Discussion Papers in Economics

ON THE EFFICIENCY OF MONETARY EQUILIBRIUM WHEN AGENTS ARE WARY

By

Aloisio Araujo
(IMPA and EPGE/FGV, Brazil)
Juan Pablo Gama-Torres
(IMPA, Brazil)
Rodrigo Novinski
(Faculdades Ibmec, Brazil)
&
Mario R. Pascoa
(University of Surrey)

DP 04/16

School of Economics
University of Surrey
Guildford
Surrey GU2 7XH, UK
Telephone +44 (0)1483 689380
Facsimile +44 (0)1483 689548
Web www.econ.surrey.ac.uk
ISSN: 1749-5075

ON THE EFFICIENCY OF MONETARY EQUILIBRIUM WHEN AGENTS ARE WARY[1]

Aloisio Araujo

IMPA and EPGE/FGV, Rio de Janeiro, Brazil

aloisio@impa.br

Juan Pablo Gama-Torres

IMPA, Rio de Janeiro, Brazil

jpgamat@impa.br

Rodrigo Novinski

Faculdades Ibmec, Rio de Janeiro, Brazil

rodrigo.novinski@ibmecrj.br

Mario R. Pascoa

University of Surrey, U.K.

m.pascoa@surrey.ac.uk

March 7, 2016

[1]We gratefully acknowledge the financial support from CAPES, CNPq and FAPERJ in Brazil and from FCT/FEDER (project EGE-ECO/114496/2009) and NOVA FORUM in Portugal. We are grateful to comments by seminar audiences at UCLA, the University of Chicago Money Workshop, the Harvard-MIT Theory seminar, 2nd BWGT at USP and Sorbonne Workshop in Ec. Theory 2014.

Abstract

Wary agents tend to neglect gains at distant dates but not the losses that occur at those far away dates. For these agents, Ponzi schemes are not the only improving schemes compatible with non-arbitrage pricing. However, efficient allocations can be sequentially implemented by allocating money and then, at subsequent dates, taxing savings plans whose open end benefits to wary agents outweigh the cost of carrying on cash. The allocative role of money does not disappear over time and the transversality condition allows for consumers to have limiting long positions. Money supply does not have to go to zero and, actually, there are equilibria where it does not. We address also why fiat money does not lose its value when Lucas trees are available.

1 Introduction

This paper reexamines some core questions in monetary economics in the light of a reformulation of the way infinite lived agents discount the future. We depart from the classical impatience assumption and allow for *wary* consumers, who are willing to ignore distant gains but not distant losses. Our examples focus on two interesting classes of preferences. First, *precautionary preferences*, paying a particular attention to the worst lifetime outcome (Example 1). Second, *habit persistence*, in the form of a deslike of descending (on average, over time) standards of living (Example 2).

In general, wariness can be defined in terms of consumers being upper but not lower semi-impatient[1]. Mathematically, preferences are upper but not lower semi-continuous for the Mackey topology on the space of bounded sequences. For such preferences, Bewley [1972] established existence of Arrow-Debreu (AD) equilibrium prices that may fail to be a summable sequence. That is, the price functional may belong to the dual space, rather than to the pre-dual. In other words, the price may have a pure charge component, which is a linear functional that, apart from a positive scalar multiple, is a generalized limit. We address the sequential implementation of such AD allocations.

Wariness poses new problems for the implementation of efficient allocations by trading assets sequentially. Ponzi schemes are not anymore the only improvement strategies compatible with one-period non-arbitrage pricing. Even for portfolio plans with a non-negative limiting deflated cost, there could be a positive limiting benefit of reducing distant losses. When the latter exceeds the former there is an improvement opportunity that precludes existence of equilibrium with sequential budget constraints.

We chose to look at what might be done when the implementing asset is fiat money, since this is the asset for which, in a context of impatience, the sequential implementation was quite straightforward. It is well known that in economies with impatient agents, fiat money, being traded always in non-negative amounts, avoids Ponzi schemes

[1]These impatience (or myopia) notions were developed by Brown and Lewis [1981], Araujo [1985], Raut [1986] and Sawyer (1987). See also Mas-Colell and Zame [1991].

and, therefore, dispenses with the enforceability and frictions associated with borrowing constraints. Moreover, if initial money holdings are large enough, the usual no-short-sales constraint does not introduce frictions and efficiency holds. Differently from what happens with assets paying dividends, the same optimal consumption plan can be accommodated when the plan of money balances together with the initial holdings are being shifted by some positive amount. There is a catch however. Money only has a positive price, in an efficient outcome under impatience, if the money supply is retrieved over time. This prescription is usually known as Friedman's (strong) rule. Can fiat money also serve as an implementing devise when agents are non-impatient? Should the limiting money supply, after retrieving the efficient taxes, be always zero?

When agents were impatient, efficient taxes could be lump sum, as what mattered was to remove all the money without perturbing the equality of consumers' marginal rates of substitution. However, when agents are wary, the non-negativity of money balances is not enough to rule out all the long-term improvement strategies. We look at taxes that are harsher on money balances plans for which the benefits from the asymptotic dishoarding exceed the cost of carrying on cash. This raises the effective opportunity cost of holding cash and allows non-impatient consumers to find a finite optimum. By correcting a mismatch of savings benefits and costs at the individual level, taxes end up implementing efficient savings. While being non-lump-sum, taxes are impersonal and can be chosen to be recursive, levied upon the disposal of the balances (say upon the current dishoarding benefit net of the carry on cost up to now), being in a deflationary context the equivalent for money to what capital gains taxes are for all other assets[2].

Our *first implementation result* establishes that efficient allocations can always be implemented using fiat money, in non-negative balances, by introducing non-lump-sum taxes that correct for the long-term gap in the benefits and costs of saving plans. The tax schedule can be chosen so that if it were applied in a context of impatience money

[2]Actually, foreign currency balances tend to be subject to such taxes, like any other non-monetary asset. We are taking a step forward, suggesting that all money balances should be, in order for efficiency to be attained in a deflationary context where agents are wary and prone to dishoard in the long run.

supply would go to zero, or equivalently, to be non-distortionary in the sense that the Euler and transversality conditions are just as they were in an economy without taxes.

When some of the agents are non-impatient, a multiplicity of equilibrium money balances plans occurs, with different limiting supplies of money. Our *second implementation result* says that if for some non-impatient agent the loss from reverting the optimal savings plan is higher than the AD price of that plan, then there is a monetary implementation with non-vanishing money supply. Such dual appraisal is a consequence of the non-differentiability that is intrinsic to wary preferences (due to the freedom in the choice of the generalized limit for the pure charge of the supergradient). The non-differentiability implies that, for a wary consumer, the marginal loss from reverting a savings plan is greater than the marginal gain from intensifying that savings strategy. Such asymmetry is reminiscent of the asymmetrical attitude towards losses and gains proposed in prospect theory. For our second implementation result to hold, there should be at least one consumer whose left-hand-side derivative, in the direction of the equilibrium savings plan, exceeds the AD price of that plan. Next, we look for preferences and AD allocations for which this is the case.

An interesting case, illustrating the second implementation result, occurs when the endowments of non-impatient agents are subject to persistent shocks. We give examples both for the precautionary and the habit persistent preferences. For the former, we consider utility functions that depart from the usual series of discounted utilities since we add a term dealing explicitly with the lifetime infimum of utilities. These preferences are related to the Rawlsian utility (which would be just that infimum), mentioned in Araujo [1985], but now the presence of the series of discounted utility makes preferences monotonic. For the latter, there is instead a specific term dealing with the infimum of the utility averages up to each point in time. When the relevant infimum is not attained in finite time, there is at least one consumer for whom the marginal loss at infinity (from reverting the savings strategy) is higher than the AD price of the savings plan.

Both examples have a nice reinterpretation in terms of *endogenous discounting*. Not being sure how to discount the future, the consumer's objective function is the minimal series of discounted utilities, over some set of discount factors. For each consumption plan, it is the most penalizing discount factor that is being picked that set. In the precautionary example that set is generated by the ϵ-contamination capacity, as in Gilboa and Schmeidler [1989] and Dow and Werlang [1992]. However, we can allow for other sets of discount factors, as the habit persistence example illustrates.

In a broad context of endogenous discounting it is easy to see why wariness may prevail: the lower envelope of a family of continuous functions is known to be upper semi-continuous but may fail to be lower semi-continuous. That is, even if for a given discount factor we have a payoff that is Mackey continuous, once we take the infimum over discount factors, we may end up with a utility function that is just Mackey upper semi-continuous. Actually, we can say more. We establish (theorem 1) that pure charges in the supporting prices of efficient allocations can only occur when the standard series of discounted marginal utilities do not converge uniformly, across discount factors in the allowed set. The macroeconomics literature on misspecification of preferences and robust control has paid a significant attention to endogenous discounting. Examples can be found in Hansen and Sargent [2001] and Hansen and Sargent [2008].

Our two main examples are complemented by an example (Example 3) of an economy where an inefficient sequential equilibrium occurs when the taxes that we propose are absent. In the AD equilibrium, the infimum of the consumption plans is not attained in finite time, but in the inefficient sequential equilibrium that infimum is attained at infinitely many dates, different for the two consumers. The inefficient consumption plans have some supergradients without pure charges and this allows for the consumers' problems to have finite optima under the plain no-short-sales constraint on money. In other words, there are no asymptotic gains from dishoarding under such supergradients and, therefore, sequential equilibria exist in the absence of our proposed taxes. However, for such supergradients, at dates where the infimum is attained, the left and the right marginal utilities do not coincide and no-short-sales constraints have shadow values.

The paper is related to the work in Araujo, Novinski and Pascoa [2011] on sequential implementation of AD allocations using long-lived assets *paying dividends*, but differs from it in three crucial aspects. First, in what causes a bubble in the price of the implementing asset. Before, the bubble was just the value that the AD price pure charge takes at the dividends sequence, but in the case of money such value would be zero. Now, the initial holdings scaled up by the bubble can be taken to be the difference between the value that the pure charge of a supergradient can take at the net trade and the corresponding value for the AD price pure charge - the non-differentiability of preferences, inherent to wariness, plays now a crucial role[3] . Second, finite optima for consumers' sequential problems is now ensured by introducing taxes that discourage inefficient savings, rather than by imposing portfolio constraints[4]. We believe this approach is quite novel and illustrates well what can be done differently when the implementing asset is money. Third, we have now general results for wary preferences and, even when we study in detail the endogenous discounting context, we are no longer focused on the specific form driven by the ϵ-contamination capacity.

The comparison of our monetary implementation with the implementation with other assets leads us to other important issues. We are not claiming that money plays an irreplaceable allocative role[5]. A Lucas tree could also play a hedging role at infinity but the sequential market completeness might not be attained so easily. In fact, short positions in money can be easily avoided by raising the initial holdings high enough, whereas in the case of a Lucas tree such increase in initial holdings would be incompatible with the given AD endowments (which must be equal to the sequential endowments plus the returns from the initial holdings of the tree). To implement using Lucas trees, in non-negative positions, we would need the help of zero-net-supply promises that should not be secured by the trees (otherwise the markets might become incomplete due to the friction created by the collateral constraint[6]). The drawback of relying on unsecured

[3]see Lemma 4 in the Appendix

[4]Actually an implementation with persistently positive money supply could not be done as before with constraints using the AD price pure charge evaluation of the net trade. The constraints would have to use instead the limsup of the net trade.

[5]We were asked this question by Nancy Stokey at a presentation at the University of Chicago in 2012.

[6]See, for example, Gottardi and Kubler [2015] on this issue and on weaker notions of efficiency, that depart from the full efficiency we are interested in.

credit is that full commitment of debtors would have to be assumed, which might clash with incentive compatibility.

We address also a related, less demanding, classical monetary theme. We show that, in a stochastic environment, coexistence of money and other long-lived assets, paying dividends, does not make money lose its positive price or its efficient role. Money widens the hedging that the other long-lived assets can do. In fact, non-negative positions of positive-net-supply long-lived assets is ensured at no cost since the initial holdings of money can be increased so that all long-lived assets complete the markets without any short sales. Our results illustrate how a new approach to the preferences of infinite-lived consumers yields quite different answers to long-standing monetary themes.

