + (1 - \rho_r) \theta_y \log \left(\frac{Y_t}{Y_t^F} \right) + \epsilon_{M,t} \\
\log A_t - \log A &= \rho_A (\log A_{t-1} - \log A) + \epsilon_{A,t} \\
\log G_t - \log G &= \rho_G (\log G_{t-1} - \log G) + \epsilon_{G,t} \\
\log MS_t - \log MS &= \rho_{MS} (\log MS_{t-1} - \log MS) + \epsilon_{MS,t}
\end{aligned}$$

A.2 Exogenous Point Expectations

As a first step towards individual rationality we now formulate the consumption and pricing decision of the household and firms respectively in terms of current and expected future aggregate variables exogenous these agents.

A.2.1 Households

For households, solving (A.3) forward in time and imposing the transversality condition on debt we can write

$$B_{t-1}(j) = \text{PV}_t(C_t(j)) - \text{PV}_t(W_t H_t(j)) - \text{PV}_t(\Gamma_t) \tag{A.41}$$

where the present (expected) value of a series $\{X_{t+i}\}_{i=0}^{\infty}$ at time t is defined by

$$\text{PV}_t(X_t) \equiv \mathbb{E}_t \sum_{i=0}^{\infty} \frac{X_{t+i}}{R_{t,t+i}} = \frac{X_t}{R_t} + \frac{1}{R_t}\text{PV}_{t+1}(X_{t+1}) \tag{A.42}$$

where $R_{t,t+1} \equiv R_t R_{t+1} R_{t+2} \cdots R_{t+i}$ is the real interest rate over the interval $[t, t+i]$.

The forward-looking budget constraint (A.41) holds for the representative household. In aggregate there is no net debt so $B_{t-1} = 0$. Then in a symmetric equilibrium, substituting for H_t from (A.9) we have

$$\text{PV}_t(C_t) = \frac{1}{\kappa^{\frac{1}{\phi}}}\text{PV}_t\left(\frac{W_t^{1+\frac{1}{\phi}}}{C_t^{\frac{1}{\phi}}}\right) + \text{PV}_t(\Gamma_t) \tag{A.43}$$

Solving (A.8) forward in time we have for $i \geq 1$

$$\frac{1}{C_t} = \beta^i \mathbb{E}_t\left[\frac{R_{t+1,t+i}}{C_{t+i}}\right] \; ; \; i \geq 1 \tag{A.44}$$

The individually rational solution to the household optimization problem seeks a solution to its decision functions for C_t and H_t that are functions of *non-rational point expectations* $\{\mathbb{E}_t^* W_{t+i}\}_{i=1}^{\infty}$, $\{\mathbb{E}_t^* R_{t+1,t+i}\}_{i=1}^{\infty}$ and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$ treated as exogenous processes given at time t as opposed to rational model-consistent expectations $\{\mathbb{E}_t W_{t+i}\}_{i=1}^{\infty}$ etc. [10] With point expectations we use (A.44) to obtain

$$\mathbb{E}_t^* C_{t+i} = C_t \beta^i \mathbb{E}_t^* R_{t+1,t+i} \; ; \; i \geq 1 \tag{A.45}$$

$$\mathbb{E}_t^*(W_{t+i} H_{t+i}) = \frac{1}{\kappa^{\frac{1}{\phi}}} \frac{(\mathbb{E}_t^* W_{t+i})^{1+\frac{1}{\phi}}}{(\mathbb{E}_t^* C_{t+i})^{\frac{1}{\phi}}} \tag{A.46}$$

Substituting (A.45) and (A.46) into the forward-looking household budget constraint and using $\sum_{i=0}^{\infty} \beta^i = \frac{1}{1-\beta}$, we arrive at

$$\frac{C_t}{R_t(1-\beta)} = \frac{1}{R_t(\kappa C_t)^{\frac{1}{\phi}}}\left(W_t^{1+\frac{1}{\phi}} + \sum_{i=1}^{\infty}(\beta^{\frac{1}{\phi}})^{-i}\left(\frac{\mathbb{E}_t^* W_{t+i}}{\mathbb{E}_t^* R_{t+1,t+i}}\right)^{1+\frac{1}{\phi}}\right) + \sum_{i=0}^{\infty}\frac{\mathbb{E}_t^* \Gamma_{t+i}}{\mathbb{E}_t^* R_{t,t+i}}$$