The idea that money plays a crucial reserve role has captured a lot of attention in the literature. Friedman [1953, 1969] put forward the idea that consumers should not economize unnecessarily on money balances as these holdings are “a reserve against future emergencies”. The wasteful economizing of cash should be avoided by deflation or by providing money with a real rate of interest. This proposition has been often associated with the stronger recommendation of a steady contraction of the money supply. Our reappraisal of the hedging role of money resumes Bewley’s (1980) approach centered on the idea that the “devise to give money a value is infinite horizon (together with the need for insurance)”, but we take a step forward and take into account the limiting hedging role of money for non-impatient agents. Our results are reminiscent of the persistent role of money found by Samuelson [1958] in the overlapping generations model, which seemed until now incompatible with immortal agents. For impatient agents, Bewley [1980, 1983] showed that a non-vanishing money supply, together with interior consumption, had to be inefficient. Levine [1986, 1988, 1989] confirmed this under Inada’s condition but observed that efficiency might prevail under non-interior consumption[7].

Finally, it should be pointed out that time consistency is compatible with wariness. As an example, when the series of discounted utilities describes a time-consistent behav-

[7] see also Woodford [1990], Kehoe, Levine and Woodford [1992] and Pascoa, Petrassi and Torres-Martinez [2010].

ior (say, under exponential discounting), then adding a term dealing with the infimum of the utilities makes the consumer wary and could introduce an inconsistency but it does not in equilibrium as long as the infimum is not attained in finite time, which is precisely a case we are interested in.

The rest of the paper is organized as follows. Section 2 characterizes wariness and AD prices. Section 3 relates endogenous discounting and wariness, illustrating with the precautionary and the habit persistence examples. Section 4 describes the deterministic sequential monetary model and Section 5 presents the results on efficient monetary equilibrium, illustrating the implementation for the above examples. Section 6 addresses implementation when Lucas trees are also available for the deterministic economy and extends the results to stochastic economies. The proofs of the results of Section 5 are in the Appendix. All the other proofs can be found in the supplementary material.

2 Wariness

2.1 Wariness and Arrow Debreu Prices

There are I infinite lived consumers, who are endowed with quantities $\omega_t^i \geq 0$ of a single commodity at the countably many dates. We allow for consumers that neglect distant gains but not the losses at far away dates. Such attitude, which we refer to as *wariness*, consists in being upper but not lower semi-impatient. Let us formalize these concepts, presuming monotonicity of preferences $\succeq^i$ over sequences of consumption of the single good. For any sequence $v \in \ell^\infty$ we denote by $v(n)$ the sequence such that $v(n)_t = v_t$ for $t \geq n$ and $v(n)_t = 0$ otherwise.

Consumer i is said to be *upper semi-impatient* at a bundle x if $x \succ^i y$ implies, for any $z \in \ell^\infty_+$, that $x \succeq^i y + z(n)$ for n large enough. Consumer i *fails to be lower semi-impatient* at a bundle x if there exists y for which $y \succ^i x$ but $x \succeq^i y - y(n)$, $\forall$ n. Losses beyond date n reverse the preference ordering, no matter how large n is.

These concepts can be formulated in terms of the Mackey topology on ℓ^∞, the finest topology on ℓ^∞ for which the dual is ℓ^1. For norm-continuous preferences $\succeq^i$, upper (lower) semi-impatience at x consists in the Mackey upper (lower) semi-continuity of $\succeq^i$ at x. A consumer whose preferences are norm continuous and Mackey upper semi-continuous, is *wary* at $x \in \ell^\infty_+$ if the preferences are not Mackey lower semi-continuous at x. If this condition holds on the norm interior of ℓ^∞_+, the consumer is said to be wary.

Assumption **A1**: *for each agent, preferences are representable by a utility function U^i that is concave, norm continuous, Mackey upper semi-continuous and such that $U^i(x) > U^i(x')$ whenever $x > x'$.*

Wariness impacts on the nature of the supporting prices of efficient allocations. An *Arrow-Debreu equilibrium* (AD) is defined as a pair (x, π) such that $x = (x^1, ..., x^I)$ is a feasible allocation, π a linear functional on ℓ^∞ and, for each i, x^i maximizes U^i in the budget set $\{a \in \ell^\infty_+ : \pi(a - \omega^i) \leq 0\}$. The natural environment where to look for prices is the norm dual of ℓ^∞. This is the space *ba*, of bounded finitely additive set functions, also called *charges*, on $\mathbb{N}$, equipped with the total variation norm (given $b \in ba$, its norm is $\|b\| = |b|(\mathbb{N})$). Let us rephrase a well-known result by Bewley [1972].

Proposition 1. *If A1 holds and $\sum_i \omega^i \ggg 0$, there exists an AD equilibrium (x, π), with the price π in ba. Some consumer being wary at x^i is a necessary condition for AD equilibrium prices to be outside of ℓ^1.*

Notice that *ba* contains strictly ℓ^1, the space of absolutely summable sequences, since each $y \in \ell^1$ induces an element μ in the space *ca* of countably additive set functions on $\mathbb{N}$ (by setting $\mu(\{t\}) = y_t$). We start by examining how do supporting prices look like when they are charges that are outside of ℓ^1.

2.2 On Charges as Supporting Prices

By the Yosida-Hewitt Theorem, any $\pi \in ba$ can be decomposed uniquely in the form $\pi = \mu + \nu$ where $\mu \in ca$ whereas ν is a pure charge. For any finite subset B of $\mathbb{N}$, if ν is a positive pure charge, then $\nu(B) = 0$. Denote by *pch* the set of pure charges on $(\mathbb{N}, 2^{\mathbb{N}})$. Let us see characterize the pure charge components of a supporting price.

Recall that $T \in ba$ is a supporting price for an allocation $(x^i)_i$ if $U^i(z) \geq U^i(x^i)$ implies that $T(z - x^i) \geq 0$ for any i. A supporting price is, up to a positive scalar multiple, a supergradient[8] of U^i at x^i. Denote by $\partial U^i(x)$ the set of supergradients of U^i at x. For any $v \in \ell^\infty$ we denote by $\delta^+ U^i(x; v)$ and $\delta^- U^i(x; v)$ the right and left derivatives of U^i at x along the v-direction[9].

Lemma 1. *If $\nu \in pch_+$, then $\nu(x)/\|\nu\|_{ba} \in [\liminf x, \limsup x]$, $\forall x \in \ell^\infty$. If for some $\mu \in ca$, $\mu + \nu \in \partial U^i(x)$, then $\|\nu\|_{ba} \in [\lim_n \delta^+ U^i(x; \mathbb{1}(n)), \lim_n \delta^- U^i(x; \mathbb{1}(n))]$*

We use the notation LIM to represent a linear functional taking on each $x \in \ell^\infty$ a value in $[\liminf x, \limsup x]$. Notice the *real indeterminacy in AD equilibrium* resulting from the choice of LIM for the pure charge ν in the support price (if we pick another LIM, the AD budget equation will not hold for the same bundle x^i, except when the supergradients at x^i have also multiple ca components). Wariness is necessary for the occurrence of pure charges supporting interior bundles:

Lemma 2. *Under A1, if U^i is Mackey continuous, then $\partial U^i(x) \subset \ell^1$ for $x \ggg 0$.*

3 Endogenous discounting

We focus on preferences for which wariness is induced by an aversion to the ambiguity in the discount factor. That is, consumers have a collection of possible discount factors and, not being sure which one to pick, end up choosing for each consumption plan the discount factor that gives the lowest sum for the series of discounted utilities. Such attitude implies *endogenous discounting*, a feature that has received significant attention in the recent literature on the (mis)specification of macro models (see Hansen and Sargent [2001, 2008]). To be quite general, preferences are described by

$$U(x) = \inf_{\delta \in C} \sum_{t=1}^{\infty} \delta_t u(x_t), \tag{1}$$

[8] $T \in ba$ is a supergradient of U^i at x if $U^i(x + h) - U^i(x) \leq Th$ for any $h \in \ell^\infty$.

[9] $\delta^+ U(x; v) = \lim_{h \downarrow 0} \frac{U(x+hv) - U(x)}{h}$ and $\delta^- U(x; v))$ is defined with $h \uparrow 0$ instead.

where C is a subset of $\ell^1_+ \cap B_1(0)$ and $B_1(0)$ is the unit ball of ℓ^1. Such preferences have an analogy with the ambiguity aversion attitude in a context of uncertainty, modeled, as in Gilboa and Schmeidler [1989], by considering a functional which is the minimum of the expected utilities over a collection of beliefs described by finitely additive set functions. More precisely and for a general set X of objects (in our context $X = \mathbb{N}$),

$$U(x) = \min_{\eta \in \tilde{C}} \int_X u \circ x \, d\eta, \tag{2}$$

where $\tilde{C}$ is a convex and weak* closed subset of *ba* (the referred papers discussed axiomatically this representation)[10]. The minimal integral over beliefs represents a precautionary or pessimistic behavior. The minimization solution η^* puts more weight on sets where u attains its lowest values. To see that (1) can be reformulated in terms of (2), take the closure in the weak* topology of the convex hull of C.

3.1 Endogenous discounting and wariness

The following result tells us the most that one can say without imposing more structure on the set C of possible discount factors.

Lemma 3 (Aversion to ambiguity and wariness)**.** *Under aversion to ambiguity in discounting, that is, when preferences are given by (1), preferences are Mackey upper semi-continuous but may fail to be Mackey lower semi-continuous.*

This is a consequence of the fact that the lower envelope of a family of upper semi-continuous functions (on any topological space) is still upper semi-continuous. Other results can be obtained by specifying the set C. We start by examining the case of a set C generated by a *capacity*, that is, a function $\nu : 2^{\mathbb{N}} \to I\!R$ such that $\nu(\emptyset) = 0$ and $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$. A capacity ν is *convex* when $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)\ \forall A, B \subset \mathbb{N}$. We normalize $\nu(\mathbb{N}) = 1$. The set core(ν) is defined as $\{\eta \in ba : \eta \geq \nu,\ \eta(\mathbb{N}) = 1\}$. When $C = \text{core}(\nu)$ for a convex capacity ν[11], more can be said about the absence of Mackey lsc. A capacity ν is said to be *continuous at certainty*

[10] See Dunford and Schwartz [1958], ch. III.2, for the definition of integral with respect to a charge η.
[11] In this case, the utility function is a Choquet integral (see Schmeidler [1989]).

if, for any sequence $(A_n) \subset 2^{\mathbb{N}}$ such that each $A_n \subset A_{n+1} \subset \mathbb{N}$ and $\cup_n A_n = \mathbb{N}$, we have $\lim \nu(A_n) = \nu(\mathbb{N})$. Now, U is Mackey lsc if and only if the capacity is continuous at certainty (by Theorem 2.1 in Araujo [1985]). The discontinuity at certainty can be interpreted as if there were a missing state. In Araujo, Novinski and Pascoa [2011], the focus was on a well-known example of a convex capacity, which we recall next.

Example 1

Given a probability measure μ, let ν_ϵ be the convex capacity obtained by a linear distortion of μ with coefficient $(1-\epsilon) \in (0,1]$, i.e., taking $\nu_\epsilon(A) = (1-\epsilon)\,\mu(A)$ for $A \subsetneqq \mathbb{N}$ and $\nu_\epsilon(\mathbb{N}) = 1$. This is called the *$\epsilon$-contamination capacity* with respect to μ and allows us to rewrite (2) as[12]

$$U(x) = (1-\epsilon) \int_{\mathbb{N}} u \circ x \, d\mu + \epsilon \inf u \circ x. \tag{3}$$

In this case, the minimum over normalized dominating charges coincides with the infimum over dominating probability measures. That is, $U(x) = \inf\{\int_{\mathbb{N}} u \circ x \, d\eta : \eta \in ca \cap \mathrm{core}(\nu_\epsilon)\}$. Clearly, ν_ϵ is discontinuous at certainty and, therefore, this utility represents wary preferences at some point[13]. Actually, for some (ζ, β) proportional to $((1-\epsilon)\mu, \epsilon)$, the utility can be rewritten (up to a scalar multiple) as

$$U(x) = \sum_{t=1}^{\infty} \zeta_t u\,(x_t) + \beta \inf_{t \geq 1} u(x_t) \tag{4}$$

Under (4) time-consistency may not hold, but it does if the infimum of consumption is not attained. This will be the case when we resume Example 1.

Let us denote the infimum of a bundle x by $\underline{x}$. Any supergradient T of U^i at x must be of the form[14] $T(a) = \sum_{t+1}^{\infty} u'(x_t)(\zeta_t^i + \gamma_t \beta^i) a_t + \sigma \beta^i u'(\underline{x}) \mathrm{LIM}^T(a)$, for any $a \in \ell^\infty$, where (i) $\gamma_t \geq 0$, (ii) $\gamma_t = 0$, if $x_t > \underline{x}$, (iii) $\sigma \geq 0$ is zero when $\underline{x}$ is not a cluster point of x and (iv) $\sum_{t=1}^{\infty} \gamma_t + \sigma = 1$. That is, there is a supergradient with a pure charge only if $\underline{x}$ is a cluster point of x and all supergradients will have pure charges if $\underline{x}$ is not attained.

Araujo [1985] showed that the Mackey topology is the finest topology of continuity in order for AD equilibrium to exist (with prices in ba), if no further assumptions are

[12] As was shown already by Dow and Werlang [1992].
[13] Actually, wariness holds at every $x \ggg 0$ as the lower contour set of x is not Mackey closed.
[14] For a proof see Araujo, Novinski and Pascoa [2011].

imposed, except for the convexity of preferences. In the same article, it was also claimed that for $U(x) = \inf_t u(x_t)$ equilibrium does not exist. However, when monotonicity with respect to increments at finitely many dates is added (which is satisfied by (4)) AD equilibrium (with prices in *ba*) exist, in spite of the failure of Mackey lower semi-continuity, as Bewley [1972] established. The other case of pessimistic preferences, with $U(x) = \liminf_t u(x_t)$, mentioned in Araujo [1985] has the drawback that the upper Mackey semi-continuity does not hold and, even if a series of discounted utilities would be added, AD equilibrium might not exist.

To go beyond the epsilon-contamination case but still have wariness we need to gain some intuition on when do preferences of the form (1) exhibit wariness and actually have supporting prices outside of ℓ^1. Let us examine what happens with the family $\{\sum_t \delta_t u'(x_t)\}_{\delta\in C}$ of marginal utilities, for deflators δ in C.

Theorem 1. ***(Pure charges and the non-uniform convergence of marginal utilities)*** *Let $C \subseteq \ell^1_{++}$ and $(x_t)_{t\in\mathbb{N}} \ggg 0$. If the series of marginal utilities at $(x_t)_{t\in\mathbb{N}}$ converges uniformly on the set C, in the sense that $\lim_t \sup_{\delta\in C}\left\{\sum_{s\geq t}\delta_s u'(x_s)\right\} = 0$, then there is no pure charge in any supergradient of U at x.*

We introduce another example of wary preferences in which the concern about the infimum is weaker than in Example 1. The non-additively separable part of the utility function will be related to the Polya index introduced by Marinacci [1998] to described patience. To do so, we define a countable set of priors in ℓ^1_+ which induces a smaller set of priors in *ba* than the one obtained from the ϵ-contamination capacity.

Example 2

The agent set of beliefs C^i is defined by $C^i := \text{core}(\nu_\epsilon) \cap \hat{C}^i$ where $\hat{C}^i$ is the closed convex hull of $\{(\delta_m)_{m\in\mathbb{N}} : \delta_m(t) = \zeta^i_t + \beta^i/m \text{ for } 1 \leq t \leq m, \delta_m(t) = \zeta^i_t \text{ elsewhere}\}$ in the weak* topology of *ba*. With this set of priors, the utility function in (1) becomes, by multiplying by a suitable scalar,

$$U^i(x) = \sum_t \zeta^i_t u^i(x_t) + \beta^i \inf_t \left(\frac{1}{t}\sum_{k=1}^{t} u^i(x_k)\right). \qquad (5)$$

Marinacci [1998] proposed a notion of complete patience using the Polya index $\lim_t \left(\frac{1}{t}\sum_{k=1}^{t} u^i(x_k)\right)$, which has a similarity with the last term in Equation (5), although the replacement of limit by the infimum implies that the agent now cares about small consumption in the first dates and, therefore, the agent has some degree of impatience. The time separable component of U^i also enhances the level of impatience, but some patience prevails, as the agent is worried about *mean losses, for means computed up to any distant date*. This is a form of habit persistence.

For any $(x_t)_t$ such that the infimum of $\left(\frac{1}{t}\sum_{k=1}^{t} u^i(x_k)\right)_t$ is not attained, the supporting prices have the following form $\pi(c) = \sum_{t=0}^{\infty} \zeta_t^i (u^i)'(x_t)c_t + \beta^i \text{LIM}\,(\phi(c))$ where $\phi : \ell_+^\infty \to \ell_+^\infty$ such that $\phi(c)_t = \frac{1}{t}\sum_{k=1}^{t} c_k$. The presence of a pure charge is related to the non–uniform (on m) convergence of the series of marginal utilities.

Another example, presented in the supplementary material, illustrates a consumer that cares about worst cycles (say one year) of consumption.

4 A Sequential Economy with Fiat Money

4.1 Money and Taxes

The set of trading dates is $\mathbb{N} \equiv \{1, 2, ...\}$. Before the initial date, the government allocates non-negative initial holdings $y_0^i \geq 0$ of money to each consumer and then, at each trading date t, money holdings y_t^i may be taxed. The taxes levied at each date, denoted by $\tau_t(y^i)$, may depend on the whole individual plan y^i of money holdings, but through an *impersonal tax schedule* $\tau_t(.)$.

The consumption good is the numeraire and we denote by $q = (q_t)_{t\in\mathbb{N}}$ the sequence of prices of money. Every consumer i faces at eacc date $t \in \mathbb{N}$, the following constraints:

$$y_t \geq 0 \tag{6}$$

$$x_t - \omega_t^i \leq q_t(y_{t-1} - y_t - \tau_t(y)) \tag{7}$$

Observe that $\tau_t(y)$ just has an impact at date t when $q_t > 0$.

Let us denote by $B(q, y_0^i, \omega^i, \tau)$ the set of couples $(x, y) \in \ell_+^\infty \times I\!R_+^\infty$ of consumption and money holdings plans satisfying constraints (6) and (7). The goal of agent i is to maximize U^i under $B(q, y_0^i, \omega^i, \tau)$. We denote the set $\{1, ..., I\}$ of agents by $\mathcal{I}$.

The initial money supply M_0 is given, equal to $\sum_{i=1}^I y_0^i$ and assumed to be positive. However, at each trading date, the *money supply* M_t is endogenous, satisfying

$$M_t(y^1, ..., y^I) = M_{t-1}(y^1, ..., y^I) - \sum_{i=1}^I \tau_t(y^i) = \sum_{i=1}^I \left(y_0^i - \sum_{s=1}^t \tau_s(y^i) \right),$$

Definition 1. $(q, (x^i, y^i)_{i\in\mathcal{I}}) \in I\!R_+^\infty \times (\ell_+^\infty \times I\!R_+^\infty)^I$ is an *equilibrium* for the economy with initial money holdings $(y_0^1, ..., y_0^I)$ and a tax policy τ if (a) $(x^i, y^i) \in \text{argmax}\{U^i(x) : (x, y) \in B(q, y_0^i, \omega^i, \tau)\}$; (b) $\sum_{i=1}^I (x^i - \omega^i) = 0$; (c) $M_t(y^1, ..., y^I) = \sum_{i=1}^I y_t^i \ \ \forall t \in \mathbb{N}$.

Definition 2. An equilibrium $(q, (x^i, y^i)_{i\in\mathcal{I}})$ is a *monetary equilibrium* if $q \neq 0$.

If $q_{t_0} > 0$ for some date t_0, it will be true by non-arbitrage that $q_t > 0 \ \forall t$. Note that, under A1, (7) holds as equality, which summed over i, make (b) imply (c).

4.2 Sequential equilibrium and improvement opportunities

Observe first that in the absence of any taxes, sequential budget constraints are as follows

$$x_t - \omega_t^i \leq q_t(z_{t-1} - z_t) \quad \forall t \in \mathbb{N}, \tag{8}$$

A very useful sufficient condition for individual optimality is given as follows. Let $x(z)$ be defined by $x_t(z) = q_t(z_{t-1} - z_t)$.

Proposition 2. *Let z^* be portfolio satisfying (6) and (8) and let $x^* = x(z^*)$. (i) Suppose there exists $T \in \partial U(x^*)$ with $T = \mu + \nu$, $\mu \in \ell_+^1$ and $\nu \in \text{pch}_+$ such that, for every t,*

$$\mu_t q_t \geq \mu_{t+1} q_{t+1} \qquad (\mu_t q_t - \mu_{t+1} q_{t+1}) z_t^* = 0 \tag{9}$$

and

$$\lim \mu_t q_t z_t^* = \nu(x^* - \omega). \tag{10}$$

(ii) Suppose also that every feasible portfolio z satisfies the condition

$$\lim_t \mu_t q_t z_t \geq \nu(x(z) - \omega), \tag{11}$$

Then z^* is an optimal solution for the problem with constraints (6) and (8).

Proof. Given a feasible portfolio z, $U(x(z)) - U(x^*) \leq T(x(z) - x^*) = T(x(z) - \omega) + T(\omega - x^*)$. Moreover, $\mu(x(z) - \omega) = \sum_{t=1}^{\infty} \mu_t q_t (z_{t-1} - z_t)$. By (9) and $z \geq 0$, $\mu(x(z) - \omega) \leq \mu_1 q_1 z_0 - \lim_t \mu_t q_t z_t$. Also, $\mu(x^* - \omega) = \mu_1 q_1 z_0 - \lim_t \mu_t q_t z_t^*$. Now by (10), $U(x(z)) - U(x^*) \leq \nu(x(z) - \omega) - \lim_t \mu_t q_t z_t$. Now, by (11), $U(x(z)) - U(x^*) \leq 0$. □

Remark 1:

Notice that (9) is necessary for individual optimality. For $(x^i)_i \gg 0$ efficient, the inequalities in (9) hold as equalities for every i and t (otherwise agents holding money would have marginal rates of substitution different from those of other agents), so $\mu_t^i q_t$ is constant and, for the *ca* component p of the AD price, $p_t q_t$ is also constant. Hence, an efficient monetary equilibrium with $(x^i)_i \gg 0$ is deflationary at infinitely many dates and Friedman's weak rule, prescribing a zero nominal interest rate, holds[15].

Remark 2:

When agents are not impatient, *long-run improvement opportunities* are not ruled out by no-short-sales constraints. Suppose $((x^i)_i, \pi)$ is an AD equilibrium for $(\omega^i)_i$, where $\pi = p + a\text{LIM}^{AD}$, $a > 0$ and LIM^{AD} is a generalized limit. Proposition 2 is the route for its sequential implementation. To get some intuition on the role of condition (11), let $(z^i)_i \geq 0$ be money balances which accommodate $(x^i)_i$ in (8), given some initial holdings $(y_0^i)_i$ and money prices q. Let us use the normalization $p_t q_t = 1$ and rescale the utilities so that $\pi \in \partial U^i(x^i)$ for all i.

Take any non-negative real sequence z. If consumer i replaces z_t^i by $z_t^i + hz_t$, with $h > 0$, from some date n onward, there will be no utility gain *along this direction* if (i) $\text{LIM}^{AD}(q_t(z_{t-1} - z_t)) \leq \lim z_t$, which clearly holds when (ii) $\limsup(q_t(z_{t-1} - z_t)) \leq \lim z_t$. The resulting change in consumption is $c(z(n))_t = 0$ if $t < n$, $c(z(n))_n = -q_n z_n$ and $c(z(n))_t = q_t(z_{t-1} - z_t) = x_t - \omega_t^i$ if $t > n$. Moving on the right along this direction, we hoard more at date n and at subsequent dates for which $\omega_t^i > x_t$, in order to consumer more at subsequent dates where $\omega_t^i < x_t$.

Conditions (i) or (ii) say that the asymptotic dishoarding (evaluated using the AD

[15] There is no room in the Euler equations to replace q_{t+1} by $q_{t+1}(1 + i)$ with a positive interest rate i. That is, the deflation rate should be equal to the consumers' optimal rate of time preference.

generalized limit or the limsup) should not exceed the cost of carrying on cash. This cost is the absolute value of $-p_n q_n z_n + \sum_{t>n} p_t q_t (z_{t-1} - z_t)$, which by (9) reduces to $-\lim p_t q_t z_t$ where $p_t q_t$ is constant and can be set equal to 1.