[10] With point expectations agents treat $E_t^*(\cdot)$ as certain, although the environment is stochastic (see Evans and Honkapohja (2001), page 61). Since $E_t f(X_t) \approx f(E_t(X_t))$ and $E_t f(X_t Y_t)) \approx f(E_t(X_t Y_t))$ up to a first-order Taylor-series expansion, assuming point expectations is equivalent to using a linear approximation of (A.43) and (A.44) as is usually done in the literature.

$$(A.47)$$

$$H_t = \left(\frac{W_t}{\kappa C_t}\right)^{\frac{1}{\phi}} \tag{A.48}$$

Writing $\mathbb{E}_t^* R_{t,t+i} = R_t \mathbb{E}_t^* R_{t+1,t+i}$ for $i \geq 1$ (A.47) becomes

$$\frac{C_t}{(1-\beta)} = \frac{1}{(\kappa C_t)^{\frac{1}{\phi}}} \left(W_t^{1+\frac{1}{\phi}} + \sum_{i=1}^{\infty} (\beta^{\frac{1}{\phi}})^{-i} \left(\frac{\mathbb{E}_t^* W_{t+i}}{\mathbb{E}_t^* R_{t+1,t+i}} \right)^{1+\frac{1}{\phi}} \right) + \Gamma_t + \sum_{i=1}^{\infty} \frac{\mathbb{E}_t^* \Gamma_{t+i}}{\mathbb{E}_t^* R_{t+1,t+i}} \tag{A.49}$$

This can be written as

$$\frac{C_t}{(1-\beta)} = \frac{1}{(\kappa C_t)^{\frac{1}{\phi}}} \left(W_t^{1+\frac{1}{\phi}} + \Omega_{1,t} \right) + \Gamma_t + \Omega_{2,t} \tag{A.50}$$

$$\Omega_{1,t} \equiv \sum_{i=1}^{\infty} (\beta^{\frac{1}{\phi}})^{-i} \left(\frac{\mathbb{E}_t^* W_{t+i}}{\mathbb{E}_t^* R_{t+1,t+i}} \right)^{1+\frac{1}{\phi}} = (\beta^{\frac{1}{\phi}})^{-1} \left(\frac{\mathbb{E}_t^* W_{t+1}}{\mathbb{E}_t^* R_{t+1,t+1}} \right)^{1+\frac{1}{\phi}} + (\beta^{\frac{1}{\phi}})^{-1} \Omega_{1,t+1}$$

$$\Omega_{2,t} \equiv \sum_{i=1}^{\infty} \frac{\mathbb{E}_t^* \Gamma_{t+i}}{\mathbb{E}_t^* R_{t+1,t+i}} = \frac{\mathbb{E}_t^* \Gamma_{t+1}}{\mathbb{E}_t^* R_{t+1,t+1}} + \Omega_{2,t+1}$$

With a non-zero balanced growth steady state β is simply replaced with β_g in the expressions for C_t and $\Omega_{1,t}$ above.

(A.48) and (A.50) and constitute the *consumption and hours decision rules given point expectations of* $\{\mathbb{E}_t^* W_{t+i}\}_{i=1}^{\infty}$, $\{\mathbb{E}_t^* R_{t+1,t+i}\}_{i=1}^{\infty}$ *and* $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$.

A.2.2 Retail Firms

Turning next to price-setting by retail firms, write (A.21) and (A.22) as

$$J_t = \left(\frac{1}{1-\frac{1}{\zeta}}\right) Y_t MC_t MS_t + \mathbb{E}_t \sum_{k=1}^{\infty} \xi^k \Lambda_{t,t+k} \Pi_{t,t+k}^{\zeta} Y_{t+k} MC_{t+k} MS_{t+k} \tag{A.51}$$

$$JJ_t = Y_t + \mathbb{E}_t \sum_{k=1}^{\infty} \xi^k \Lambda_{t,t+k} \Pi_{t,t+k}^{\zeta-1} Y_{t+k} \tag{A.52}$$