Remark 3:

Moreover, (10) is a particular form of another necessary condition, the transversality condition. When marginal utilities μ_t^i, at each t, are well defined[16], if we move in the $z^i(n)$ direction (that is, in the direction of z^i from date n onwards) either along the right (multiplying z_t^i by $1+h>1$, for $t \geq n$) or along the left (multiplying z_t^i by $1+h \in (0,1)$ for $t \geq n$), we have the following (irrespective of the presence of constraints of the form (11))[17]: there is no utility gain

- by moving on the right along $z^i(n)$ **only if** $\lim \mu_t^i q_t z_t^i \geq \nu^{i1}(x^i - \omega^i)$
- by moving on the left along $z^i(n)$ **only if** $\lim \mu_t^i q_t z_t^i \leq \nu^{i2}(x^i - \omega^i)$

for some $\mu^i + \nu^{i1}$, $\mu^i + \nu^{i2} \in \partial U^i(x^i)$.

These transversality conditions do not imply that efficient individual money balances must go to zero. The latter would hold **if** $\sum_i \nu^{i2}(x^i - \omega^i) = 0$, which is the case when all net trades converge (as $\nu^{i2}(x^i - \omega^i) = \nu^{AD}(x^i - \omega^i)$).

We would like to design a fiscal policy that guarantees conditions (i) or (ii) in Remark 2 and, therefore (by making (11) hold), eliminates long-run improvement opportunities.

4.3 Taxes that eliminate the marginal benefit - marginal cost gap

The tax $\tau_t(y)$ levied at date t upon a plan y of money holdings consists of a fixed summable component θ_t and another component that eliminates the above long-run improvement opportunities. Funds that are put aside at each date are $z_t = y_t + \sum_{s \leq t} \tau_s(y)$. The gap between the asymptotic dishoarding and the cost of carrying on cash is now bounded by $\text{LIM}^{AD} q_t(y_{t-1} - y_t) - \lim y_t - \sum_{t=1}^{\infty} \tau_s(y)$, which we want to be non-positive. This is achieved if we require the following

[16] This is not a technical assumption, it is instead a property that depends on the asymptotic behavior of the consumption sequence, as was illustrated is Example 1.

[17] This follows from Proposition 4 in Araujo, Novinski and Pascoa [2011], as $z^i(n)$ is both left and right admissible for no-short-sales constraints and also under (11).

(i') $\sum_{t=1}^{\infty} \tau_s(y) \geq \mathrm{LIM}^{AD} q_t(y_{t-1} - y_t) - \lim y_t$

or the stronger requirement that

(ii') $\sum_{t=1}^{\infty} \tau_s(y) \geq \limsup q_t(y_{t-1} - y_t) - \lim y_t$.

An example are the following taxes that spread the fiscal burden over all dates

$$\tau_t(y) = \theta_t + \tilde{p}_t[\limsup q_t(y_{t-1} - y_t) - \lim y_t]^+. \tag{12}$$

We assume that $\tilde{p}$ and the lump-sum component θ are such that $\sum_{t=1}^{\infty} \theta_t < \infty$, $\tilde{p} \in \ell^1_{++}$, $||\tilde{p}||_1 = 1$. Moreover, we require $\lim q_t \tau_t(y) = 0$. This is achieved if $\lim \tilde{p}_t q_t = 0$ ($\tilde{p}_t$ tends to zero faster than p_t) and $\lim q_t \theta_t = 0$. The presence of the lump-sum tax θ may be useful to withdraw additional initial holdings that allow for an implememtation with non-negative money balances.

Observe that both $\lim y_t < \infty$ and $[\limsup q_t(y_{t-1} - y_t) - \lim y_t]^+ < \infty$ for a plan y that was already accommodating a bounded consumption plan $x(y)$ in the sequential budget set when taxes were not levied (as $y_0^i - \lim y_t = p(x(y) - \omega^i) < \infty$, since $p_t q_t = 1$).

These non-lump-sum taxes are invariant to changes in y at a finite set of dates and, therefore, Euler conditions (9) hold. However, it may be hard to accept that the tax authorities would have such a perfect foresight and we can suppose instead that the accumulated taxes up to each date, $\sum_{s \leq t} \tau_s(y)$ to depend just on $(y_1, ..., y_t)$.

Recursive tax schedules are of the form $\tau_t(y) = \theta_t + \tilde{\tau}(y)$ where θ is lump-sum (satisfying again $\lim q_t \theta_t = 0$) and the non-lump-sum component is (again for $\tilde{p}$ such that $\lim \tilde{p}_t q_t = 0$) given by

$$\sum_{s \leq t} \tilde{\tau}_s(y) \equiv [\Phi_t - y_t - a_t]^+ \sum_{s \leq t} \tilde{p}_s \tag{13}$$

where

(a) $\Phi_t = \sup_{s \leq t} q_s(y_{s-1} - y_s)$ or

(b) $\Phi_t = [q_t(y_{t-1} - y_t)]^+$ or

(c) $\Phi_t = \frac{1}{t} \sum_{s \leq t} q_s(y_{s-1} - y_s)$.

The sequence a is chosen so that $\tilde{\tau}(y^i) = 0$ for some reference allocation $(y^i)_i$. Let $a_t \equiv \max_i[\Phi_t - y_t^i]^+ + 1/t$. Notice that the function $\tilde{\tau}_t$ depends only on money balances at finitely many dates but at y^i, it has null derivative with respect to these positions. A monetary equilibrium $(q, (x^i, y^i)_i)$ should fulfill the additional requirement that the equilibrium money balances allocation is the reference allocation. Then, Euler conditions (9) hold. Cases (a), (b) or (c) allow for *tax rebates*: $\tilde{\tau}_t(y)$ can be negative if $|\tilde{\tau}_t(y)| \leq \sum_{s \leq t-1} \tilde{\tau}_s(y)$, where the latter is always non-negative[18].

Schedule (b) tends to tax, at a non-early date t, an agent that dishoards *at that date* more than the whole cost of carrying on cash *up to that date* (which is the cost of carrying on cash on top of the initial holdings $(-\sum_{s \leq t} p_s q_s (y_{s-1} - y_s))$ plus the cost of the initial holdings $(p_1 q_1 y_0^i)$, so the sum is $p_t q_t y_t = y_t$). Recursive taxes are in fact taxes on the use of savings rather than on savings per se and, therefore, have *a flavor of a capital gains tax*: the benefit from disposing of money balances is being taxed if it exceeds the cost of carrying on cash up to then.

Plans that end up being taxed are those for which the asymptotic dishoarding is too appealing, relative to the accumulated sacrifice made so far. Sequential budgets fail to price such mismatch, but the AD budget does since the asymptotic dishoarding benefit is captured by the price pure charge. It was well known that sequential budget constraints may fail to constraint properly the open end optimization problem and Ponzi schemes may occur. What is new, for an asset that cannot be shorted, is the open end utility gain that wary agents have by reducing distant losses.

Once recursive taxes have been imposed, we see that the plans y accommodating budget feasible bounded $x(y)$ are: in case (a) (as for (12)) those for which $\lim y_t$ exists, in case (b) those for which $\lim y_t$ and $\lim[x_t(y) - \omega_t^i]^+$ exist and in case (c) those for which $\lim y_t$ and $\lim \frac{1}{t} \sum_{s \leq t}(x_s(y) - \omega_s^i)$ exist.

If we add lump-sum taxes $\theta_t \geq \tilde{p}_t \lim_t a_t$ (and give each consumer an additional

[18]The non-convexities in (a), (b) or (c) do not constitute a difficulty since the budget set is mapped (non-linearly) into the convex set of an auxiliary economy without taxes and constraints (i) or (ii) (see the Appendix).

money holding equal to $\sum_{t=1}^{\infty} \theta_t$), we see that a plan y inducing bounded $x(y)$ has a finite $\sum_{t=1}^{\infty} \tau_t(y)$ which is greater or equal to, in case (a) $[\sup_t q_t(y_{t-1} - y_t) - \lim y_t]^+$, in case (b) $[\lim[q_t(y_{t-1} - y_t)]^+ - \lim y_t]^+$ and in case (c) $[\lim \frac{1}{t} \sum_{s \leq t} q_s(y_{s-1} - y_s) - \lim y_t]^+$.

Cases (a) and (b) satisfy condition (ii') and case (c) satisfies condition (i') when LIM^{AD} is the Banach limit[19]. Taxes defined by (a) or (b) manage to implement any AD equilibrium allocation, whereas taxes defined by (c) introduce an equilibrium selection: out of the multiple AD allocations (associated with different generalized limits in the AD price), (c) picks the one supported by the Banach limit.

Our general assumption on taxes is formalized as follows:

We say that taxes $\tau(.)$ satisfy Assumption **A2(i)** or **A2(ii)** *if (A) conditions (i') or (ii') hold, respectively, (B)* $\lim_t q_t \tau_t(y) = 0$ *for any plan* y *inducing bounded* $x(y)$ *and (C) at any monetary equilibrium allocation* $(y^i)_i$ *the derivative of* τ_t *is null for any direction involving just changes in money balances at finitely many dates.*

We may want to strengthen our assumptions on the tax schedule.

Assumption **A3**: *taxes are such that if all agents are impatient, an efficient monetary equilibrium must have money supply going to zero.*

This assumption holds for taxes given by (12) or the recursive taxes (a), (b) or (c), but would not hold if $A > 0$ were added to the expressions inside square brackets in (12) or (13) (although assumption A2 would still hold). Under A3, any additional money holdings that might be needed to avoid short sales should be retrieved through appropriate lump-sum taxes[20]. We already knew that under A2 taxes are non-distortionary in terms of short-run actions, in the sense that the Euler conditions that would hold without taxes (given by (9)) are still necessary for optimality when taxes are introduced. When A3 holds, on top of A2, taxes are non-distortionary in terms of long-run actions, in the sense that the transversality conditions that would hold without taxes (given in

[19] A generalized limit B is a Banach limit if $B(c) = \lim_n \frac{1}{n} \sum_{t=1}^{n} c_t$ whenever this limit exists.

[20] Whereas by adding $A > 0$ we could dispense with the lump sump taxes but would introduce a floor on money holdings.

Remark 3, for efficient allocations), together with its implications for the asymptotics of money supply (idem), must still hold when taxes are introduced.

Proposition 3. ***(Non-distortionary taxes)*** *Let $((x^i, y^i)_i, q)$ be a monetary equilibrium, for given initial holdings $(y_0^i)_i$ and taxes $\tau(.)$ satisfying A2. The transversality conditions, $\delta^- U^i(x^i, c(y^i(n)) \geq 0$ and $\delta^+ U^i(x^i, c(y^i(n)) \leq 0$, are the same that would hold in the absence of taxes (and in the presence or not of constraints (i) or (ii)) if and only if taxes satisfy assumption A3.*

Example 3 will show that in the absence of taxes satisfying A2 inefficient equilibria exist, where consumers' sequential problems have finite optimum since the allocations are implemented (using Proposition 2) with supergradients for which there are no asymptotic gains from dishoarding. The taxes proposed in A2 should not be seen as being required for the existence of any sequential equilibria. They are instead the *efficient taxes on savings*, in the sense that they implement efficient allocations.

5 Monetary Implementation of Efficient Allocation

5.1 Main Results

Theorem 2. ***(Efficient monetary equilibrium)***
Suppose preferences satisfy A1 and taxes satisfy A2(ii). Let (x, π) be an AD equilibrium for $(\omega^i)_i$. If $x \gg 0$, there exist initial money holdings $(y_0^i)_i$ so that x can be implemented as the consumption allocation of a monetary equilibrium with taxes.
For taxes satisfying A2(i), implementation holds for the AD equilibrium selector implied by the tax formula.

The same results hold when taxes satisfy also A3. When agents are wary there may be multiple monetary implementations. Let us denote by $\mu^i + \nu^{iL}$ the pure charge of the supergradient of U^i at x^i which takes the highest value on the net trade of agent i. It is shown in the proof of Theorem 2 that, if μ^i is collinear with the countably additive component p of the AD price π and A3 holds, but (after rescaling utility so that $||\mu^i|| = ||p||$) we have $\nu^{AD}(x^i - \omega^i) < \nu^{iL}(x^i - \omega^i)$, then there is a monetary equilibrium with a positive limiting money supply. Let us explore this. To simplify, let us assume that preferences are differentiable along the canonical directions (and let us rescale the utility functions so that these canonical marginal derivatives μ_t^i are such that $||\mu^i|| = ||p||$).