Then assuming point expectations (as for households)

$$J_t = \left(\frac{1}{1-\frac{1}{\zeta}}\right) \left(Y_t MC_t MS_t + \sum_{k=1}^{\infty} \xi^k \mathbb{E}_t^* \Lambda_{t,t+k} (\mathbb{E}_t^* \Pi_{t,t+k})^{\zeta} \mathbb{E}_t^* Y_{t+k} \mathbb{E}_t^* MC_{t+k} \mathbb{E}_t^* MS_{t+k} \right)$$

$$= \left(\frac{1}{1-\frac{1}{\zeta}}\right) (Y_t MC_t MS_t + \Omega_{3,t}) \tag{A.53}$$

$$JJ_t = Y_t + \sum_{k=1}^{\infty} \xi^k \mathbb{E}_t^* \Lambda_{t,t+k} (\mathbb{E}_t^* \Pi_{t,t+k})^{\zeta-1} \mathbb{E}_t^* Y_{t+k}$$

$$= \quad Y_t + \Omega_{4,t} \tag{A.54}$$

where noting that $\mathbb{E}_t^* \Lambda_{t,t+1} = \frac{1}{\mathbb{E}_t^* R_{t+1}}$ and $\Pi_{t,t+1} = \Pi_{t+1}$ we have

$$\Omega_{3,t} = \xi \frac{(\mathbb{E}_t^* \Pi_{t+1})^{\zeta} \mathbb{E}_t^* Y_{t+1} \mathbb{E}_t^* MC_{t+1} \mathbb{E}_t^* MS_{t+1}}{\mathbb{E}_t^* R_{t+1}} + \xi \frac{\mathbb{E}_t^* \Pi_{t+1}^{\zeta}}{\mathbb{E}_t^* R_{t+1}} \Omega_{3,t+1} \tag{A.55}$$

$$\Omega_{4,t} = \xi (\frac{\mathbb{E}_t^* \Pi_{t+1})^{\zeta - 1} \mathbb{E}_t^* Y_{t+1}}{\mathbb{E}_t^* R_{t+1}} + \xi \frac{\mathbb{E}_t^* \Pi_{t+1}^{\zeta - 1}}{\mathbb{E}_t^* R_{t+1}} \Omega_{4,t+1} \tag{A.56}$$

Recalling that the optimal price re-setting decision rule is given by $\frac{P_t^O}{P_t} = \frac{J_t}{JJ_t}$, (A.53) and (A.54) now give us the this rule given exogenous expectations of $\{\mathbb{E}_t^* \Pi_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* Y_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* MC_{t+i}\}_{i=0}^{\infty}$ and $\{\mathbb{E}_t^* MS_{t+i}\}_{i=0}^{\infty}$

Recalling that the optimal price re-setting decision rule is given by $\frac{P_t^O}{P_t} = \frac{J_t}{JJ_t}$, (A.53) and (A.54) now give us the this rule given exogenous expectations of $\{\mathbb{E}_t^* \Pi_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* Y_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* MC_{t+i}\}_{i=0}^{\infty}$ and $\{\mathbb{E}_t^* MS_{t+i}\}_{i=0}^{\infty}$

A.3 Individual Rationality in the NK Model

The final step to complete the IR equilibrium is to choose the learning rule for $\{\mathbb{E}_t^* W_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$ and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$ for households and $\{\mathbb{E}_t^* \Pi_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* Y_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* MC_{t+i}\}_{i=0}^{\infty}$ and $\{\mathbb{E}_t^* MS_{t+i}\}_{i=0}^{\infty}$ for retail firms.

We assume general adaptive expectations rules so that

$$\mathbb{E}_t^*[W_{t+i}] = \mathbb{E}_t^*[W_{t+1}] \text{ for } i \geq 1 \tag{A.57}$$

and similarly for $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* \Pi_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* Y_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* MC_{t+i}\}_{i=0}^{\infty}$ and $\{\mathbb{E}_t^* MS_{t+i}\}_{i=0}^{\infty}$ whilst with point expectations

$$\begin{aligned} \mathbb{E}_t^* R_{t+1,t+i} &= \mathbb{E}_t^* \left[\frac{R_{n,t}}{\Pi_{t+1}} \frac{R_{n,t+1}}{\Pi_{t+2}} \cdots \frac{R_{n,t+i-1}}{\Pi_{t+i}} \right] \\ &= \frac{R_{n,t}}{\mathbb{E}_t^* R_{n,t+1}} \left(\frac{\mathbb{E}_t^* R_{n,t+1}}{\mathbb{E}_t^* \Pi_{t+1}} \right)^i \text{ for } i \geq 1 \end{aligned} \tag{A.58}$$

which takes into account the observation of $R_{n,t}$ at time t.

One-period ahead forecasts are given by

$$\begin{aligned} \mathbb{E}_t^*[W_{t+1}] &= \mathbb{E}_{t-1}^*[W_t] + \lambda_W \left(W_t - \mathbb{E}_{t-1}^*[W_t] \right) \ ; \ \lambda_W \in [0,1] \\ \mathbb{E}_t^*[\Gamma_{t+1}] &= \mathbb{E}_{t-1}^*[\Gamma_t] + \lambda_\Gamma \left(\Gamma_t - \mathbb{E}_{t-1}^*[\Gamma_t] \right) \ ; \ \lambda_\Gamma \in [0,1] \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}_t^*[R_{n,t+1}] &= \mathbb{E}_{t-1}^*[R_{n,t}] + \lambda_{R_n}\left(R_{n,t} - \mathbb{E}_{t-1}^*[R_{n,t}]\right) \;;\; \text{(households)} \\
\mathbb{E}_{h,t}^*[\Pi_{t+1}] &= \mathbb{E}_{t-1}^*[\Pi_t] + \lambda_{h,\Pi}\left(\Pi_t - \mathbb{E}_{t-1}^*[\Pi_t]\right) \;;\; \lambda_{h,\Pi} \in [0,1] \text{ (households)} \\
\mathbb{E}_{f,t}^*[\Pi_{t+1}] &= \mathbb{E}_{t-1}^*[\Pi_t] + \lambda_{f,\Pi}\left(\Pi_t - \mathbb{E}_{t-1}^*[\Pi_t]\right) \;;\; \lambda_{f,\Pi} \in [0,1] \text{ (firms)} \\
\mathbb{E}_t^*[Y_{t+1}] &= \mathbb{E}_{t-1}^*[Y_t] + \lambda_Y\left(Y_t - \mathbb{E}_{t-1}^*[Y_t]\right) \;;\; \lambda_Y \in [0,1] \\
\mathbb{E}_t^*[\tilde{MC}_{t+1}] &= \mathbb{E}_{t-1}^*[\tilde{MC}_t] + \lambda_{MC}\left(\tilde{MC}_t - \mathbb{E}_{t-1}^*[\tilde{MC}_t]\right) \;;\; \lambda_{MC} \in [0,1] \\
\text{where } \tilde{MC}_t &\equiv MC_t MS_t
\end{aligned}
$$

noting that we assume that $\tilde{MC}_t$ is observed by the firm, but it cannot disentangle MC_t and MS_t.

With adaptive point expectations (A.49) now becomes

$$
\frac{C_t}{(1-\beta)} = \frac{1}{(\kappa C_t)^{\frac{1}{\phi}}}\left(W_t^{1+\frac{1}{\phi}} + \frac{\left(\left(\frac{\mathbb{E}_t^* R_{n,t+1}}{R_{n,t}}\right)\mathbb{E}_t^* W_{t+1}\right)^{1+\frac{1}{\phi}}}{\beta^{\frac{1}{\phi}}(\mathbb{E}_t^* R_{t+1}^{ex})^{1+\frac{1}{\phi}} - 1}\right) + \Gamma_t + \frac{\left(\frac{\mathbb{E}_t^* R_{n,t+1}}{R_{n,t}}\right)\mathbb{E}_t^*\Gamma_{t+1}}{\mathbb{E}_t^* R_{t+1}^{ex} - 1}
$$

where we have defined the ex post real interest rate as

$$
R_t^{ex} \equiv \frac{R_{n,t}}{\Pi_t} \tag{A.59}
$$

whilst (A.55) and (A.56) now become

$$
\begin{aligned}
\Omega_{3,t} &= \frac{\xi(\mathbb{E}_t^*\Pi_{t+1})^{\zeta}\mathbb{E}_t^* Y_{t+1}\mathbb{E}_t^* MC_{t+1}\mathbb{E}_t^* MS_{t+1}}{\mathbb{E}_t^* R_{t+1} - \xi(\Pi_{t+1})^{\zeta}} \\
\Omega_{4,t} &= \frac{\xi(\mathbb{E}_t^*\Pi_{t+1})^{\zeta-1}\mathbb{E}_t^* Y_{t+1}}{\mathbb{E}_t^* R_{t+1} - \xi(\mathbb{E}_t^*\Pi_{t+1})^{\zeta-1}}
\end{aligned}
$$

This completes the individually equilibrium with point adaptive expectations.