Definition 3. We say that agent i is *particularly wary* at the AD allocation $(x^i)_i$ if agent's i marginal loss from reverting the savings policy (the left derivative of U^i at x^i along the direction $x^i - \omega^i$) is greater than the value that the AD support price assigns to this savings policy ($\pi(x^i - \omega^i)$).

This non-differentiability is intrinsic to wary preferences, due to the diversity of generalized limits that can be taken for the pure charges of the supergradients. When $\delta^- U^i(x^i, x^i - \omega^i) > \pi(x^i - \omega^i)$, the consumer values the impact of the reversal of the savings policy differently from the way the AD supporting price does. Figure 1 illustrates what that gap would look like in a 2-date economy, although the non-differentiability is somehow ad-hoc in finite horizon but has a sound reason to occur for wary infinite lived agents. AD equilibria requires the support prices to be in the intersection of the superdifferentials of all agents. The sequential monetary implementation can be achieved by having each agent checking first order effects according to some supergradient, and there is no reason why agents should be coordinating to have collinear supergradients.

Theorem 3. ***(Equilibrium with non-vanishing money supply)*** *Under the assumptions of Theorem 2, suppose that marginal utilities are well defined at each date and that some consumer i is* particularly wary *at x^i, then, even for taxes satisfying also assumption A3, there is a monetary implementation of the AD equilibrium for which the money supply does not go to zero.*

Under impatience and strictly positive consumption, what has been known as the *strong version of Friedman's rule* still holds for taxes satisfying A2 and A3 (just like it

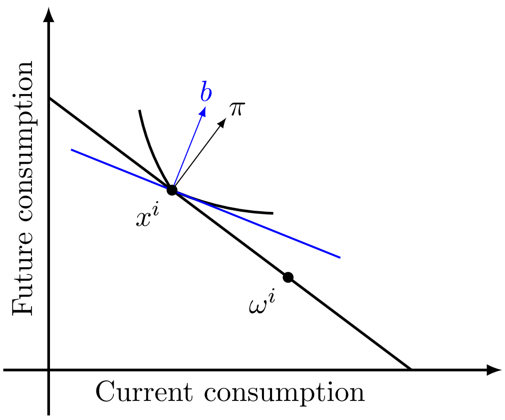

Figure 1: The marginal loss from undoing the savings $(b.(x^i - \omega^i))$ exceeds the AD price of the savings $(\pi.(x^i - \omega^i) = 0)$

did for lump-sum taxes[21]). On the other hand, Theorem 3 says that when agents are wary the money supply does not have to go to zero at an efficient monetary equilibrium, which contradicts Friedman's strong rule. Theorem 3 does not imply, however, that any monetary implementation, under its assumptions, must have non-vanishing money supply: given an equilibrium (x, y, q), there is $r \in \ell^1_+$, with $||r|| = 1$ and $\lim r_t q_t = 0$, such that if $\tilde{y}^i_t = y^i_t - (\sum_{s \le t} r_s) \lim y^i$, then $(x, \tilde{y}, q)$ is also an equilibrium, due to the automatic adjustment in the non-lump-sum taxes. The same consumption bundles are attained without carrying on cash to infinity but by putting aside from consumption the same amount, now in the form of taxes.

An interesting case where the assumptions of Theorem 3 holds is described next. The pure charges of all supergradients of U^i at x^i have the same norm if $\lim_n \delta^- U^i(x^i, 1^n) = \lim_n \delta^+ U^i(x^i, 1^n)$, where $1^n_t = 0$ for $t < n$ and $1^n_t = 1$ otherwise. In such case, the pure charge of the left derivative along the direction $(x^i - \omega^i)$ has the largest generalized limit of $(x^i - \omega^i)$, which is $\limsup(x^i - \omega^i)$. This will be illustrated as we resume Examples 1 and 2. Then, some consumer will be particularly wary at the AD allocation (and the assumptions of Theorem 3 hold) if $\limsup(x^i - \omega^i) \neq LIM^{AD}(x^i - \omega^i)$ for some agent,

[21]In fact, lump-sum taxes do not affect the necessary conditions for individual optimality and the result follows as in Proposition 5 in Pascoa, Petrassi and Torres-Martinez [2010].

which is the case when not all net trades are converging.

Corollary 1. *Under the assumptions of Theorem 2, if for all agents i, the marginal derivatives at infinity* $\lim_n \delta U^i(x^i, 1^n)$ *and at each date* $\delta U^i(x^i, e_t)$ *exist, then there is a monetary implementation of the AD equilibrium for which the money supply does not tend to zero if and only if agents' net trades are not all converging.*

This corollary is also proven in the Appendix. The necessity is actually quite general and was known from Remark 3; sufficiency depends on the assumptions that were made. Let us compare our results with what had been established in the literature. What is designated as Friedman's strong rule is a variation upon a claim made by Friedman (1969), although his claim actually just required a zero nominal interest rate and that, for that purpose, money supply should contract at a rate equal to the equilibrium real interest rate. Bewley's work (1980,1983) on impatient preferences not satisfying Inada had already shown that when consumption is always positive, a constant money supply is inefficient, whereas a money supply decreasing to zero at a constant rate can be made efficient when combined with lump-sum taxes. Levine (1986) gave interesting examples of efficient non-vanishing money supply for impatient agents with linear utilities, where corner solutions were crucial for building up large money balances[22]. Wary agents have an incentive to keep large money balances for a long-run hedging effect, and, in this case, Inada's conditions will not prevent the implementation of efficient monetary equilibrium with constant money supply, as our theorems assert and the examples will illustrate.

5.2 Monetary equilibrium for endogenous discounting

Resuming Example 1

We consider two-agent economies where preferences are as in Example 1 and endowments suffer shocks that alternate in sign along time but are not of the same magnitude. When one consumer gets a positive shock, the other suffers the symmetric negative shock. Money can be used to hedge against these shocks. Consumers would like to hold

[22] See also Levine's (1989) later results under differentiable preferences not satisfying Inada.

money forever (or at least, along some subsequence) in order to find a consumption path in between the upper and the lower endowment subsequences.

The utility function is as in (4) with $u^i(.) = \sqrt{.}$ and $\beta = 6$. Take, for both agents, $\zeta_t = (1/2)^{t-1}\sqrt{1+1/t}$. Endowments are $\omega_t^i = 16\frac{t+1}{t} + G_t^i$, where G_t^1 is given by $G_t^1 = 13$ if t is even and $G_t^1 = -11$ if t is odd, and $G_t^2 = -G_t^1$. Recall that the indeterminacy in the generalized limit considered in the AD price leads to a real indeterminacy in AD equilibrium allocations. Take the equilibrium allocation that results from using a Banach limit B. Consider the allocation $x_t^i = 16\frac{t+1}{t}$ and supergradients of the form $\pi^i c = \sum_{t=1}^{\infty}(\frac{1}{2})^{t+2}c_t + \frac{3}{4}B(c)$. We normalize prices so that the coefficient of the Banach limit is one: $\pi = \frac{4}{3}\pi^i$ (AD Lagrange multipliers are 3/4). Denote by p the summable component of π, the deflator $p_t = \frac{4}{3}2^{-t-2}$. The pair $((x^i)_i, \pi)$ is an AD equilibrium, as AD budget equations hold since $\pi(G^1) = 0$ follows from $B(G^1) = 1$ and $p(G^1) = -1$.

For $y_0^i = 9$, make $q_t = \frac{3}{4}2^{t+2}$, the inverse of the deflator p_t. Let z_t be the funds put aside by a consumer at date t, which will be decomposed as a sum of his money balances and the cumulated taxes on his money balances: $z_t = y_t + \sum_{s\leq t}\tau_s^i(y)$.

As the infimum $\underline{x}^i$ of consumption is never attained, the marginal utility at infinity $\lim_n \delta U^i(x^i, 1^n)$ exists (see part C of the Supplementary Material) and, therefore, the assumptions of Corollary 1 are satisfied. The implementation is achieved (as explained in detail in Appendix A) with $(z^i)_i$ if we (I) make $\lim_t z_t^i = \limsup(x^i - \omega^i)$, that is, the limiting cost of carrying on cash equals the marginal gain of hedging at infinity, given by the highest possible value that any pure charge of a supporting price can take on the net trade and (II) require all other plans $\hat{z}$ to satisfy $\lim_t \hat{z}_t^i \geq \limsup(x(\hat{z}^i) - \omega^i)$ (a limiting cost of funds not below the marginal gain at infinity).

Taxes are designed so that Condition (II) holds. A money holdings plan y pays accumulated taxes $\sum_{t=1}^{\infty}\tau_t^i(y) = \limsup(x(\hat{y} - \omega^i) - \lim_t y_y$, which ensures (II).

The AD budget equation holds if $z_0^i = \lim_t z_t^i - B(x^i - \omega^i)$. Then, (I) implies that $z_0^i = \limsup(x^i - \omega^i) - B(x^i - \omega^i)$, that is, $\limsup(-G^1) - B(-G^1) = z_0^1$ and $\limsup(G^1) - B(G^1) = z_0^2$, where $B(G^1) = 1$. Since $\limsup(-G^1) = 11$ and $\limsup(G^1) = 13$ we must have $z_0^1 = z_0^2 = 12$. So $z_t^i = 12 + \sum_{s=1}^{t} p_s G_s^i$ and short-sales are never done in equilibrium[23]. Then, $\lim z_t^1 = 11$ whereas $\lim z_t^2 = 13$.

Now, take $\theta = 0$ and $y^i = z^i$ so that equilibrium cash balances are not taxed and money supply remains constant, which illustrates Theorem 1.

Actually, as $\lim z_t^1$ is different from $\lim z_t^2$ we could not make $\sum_{t=1}^{\infty} \theta_t = \lim z_t^i$ for all i, so that money supply would tend to zero. Impersonal taxes are incompatible with a limiting zero money supply, except in the symmetric case where $\limsup(x^i - \omega^i)$ is the same for all agents, as implied by Theorem 2.

This example can be modified so that aggregate resources are not decreasing but, for any t, there exists some subsequent date where the aggregate endowment is lower than in t. Suppose that at even dates endowments follow increasing sequences and that at odd dates endowments are oscillating around a decreasing trend.

As a second remark, notice that the discount factors are a product of exponential and hyperbolic discounting. Preferences fail to be time-consistent, not as a consequence of β^i being positive, but as a result of the somehow hyperbolic discounting that was assumed for convenience reasons. The example could be redone with longer computations (along the lines of Example 1 in (Araujo, Novinski and Pascoa [2011])) under exponential discounting and consumption plans that differ from the trend endowment.

Resuming Example 2

Consider two agents with utility functions of the form in Example 2 with $u^i(y) = \sqrt{y}$, $n^i = 2$, $\eta_t^i = \left(\frac{1}{2}\right)^{t-1} \sqrt{1 + 1/t}$ and

$$\beta^i = 3\left(8\sum_{k=0}^{\infty}\frac{1}{2^{4^k}} - 8\sum_{k=0}^{\infty}\frac{1}{2^{4^k+2^k}} - 12\sum_{k=0}^{\infty}\frac{1}{4^{4^k}} + 12\sum_{k=0}^{\infty}\frac{1}{2^{2\cdot 4^k+2^k}}\right). \tag{14}$$

The consumptions are defined as $\omega^i := 16\frac{1+t}{t} + G_t^i$ where $G_t^1 = 6$ if $2^{2k+1} \leq t \leq 2^{2(k+1)} - 1$ for $k = 0, 1, \ldots$ and $G_t^1 = -4$ if $2^{2k} \leq t \leq 2^{2k+1} - 1$ for $k = 0, 1, \ldots$, and $G_t^2 = -G_t^1$ for all $t \in \mathbb{N}$. Take the equilibrium allocation $x_t^i = 16\frac{1+t}{t}$ that results from using in the AD price the pure charge, LIM, that in the net trade

[23] In this example we did not need to increase initial money holdings by some amount A to avoid short-sales.

of the agent 1, takes the lowest value, $\liminf_t \left(x_t^1 - \omega_t^1\right)$. The AD price is given by $\pi(c) = \sum_{t=0}^{\infty} \left(\frac{1}{2}\right)^{t+2} + \frac{\beta}{8}\mathrm{LIM}\left(\phi(c)\right)$ where $\phi : \ell_+^\infty \to \ell_+^\infty$ such that $\phi(c)_t = \frac{1}{t}\sum_{k=1}^{t} c_k$. Since ϕ is Frechet, using the result of the chain rule of the Clark subdiferential, we have that all pure charges in the subdifferential of the agent have the same norm, and also that the left derivative in the direction of the net trade coincides with the $\limsup$ of the net trade, since the value of ϕ in the net trade of the first agent is $\phi(4, -6, -6, 4, 4, 4, 4, -6, -6, -6, -6, -6, -6, -6, -6, 4, 4, 4, 4, 4, 4, \dots) = (4, -1, -8/3, -1, 0, 2/3, 8/7, 1/4, -4/9, -1, -11/6, -28/13, -17/7, -8/3, \dots)$. This implies that we can implement with $z^1 = 10$ and $z_0^2 = 0$ (by the same argument as in Example 1) and taxes defined as in Example 1.