A.4 Heterogeneous Expectations

The behavioural model consists of a proportions $n_{h,t}$ and $n_{f,t}$ of rational households and firms respectively. Denote the outcome of for the IR agents of X_t by X_t^{IR} given above and the corresponding outcome for RE agents who adopt composite-model consistent expectations by X_t^{RE}. Then aggregate consumption labour supply and wholesale output are given by

$$
C_t = n_{h,t}C_t^{RE} + (1-n_{h,t})C_t^{IR} \tag{A.60}
$$

$$
H_t = n_{h,t}H_t^{RE} + (1-n_{h,t})H_t^{IR} \tag{A.61}
$$

$$
Y^W = A_t H_t^{\alpha} \tag{A.62}
$$

For retail firms the aggregate re-optimized price is given by

$$\frac{P_t^O}{P_t} = n_{f,t}\left(\frac{P_t^O}{P_t}\right)^{RE} + (1-n_{f,t})\left(\frac{P_t^O}{P_t}\right)^{IR} \tag{A.63}$$

with aggregate and dispersion given by

$$1 = \xi\Pi_t^{\zeta-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{1-\zeta} \tag{A.64}$$

$$\Delta_t = \xi\Pi_t^{\frac{\zeta}{\alpha}}\Delta_{t-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{-\frac{\zeta}{\alpha}} \tag{A.65}$$

Then the proportions of rational households and firms is given by

$$n_{h,t} = \frac{\exp(\gamma\Phi_{h,t}^{RE})}{\exp(\gamma\Phi_{h,t})^{RE} + \exp(\gamma\Phi_{h,t}^{IR})} \tag{A.66}$$

$$n_{f,t} = \frac{\exp(\gamma\Phi_{f,t}^{RE})}{\exp(\gamma\Phi_{f,t}^{RE}) + \exp(\gamma\Phi_{f,t}^{IR})} \tag{A.67}$$

where payoffs $\Phi_{h,t}^{RE}$, $\Phi_{h,t}^{RE}$ and $\Phi_{h,t}^{IR}$, $\Phi_{h,t}^{IR}$ are expressed on terms of a discounted sum of past weighted forecast errors, $\Phi_{h,t}$ say, starting at $t=0$ for with rational and non-rational households respectively:

$$\begin{aligned}
\Phi_{h,t}^{RE} &= \mu_h^{RE}\Phi_{h,t-1}^{RE} - \Big(w_W((W_t - E_{h,t-1}W_t)/W)^2 + w_{h,\Pi}((\Pi_t - E_{h,t-1}\Pi)/\Pi)^2 \\
&+ w_\Gamma((\Gamma_t - E_{h,t-1}\Gamma_t)/\Gamma)^2 + w_R((R_{n,t} - E_{t-1}R_{n,t})/R_n)^2 + C_h\Big) \\
\Phi_{h,t}^{IR} &= \mu_h^{IR}\Phi_{h,t-1}^{IR} - \Big(w_W((W_t - E_{h,t-1}^*W_t)/W)^2 + w_{h,\Pi}((\Pi_t - E_{h,t-1}^*\Pi)/\Pi)^2 \\
&+ w_\Gamma((\Gamma_t - E_{h,t-1}^*\Gamma_t)/\Gamma)^2 + w_R((R_{n,t} - E_{t-1}R_{n,t})/R_n)^2\Big)
\end{aligned}$$

The parameter C_h is a fixed cost of being rational for households. For firms this becomes

$$\begin{aligned}
\Phi_{f,t}^{RE} &= \mu_f^{RE}\Phi_{f,t-1}^{RE} - \Big(w_Y((Y_t - E_{f,t-1}Y_t)/Y)^2 + w_{f,\Pi}((\Pi_t - E_{f,t-1}\Pi)/\Pi)^2 \\
&+ w_{MC}((\tilde{M}C_t - E_{f,t-1}\tilde{M}C_t)/MC)^2 + C_f\Big) \\
\Phi_{f,t}^{IR} &= \mu_f^{IR}\Phi_{f,t-1}^{IR} - \Big(w_Y((Y_t - E_{f,t-1}^*Y_t)/Y)^2 + w_{f,\Pi}((\Pi_t - E_{f,t-1}^*\Pi)/\Pi)^2 \\
&+ w_{MC}((\tilde{M}C_t - E_{f,t-1}^*\tilde{M}C_t)/MC)^2\Big)
\end{aligned}$$

where the parameter C_f is a fixed cost of being rational for firms and we allow for the possibility that $C_h \neq C_f$.

Thus the proportion of rational agents in the steady state is given by

$$n_h = \frac{\exp(-\gamma C_h)}{\exp(-\gamma C_h) + 1} \tag{A.68}$$

$$n_f = \frac{\exp(-\gamma C_f)}{\exp(-\gamma C_f) + 1} \tag{A.69}$$

which is pinned down by the cost parameters (C_h, C_f) (which can be positive or negative).