5.3 On inefficient equilibria when there are no taxes

Example 3

Consider n economy economy with two agents $i = 1, 2$ whose preferences are given by (4), where $u^i(x) := \log(x)$, $\delta_t = 1/2^t$ and β will be specified below.

$$\omega_t^i = \begin{cases} 8 + 2^{-t} & \text{if } t \text{ and } i \text{ are even or odd simultaneously,} \\ 2 & \text{if } t \text{ is even and } i \text{ is odd, or conversely.} \end{cases}$$

There is an AD equilibrium (π, x) where $x_t^i = 5 + 2^{-t-1}$ for each i, and $\pi(c) := \sum_{t=1}^{\infty} \frac{1}{2^t} \frac{c_t}{5+2^{-t-1}} + \nu(c)$, for a pure charge ν defined by $\nu(\cdot) = \frac{\beta}{5}\mathrm{LIM}(\cdot)$ where LIM is a generalized limit such that $\mathrm{LIM}\left(x^1 - \omega^1\right) = \limsup\left(x^1 - \omega^1\right)$, provided that β is given as follows

$$\beta := \frac{5}{3}\left(\sum_{k\geq 1} \frac{2}{4^k}\left(5 + 4^{-k}\right)^{-1}\left(3 + 4^{-k}\right) - \sum_{k\geq 1} \frac{1}{4^k}\left(5 + \frac{4^{-k}}{2}\right)^{-1}\left(3 + \frac{4^{-k}}{2}\right)\right) > 0$$

Now, let us construct an inefficient equilibrium $((\tilde{x}^i, \tilde{x}^i)_i, q$ without deflation, $q_1 = \cdots = q_t = q$, where agent 1 holds no money in all even dates ($\tilde{z}_{2k}^1 = 0$ for $k \in \mathbb{N}$) and agent 2 holds no money in all odd dates ($\tilde{z}_{2k-1}^1 = 0$ for $k \in \mathbb{N}$), $\tilde{z}_0^1 = 0$ and $\tilde{z}_0^2 = 6\beta$.

Let $\left(\mu_t^i\right) \in \partial U^i(\tilde{x}^i)$ and satisfying the FOC of each agent at $\tilde{x}^i$. That is, for each t and i even or odd simultaneously,

$$\delta_t {u^i}' \left(\tilde{x}_t^i\right) = \mu_t^i = \mu_{t+1}^i = \delta_{t+1} u' \left(\tilde{x}_{t+1}^i\right) + \gamma_{t+1} \beta {u_i}' \left(\underline{\tilde{x}}^i\right) = \left(\delta_{t+1} + \gamma_{t+1}\beta\right) {u^i}' \left(\underline{\tilde{x}}^i\right) \quad (15)$$

Therefore, $\left(\tilde{x}_t^i\right)_{i,t}$, $\left(\gamma_t^i\right)_{i,t}$ and q are such that satisfy:

- $\sum_{t=1}^{\infty} \gamma_t^i = 1$ where $\gamma_t^i \geq 0$ with $\gamma_t^i = 0$ for t and i both odd or both even;
- $\inf_t \tilde{x}_t^1 \equiv \underline{\tilde{x}}^1 = \tilde{x}_{2k}^1$ for all $k \in \mathbb{N}$ and for agent 1, and $\inf_t \tilde{x}_t^2 \equiv \underline{\tilde{x}}^2 = \tilde{x}_{2k-1}^2$ for all $k \in \mathbb{N}$ and for agent 2, by choosing $q < 1/(2\beta)$;
- $\tilde{x}_t^i = \omega_t^i - 6\beta q$ and $\tilde{x}_{t+1}^i = \omega_{t+1}^i + 6\beta q$ for each t and i even or odd simultaneously, then, using Equation (15), we have that $\delta_t {u^i}' \left(\omega_t^i - 6\beta q\right) = \left(\delta_{t+1} + \gamma_{t+1}\right) {u^i}' \left(\omega_{t+1}^i + 6\beta q\right)$ for each t and i even or odd simultaneously.

The plan $\left(\tilde{x}^i, \tilde{z}^i\right)$ is optimal for agent i. To see this, notice that this infimum attained infinitely many times (in all t for which $\tilde{z}_t^i = 0$) and the plan satisfies the sufficient conditions mentioned in Lemma 2, for a supergradient whose pure charge is zero. In fact, the transversality condition (10) is satisfied: $\lim_t \mu_t^i q_t \tilde{z}_t^i = q \lim_t \mu_t^i \tilde{z}_t^i = 0$, as there is no deflation and $\tilde{z}^i$ takes a positive value, 6β, only at dates where the respective subsequence of μ^i is falling to zero. On the other hand, the condition (11) becomes $\lim_t \mu_t^i q_t z_t \geq 0$, which is trivially satisfied for any $z \geq 0$.

However, it can be noticed that the sequential equilibrium just constructed is inefficient since the marginal rates of substitution are not equal for the two agents in all pairs of dates (more precisely, no supergradient of one agent is collinear with a supergradient of the other agent). If we impose the taxes mentioned in Subsection 4.3, it is possible to implement the AD equilibrium mentioned above which is clearly Pareto efficient. If such taxes, of the recursive form (b) defined in subsection 4.3, were levied upon the inefficient equilibrium plans, agent 1 would pay a tax $x_t^1 - \omega_t^1 - y_t = 6\beta q$ on even dates (dates when the agent dishoarded more than the cost of carrying on cash up to that date) and zero taxes on odd dates. These taxes would displace the inefficient plans and guide the consumers toward efficient savings plans. However, if we just add lump-sum taxes, the outcome will not be Pareto efficient.

6 On the Implementation in Other Sequential Economies

6.1 Can Efficient Allocations be Implemented using a Lucas Tree?

Could fiat money be replaced by another long lived asset such as a Lucas tree? We consider an economy with a single asset that can not be shorted and has non-negative returns in the consumption good given by $(R_t)_{t\in\mathbb{N}} \in \ell_+^\infty \setminus \{0\}$. We call this asset a Lucas tree. In the absence of money, the government will now tax in the numeraire since is not plausible to tax directly in a private asset as the Lucas tree. The sequential budget set is the set $B(Q, y_0^i, \omega^i, \tau)$ of plans (x, y) satisfying $x \in \ell_+^\infty$ and, for each $t \in \mathbb{N}$,

$$x_t - \omega_t^i \leq Q_t\left(y_{t-1} - y_t\right) + R_t y_{t-1} - \tau_t(y) \tag{16}$$

where $Q = (Q_t)_{t\in\mathbb{N}}$ is the sequence of Lucas tree prices and τ is the tax schedule that depends on the plan y that the agent may choose. Notice that $W^i = \omega^i + Ry_0^i$.

Definition 4. A vector $\left(Q, \left(x^i, y^i\right)_{i\in\mathcal{I}}\right)$ is an equilibrium for the economy with initial Lucas tree holdings $\left(y_0^i\right)_{i\in\mathcal{I}}$ and fiscal policy τ if $\left(x^i, y^i\right) \in \text{argmax}\{U^i(x) : (x, y) \in B\left(Q, y_0^i, \omega^i, \tau\right)\}$ and, for every date t, we have $\sum_{i\in\mathcal{I}} x_t^i = \sum_{i\in\mathcal{I}} \omega_t^i + R_t \sum_{i\in\mathcal{I}} y_0^i$ and $\sum_{i\in\mathcal{I}} y_t^i = \sum_{i\in\mathcal{I}} y_0^i$.

Note that for non-negative taxes, taxes must be zero in equilibrium due to the market clearing condition in the numeraire. We assume that preferences are described by (4) and we add a condition that allows us to implement with no taxes in equilibrium.

Assumption A4: *The consumption plan* $(x^i)_i$ *of agent* i *is such that* $x^i \ggg 0$, $\underline{x}^i$ *is never attained and there is a subsequence* S *of dates such that* $x_t - \omega_t^i > 0$ *on* S, $\lim_S x = \underline{x}^i$ *and* $\limsup_S(x^i - \omega^i) = \limsup(x^i - \omega^i)$.

This assumption holds in Example 1 of section 5.2. It says that the infimum of consumption is not attained in finite time and that the dishoarding that occurs as that infimum is approached is actually the highest asymptotic dishoarding. Under A4, the marginal disutility from reverting the savings plan at an arbitrarily far away date is given by the limsup of the AD net trade (that is, $\nu^{iL}(x^i - W^i)/||\nu^{iL}|| = \limsup(x^i - W^i)$). Assuming that there are no lump-sum taxes we have the following implementation result.

Proposition 4. *Let be $\left(x^i\right)_i$ be an efficient allocation such that for each i, A4 holds at x^i. (A) If $\liminf_t (R_t) > 0$, there exist initial holdings $(y_0^i)_i$ of the Lucas tree and a fiscal policy τ that would implement $\left(x^i\right)_i$ as an equilibrium with taxes if the no-short-sale constraint on the Lucas tree were ignored. (B) If $(R_t) \geq 0$ and for some agent i, $x_t^i - W_t^i$ does not converge, the same result holds.*

In the absence of other financial instruments, short sales might not be avoided. If we tried to create additional Lucas trees (increase y_0^i) to overcome such negative positions (as we did in the case of money), the commodity endowment of each agent in the sequential economy would be reduced (since $\omega_t^i = W_t^i - R_t y_0^i$), implying that the quantity of Lucas tree required to avoid short sales could make ω_t^i become negative. Short sales could be avoided by adding *one-period* promises in zero net supply (an I.O.U. promise) and in this case taxes would be levied upon the portfolio formed by the Lucas tree and the *one-period* promise. To preserve efficiency, we should not allow for the I.O.U. promises to be secured by the Lucas tree since the collateral constraint could be binding in the presence of a low amount of Lucas tree (and we already know that it might not be possible to increase the initial holdings). In the next section we address sequential implementation using Lucas trees and I.O.U.s.

6.2 Implementation in Stochastic Economies

Can fiat money, when properly coupled with other spanning instruments, still implement AD allocations in stochastic economies? Or does the coexistence with other assets make money lose its role? Take an event tree such that at each date t and at each node s_t there exist two successors of s_t denoted by $s_{t,1}$ and $s_{t,2}$, and one predecessor s_t^-. Let σ be the root of the event tree and $\mathcal{S} := \{s_t : t \in \mathbb{N}\}$. Denote by P_{s_t} the probability of the successors of s_t. The utility function for each agent i is a generalization of (4),

$$U^i(x) := \sum_t \zeta_t^i \mathbb{E}_t \left[u^i(x_t)\right] + \beta_i \inf_t \mathbb{E}_t \left[u^i(x_t)\right] \tag{17}$$

where x_t is random consumption at nodes with date t and $\mathbb{E}_t$ is the expected value on $\mathcal{S}_t$, the set of all possible nodes s_t with date t, for the probability induced by P_{s_t}.

In stochastic economies, wary agents could not be modeled literally as in (4), carrying about the worst outcome on the whole event tree[24]. But one extension that makes sense, which we follow here, is to suppose that agents are worried about the mean losses at each date, as in (17). This means that there is no aversion to uncertainty among the states, but there is an aversion to ambiguity on the discount factors, as in equation (4).

Let us start by implementing with Lucas trees and I.O.U.s. Consider two Lucas trees in positive net supply and with positions given by $y(j)$, $j = 1, 2$. We allow for trades a on one-period zero-net-supply promises paying an interest rate i_{s_t} in the nodes that immediately follow the node s_t. At node s_{t+1} such that $s_{t+1}^- = s_t$, the budget constraint and the non-negativity of the Lucas trees constraints are given respectively by:

$$\begin{gathered} x_{s_{t+1}} - \omega_{s_{t+1}} + Q_{s_{t+1}} y_{s_{t+1}} + a_{s_{t+1}} + \tau_{s_{t+1}}(y, a) \leq \left(R_{s_{t+1}} + Q_{s_{t+1}}\right) y_{s_t} + (1 + i_{s_t}) a_{s_t}, \\ y_{s_{t+1}} \geq 0, \end{gathered} \tag{18}$$

where $Q = (Q_{s_t})_{s_t \in \mathcal{S}}$, $(R_{s_t})_{s_t \in \mathcal{S}}$ and $(i_{s_t})_{s_t \in \mathcal{S}}$ are the Lucas trees prices and returns and the interest rates, whereas τ are the taxes that depends on y and a. We denote by $B\left(Q, y_0^i, \omega^i, \tau\right)$ the set of plans (x, y, a) that satisfy (18).

To define an equilibrium for the economy with Lucas trees and taxes we use a straightforward extension of Definition 4, with the interest rates i_{s_t} and the promise trades a^i as additional variables, under the condition that the promises' trades clear, $\sum_i a_{s_t}^i = 0$, at each node s_t. Let us reformulate assumption (A4) to the stochastic case.

Assumption A5(1): *The consumption plan $(x^i)_i$ of agent i is such that $x^i \ggg 0$, $\inf_s \left(\mathbb{E}_s[u^i(x_s^i)]\right) < \mathbb{E}_t[u^i(x_t^i)]$ $\forall t \geq 0$, and there is a subsequence S of dates such that*

[24] In fact, that might imply that agents would be worried about some states with arbitrarily low probability.

$\mathbb{E}_t\big[(u^i)'(x_t)\big(x^i_t - W^i_t\big)\big] > 0$ *on* S *and* $\limsup_S \mathbb{E}_t\big[(u^i)'(x_t)\big(x^i_t - W^i_t\big)\big] = \limsup \mathbb{E}_t\big[(u^i)'(x_t)\big(x^i_t - W^i_t\big)\big]$.

Assumption A5(2): $(x^i)_i$ *is such that (a)* $\liminf_{\{t\,:\,\mathbb{E}_t[u_i'(x_t)(x^i_t - W^i_t)]>0\}} \mathbb{E}_t[u^i(x^i_t)] = \inf_s \left(\mathbb{E}_s[u^i(x^i_s)]\right)$ *and (b)* $\lim_t \mathbb{E}_t\left[u^{i'}\left(x^i_t\right)\right]$ *exists for each* i[25].

The following theorem establishes what can be done with taxes both, when the trees are traded alone or together with I.O.U.s that are not secured by the trees. The idea is that equilibrium plans will not be taxed but other plans may be penalized as in Proposition 4. These taxes will eliminate the usual Ponzi schemes (in the zero-net-supply promises) and any other long-run improvement opportunities.

Theorem 4. ***(Implementability in Unsecured Credit Economies without Money)*** *For preferences given by (17), let* $(x^i)_i$ *be an efficient allocation such that (i) for each* i, x^i *satisfies A5(1) and A5(2) and (ii) for some agent* i, $\mathbb{E}_t\left[u^i\left(x^i_t\right)\left(x^i_t - W^i_t\right)\right]$ *does not converge, then, there exist initial holdings of the Lucas trees* z^i_0 *and impersonal taxes that implement* $(x^i)_i$ *as an equilibrium for the sequential economy, but possibly with trades in the zero-net-supply one-period promises (so that short sales of the trees can be avoided).*

The dependence on unsecured credit is a fragility of the implementation, due to the full commitment assumed on debtors, which might not be incentive compatible.

Finally, we observe that in stochastic economics, efficient allocations can always be implemented with fiat money. Taxes will be paid in money and markets can be completed sequentially if other assets are added, say two Lucas trees (for the above economy with two branches at each node). Money and the Lucas trees have non-negative positions in equilibrium, thanks to the fact that the initial holdings of money can be adjusted. There is no need to allow for trades in zero-net supply promises. Denoting by $y_{s_t} \in \mathbb{R}^2_+$ the positions in the Lucas trees and by z_{s_t} the money balances in state s_t, we write the

[25] Part (b) of (A5(2)) can be replaced by the following: there exists $T > 0$ such that for every $t_1,\ t_2 \geq T$ we have that $\zeta^i_{t_1}/\zeta^i_{t_2} = \zeta^j_{t_1}/\zeta^j_{t_2}$ for each pair of agents $i,\ j$.

consumer budget constraint in this state as follows:

$$x_{s_t} - \omega_{s_t} + Q_{s_t} y_{s_t} + q_{s_t} z_{s_t} \le (R_{s_t} + Q_{s_t}) y_{s_t-} + q_{s_t} z_{s_t-} - q_{s_t} \tau_{s_t}(y,z),$$
$$y_{s_t}, z_{s_t} \ge 0,$$

where $Q_{s_t}, R_{s_t} \in \mathbb{R}^2_+$ are the prices and the returns of the Lucas assets trees, $\tau^i(y,z) \in \mathbb{R}_+$ is the taxation that depends on (y,z) and q_{s_t} is the price of money. We suppose that $R = \left(R^1, R^2\right)$ is such that for each s_t there exists some s_{t+r} successor of s_t such that $R^1_{s_{t+r}} \neq R^2_{s_{t+r}}$. An equilibrium for this economy is defined analogously to the original deterministic monetary case (with market clearing for the two Lucas trees as additional conditions) and is said to be a monetary equilibrium if the price of money is non zero.

Theorem 5. ***(Coexistence of Fiat Money and Lucas Trees)***

For preferences given by (17), let $(x^i)_i$ be an efficient allocation such that $x^i \ggg 0$, then, there exist initial holdings y^i_0, z^i_0 of the fiat money and the Lucas trees that manage to implement $(x^i)_i$ as an equilibrium with taxes and non-negative portfolios $(y^i, z^i)_i$.

Under pure discounting and apart from some special cases, fiat money would lose its efficient role (and its positive price) if other long-lived assets were being added to an economy without frictions that might justify the role of money. When impatience is replaced by wariness, our results (Theorem 5) show that, coexistence of money and those assets is compatible with efficient monetary equilibria.

In stochastic sequential economies, the study of efficient bubbles and the possibility of their crashing in some parts of the tree are quite interesting things to be analyzed. Since the characterization of bubbles can be done in terms of the pure charge of the consumers' supergradients, if these pure charges become zero in a subtree, then the bubble could crash all along that subtree.

7 Concluding Remarks

In this paper we implement sequentially the efficient allocations of economies where wary agents face persistent endowments shocks. These shocks are hedged by trading fiat money (alone in a deterministic setting or together with other long lived assets in the stochastic case). Inefficient money balances (or more precisely the disposal of such balances) are taxed on the grounds of the gap between hoarding benefits and the cost of carrying on cash. Discouraging such mismatch at the individual agent level ends up leading agents toward efficient savings plans. These taxes can be recursive and have the flavor of a capital gains tax: it is the gains from a sale of (all or part of the) money balances ($q_t(y_{t-1} - y_t)$) net of the cost of carrying cash up to then ($p_1 q_1 y_0^i - \sum_{s \leq t} p_s q_s (y_{s-1} - y_s) = y_t$) that are being taxed.

If we would dispense with fiat money and implement using Lucas trees as the only long-lived assets, we could face some difficulties. Under non-negative positions in the trees, to get sequential market completeness we might also need zero-net-supply promises. The amount of unsecured credit needed to complete the markets could be quite huge and, presumably, creditors might not be willing to lend it.

Actually, if the implementing asset were a long-lived asset with real returns, there are two extensions that might seem to be natural ways to overcome the dependency on unsecured credit but end up colliding with efficiency. One extension is to allow for the asset to collateralize the short sales of the zero-net-supply promise. The other extension is to allow for short sales of the long lived asset itself in the way that short sales of shares are actually done in financial markets, by borrowing the shares first rather than doing "'naked"' short sales. In both cases, it is common to observe frictions that lead to inefficiency. In the former, the collateral constraint could be binding. In the latter, we could have a binding constraint linking the short sale of the shares to the amount of shares that were borrowed[26]. For these reasons, in this paper, by a Lucas tree, we mean the classical notion of a long-lived real asset that can not be shorted and, furthermore, we do not allow it to serve as collateral. In this context, the complementary negative hedging is done through I.O.U. promises.

Fiat money has the merit of dispensing with the problematic role of that unsecured

[26] Actually, the two cases are often (as in repo markets) two legs of the same operation and the binding constraint becomes the same.

credit (in the form of I.O.U.) in completing the markets. In fact, the initial holdings of money can always be adjusted in order to implement sequentially an efficient allocation using non-negative money balances (alone in a deterministic setting or together with non-negative Lucas tree positions in a stochastic setting). Dispensing with unsecured credit allows us to avoid modeling reputation problems and complex bankruptcy procedures.

Wariness is a lack of impatience that makes consumers care about losses at far away dates. When fiat money is being used to implement sequentially an efficient allocation, the money supply does not have to go to zero. Wary agents can use persistently positive money balances to hedge against endowments shocks at far away dates, as Theorem 3 and its Corollary point out (and our examples illustrate). An optimal positive limit in the money supply is not a consequence of imposing money floors or peculiar portfolio constraints. We just assume the usual no-short-sales constraint on money together with a fiscal policy that taxes inefficient savings plans and correct what would be an insatiable demand for precautionary liquidity in a deflationary context (an instance of a problem already noticed by Friedman and Bewley).

APPENDIX

A Proof of Theorem 2

We construct an **auxiliary economy** where inter-temporal transfers of wealth are done by trading a no-dividends asset in constant positive net supply, not subject to taxes but subject to portfolio constraints. We denote positions in this asset by z (these will be related to money balances by $z_t = y_t + \sum_{s\leq t} \tau_s^i(y)$, which implies that $z_{t-1}^i - z_t^i = y_{t-1}^i - y_t^i - \tau_t(y)$). In the auxiliary economy, budget constraints are given by (8).

Consider the supergradient whose pure charge ν^{iL} takes the highest value on the direction of the net trade[27]. That is, ν^{iL} is such that $\delta^- U^i(x^i; x^i - \omega^i) = (\mu^i + \nu^{iL})(x^i - \omega^i)$. If μ^i is collinear with the countably additive part p of the AD price π, then we use

[27] This allows us to illustrate the multiplicity of equilibria and prepare for the proof of Theorem 3.

the supergradient $\mu^i + \nu^{iL}$ in the procedure proposed by Proposition 2. Otherwise, we can always use the supergradient collinear with $\pi \equiv p + \nu^{AD}$. This suggests the following portfolio constraint:

$$\lim \mu_t^i q_t z_t \geq \nu^{iL}(x(z) - \omega^i) \tag{19}$$

Let $B^A(q, y_0^i, \omega^i)$ be the set of plans (x, z) satisfying (8) and (19).

Definition 5. A vector $(q, (x^i, z^i)_{i \in \mathcal{I}}) \in I\!R_+^\infty \times (\ell_+^\infty \times I\!R_+^\infty)^I$ is an *equilibrium for the auxiliary economy* with initial holdings $(z_0^1, ..., z_0^I)$ if $(x^i, z^i) \in \text{argmax}\{U^i(x) : (x, z) \in B^A(q, y_0^i, \omega^i)\}$; $\sum_{i=1}^I x^i = \sum_{i=1}^I \omega^i$ and $\sum_{i=1}^I z_t^i = \sum_{i=1}^I z_0^i \ \forall t \in \mathbb{N}$.

Lemma 4. *If $((x^i)_i, \pi)$ be an AD equilibrium such that $x^i \ggg 0$, there exist z_0^i that implement $(x^i)_i$ as an equilibrium for the auxiliary economy, possibly with short-sales.*

Proof. Notice that the AD budget equation holds as an equality for a plan z^i when $\lim_t p_t q_t z_t^i - \nu^{AD}(x(z^i) - \omega^i) = z_0^i p_1 q_1$. We choose z_0^i so that $\frac{1}{\rho^i}\nu^{iL}(x(z^i) - \omega^i) - \nu^{AD}(x(z^i) - \omega^i) = z_0^i p_1 q_1$, where ρ^i is the AD Lagrange multiplier of agent i. We can actually take $p_t q_t = 1$. By Proposition 2, the portfolios z^i that satisfy (19) given z_0^i and x^i, will implememt the AD equilibrium allocation $(x^i)_i$ for $(\omega^i)_i$. □

Let us **map back into the original sequential economy**. Suppose sequential implementation without taxes was achieved with short sales under the constraint (19), with $p_t q_t = 1$. If z takes negative values at some dates, we can find money holdings $Z_0^i = z_0^i + \tilde{A}$ such that the equilibrium positions z_t^i can be replaced by *non-negative* money balances. We have the freedom of either shifting up the portfolio plans by $\tilde{A}$ or introducing lump-sum taxes θ that retrieve the additional initial holdings gradually (lightly at the finitely many dates where z_t^i was negative), or a combination of both[28].

Let us proceed by introducing taxes that replace the portfolio constraints.

Lemma 5. *The non-negative plans Z^i given by $Z_t^i = z_t^i + \tilde{A} - \sum_{s \leq t} \theta_s$ for $t \geq 1$, will implement the same efficient allocation if portfolio constraints are replaced by personal*

[28]Notice that if we choose to shift z^i up by $\tilde{A}$, then Z^i does not satisfy the second transversality condition of Remark 3 (the direction $Z^i(n)$ is not left admissible for the constraint $\lim_t \mu_t q_t z_t \geq \nu(x(z) - \omega) + \tilde{A}$, which should replace (11), together with $\lim_t \mu_t q_t z_t^i \geq \nu(x^i - \omega) + \tilde{A}$ replacing (10))

taxes τ^i *satisfying, for any portfolio plan* Z, $\sum_{t=1}^{\infty} \tau_t^i(Z) = \sum_{t=1}^{\infty} \theta_t + [\nu^i(q_t(Z_{t-1} - Z_t)) - \lim Z + A]^+$, *where* $\theta_t = \tilde{p}_t(\tilde{A} - A)$.

Proof. In fact, for $x_t^i(Z) \equiv q_t(Z_{t-1} - Z_t - \tau_t^i(Z))$ we have $\sum_{t=1}^{\infty} \tau_t^i(Z) \geq \nu^i(x^i(Z) - \omega^i) - \lim Z_t + \tilde{A}$. Let $z_t = Z_t - \tilde{A} + \sum_{s \leq t} \tau_s^i(Z)$, then $\lim_t z_t \geq \nu^i(x(z) - \omega^i)$. That is, the definition of taxes ensures that any plan Z has an image z satisfying constraint (11). As we already knew that (10) holds, it follows that Z^i is optimal, for the initial holding $Z_0^i = z_0^i + \tilde{A}$, and taxes are levied in equilibrium only if we choose to have lump-sum taxes removing (all or part of) $\tilde{A}$. □

Now, in order to define *impersonal taxes* we will increase taxes and also the initial holdings of money. Let $\gamma^i := \limsup(x^i - \omega^i) - \nu^i(x^i - \omega^i) \geq 0$. We make $y_0^i = Z_0^i + \gamma^i$. Denoting by $q(y_- - y)$ the sequence with general term $q_t(y_{t-1} - y_t)$, for any portfolio plan y we define the following tax

$$\gamma_t^i(y) = \begin{cases} \big(\limsup q(y_- - y) - \nu^i(q(y_- - y))\big)\tilde{p}_t & \text{if } \lim y \leq \nu^i\,(q(y_- - y)) + A, \\ [\limsup q(y_- - y)) - \lim y + A]^+ \tilde{p}_t & \text{otherwise.} \end{cases}$$

Then, the impersonal taxes satisfy

$$\sum_{t=1}^{\infty} \tau_t(y) = \sum_{t=1}^{\infty} (\tau_t^i(y) + \gamma_t^i(y)) = \sum_{t=1}^{\infty} \theta_t + [\limsup q(y_- - y)) - \lim y + A]^+.$$

Lemma 6. *Given the equilibrium plans* $(Z^i)_i$ *for the economy with just personal taxes* τ^i, *the plans* $y_t^i = Z_t^i + \gamma^i - \sum_{s \leq t} \tilde{p}_s(\limsup(x^i - \omega^i) - \nu^i(x^i - \omega^i))$ *constitute an equilibrium for the economy with impersonal taxes.*

Proof. Let us see first that the proposed plans y^i are in the budget set with impersonal taxes. For the proposed plans, $\lim y^i = \lim Z^i = \nu^i(q(y_- - y)) + \tilde{A} - \sum_{t=1}^{\infty} \theta_t$ and therefore $\tau^i(y^i) = \sum_{t=1}^{\infty} \theta_t$. Moreover, as $\lim_t q_t \tau_t(y) = 0$ we have $\limsup q(y_-^i - y^i) = \limsup(x^i - \omega^i)$ and $\nu^i q(y_-^i - y^i) = \nu^i(x^i - \omega^i)$. So, $\gamma_t^i(y^i) = \tilde{p}(\limsup(x^i - \omega^i) - \nu^i(x^i - \omega^i))$. Then, $y_{t-1}^i - y_t^i - \tau_t(y^i) = Z_{t-1}^i - Z_t^i - \theta_t$, which implies that y^i still accommodates x^i in the sequential budget equations.

Now, $(Z^i)_i = (z^i + A - \sum_{s \leq t} \theta_s)_i$ is also an equilibrium for the auxiliary economy, with constraints (11). To show that y^i is optimal for the economy with impersonal taxes, it suffices to show that any plan y for consumer i in the economy with impersonal taxes induces a plan z for i in the auxiliary economy with constraints (11). We look for z such that $y_{t-1} - y_t - \tau_t(y) = z_{t-1} - z_t$, then we have

$$z_t = y_t - \left(\gamma^i - \sum_{s\leq t} \gamma^i_s(y)\right) + \sum_{s\leq t} \tau^i_s(y) \tag{20}$$

Then, $\lim z_t = \lim y_t + \sum_{t=1}^{\infty} \tau^i_t(y) \geq \nu^i(q(y_- - y)) + \tilde{A}$ (since $\sum_{t=1}^{\infty} \theta_t + A = \tilde{A}$). Now, $\nu^i(q(y_- - y)) = \nu^i(q(z_- - z))$ since $\lim q_t \tau_t(y) = 0$. Hence, constraint (19) holds. □

This concludes the Proof of Theorem 2.

The **proof of Proposition 3** follows from the proof of Theorem 2 since, under A2, taxes satisfy A3 if and only if $A = 0$, which is also the necessary and sufficient condition for $\lim y^i_t$ to be equal to the limit of the equilibrium portfolio z^i of the auxiliary economy (for which the assumptions in Remark 3 hold).

B Proof of Theorem 3 and Remark 4

Notice that for taxes to satisfy assumption A3 we make $A = 0$ and we get $\lim y^i = \nu^i(x^i - \omega^i)$. If there is at least one agent j such that $\nu^j(x^j - \omega^j)$ for some supergradient $\mu^j + \nu^j$, with μ^j collinear with p and $\nu^j(x^j - \omega^j)/\rho^j > \nu^{AD}(x^j - \omega^j)$, then the monetary equilibrium generated (according to Proposition 2) by using this supergradient for j and supergradients collinear with $p + \nu^{AD}$ for all other agents, will have money supply converging to $\sum_{i\neq j} \nu^{AD}(x^i - \omega^i) + \nu^j(x^j - \omega^j)/\rho^j > 0$.

Finally, for each i let $Y^i_t = y^i_t - \sum_{s\leq t} r_s \lim y^i$, where $r \in \ell^1_+$, $||r||_1 = 1$ and $r_t q_t \to 0$. Now, $[\nu^i q(Y^i_- - Y^i) - \lim Y^i + A]^+ = [\nu^i q(y^i_- - y^i) + \lim q_t r_t \lim y^i + A]^+ = \nu^i q(y^i_- - y^i) + A$ and therefore $\sum_{t=1}^{\infty} \tau_t(Y^i) = \sum_{t=1}^{\infty} \tau_t(y^i) + \lim y^i$. Actually, for taxes given by (12) it is immediate to see that $\tau_t(Y^i) = \tau_t(y^i) + \tilde{p}_t \lim y^i$. For other tax schedules (say given by (a), (b) or (c)) we pick $r_t = \frac{\tau_t(Y^i) - \tau_t(y^i)}{\lim y^i}$.

Then, $q_t(Y^i_{t-1} - Y^i_t - \tau_t(Y^i)) = q_t(y^i_{t-1} - y^i_t + r_t \lim y^i - r_t \lim y^i - \tau_t(y^i)) = q_t(y^i_{t-1-y^i_t} - \tau_t(y^i))$ and, therefore, Y^i is optimal for agent i.

C Proof of Corollary 1

We just need to rule out that $\mathrm{LIM}^{AD}(x^i - \omega^i) = \limsup(x^i - \omega^i)$, for any i. Adding across agents, $0 = \sum_i \limsup(x^i - \omega^i)$. Say it is agent 1 whose net trade $x^1 - \omega^1$ does not converge. Now, $\limsup(x^1 - \omega^1) = -\sum_{i \neq 1} \limsup(x^i - \omega^i) = \sum_{i \neq 1} \liminf(\omega^i - x^i) \leq \liminf(x^1 - \omega^1)$, a contradiction.

References

A.Araujo (1985). Lack of Pareto Optimal Allocations in Economics with Infinitely Many Commodities: the Need for Impatience. *Econometrica* **146** 455-461 (1985).

A.Araujo, R.Novinski and M.Pascoa (2011). General Equilibrium, Wariness and Efficient Bubbles. *Journal of Economic Theory* **53** 785-811 (2011).

T.Bewley (1972). Existence of Equilibrium in Economies with Infinitely Many Commodities. *Journal of Economic Theory* **4** 514-40.

T.Bewley (1980). The Optimum Quantity of Money. in *Models of Monetary Economics*, ed. by J. Kareken and N.Wallace. Federal Reserve Bank, Minneapolis.

T.Bewley (1983). A Difficulty with the Optimum Quantity of Money. *Econometrica* **51**, 1485-1504.

K.Bhaskara Rao and M.Bhaskara Rao (1983). *Theory of Charges.* Academic Press, New York.

D.Brown and L.Lewis (1981). Myopic Economic Agents. *Econometrica* **49** 359-368 (1981).

J. Dow and S. Werlang (1992). Uncertainty Aversion, Risk Aversion and the Optimal Choice of Portfolio. *Econometrica* **60**, 197-204.

N.Dunford and J.Schwartz (1958). *Linear Operators, Part I.* Interscience, New York.

M.Friedman (1953), *Essays in positive economics.* University of Chicago Press.

M.Friedman (1969), The Optimum Quantity of Money, in *The Optimum Quantity of Money and Other Essays.* Aldine, Chicago.

I.Gilboa (1989). Expectation and Variation in Multi-period Decisions. *Econometrica* **57** 1153-69.

I.Gilboa and D.Schmeidler (1989). Maxmin Expected Utility with a Non-Unique Prior. *Journal of Mathematical Economics* **110** 605-39.

C.Gilles and S.LeRoy (1992). Bubbles and Charges. *International Economic Review* **33** 323-39.

P.Gottardi and F.Kubler (2015). Dynamic competitive economies with complete markets and collateral constraints. *Review of Economic Studies*, forthcoming.

L. Hansen and T. Sargent (2001). Robust Control and Model Uncertainty. *The American Economic Review* **91**, 60-66.

L. Hansen and T. Sargent (2008). *Robustness.* Princeton University Press, New Jersey.

K. Huang and J. Werner (2000). Asset Price Bubbles in Arrow-Debreu and Sequential Equilibrium. *Economic Theory* **15**, 253-278.

T. Kehoe, D. Levine and M. Woodford(1992). The Optimum Quantity of Money Revisited, in P. Dasguptha, D. Gale, O. Hart and E. Maskin (eds.), *The Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn*, MIT Press, 501-26.

N. Kocherlakota (2000). Injecting rational bubbles. *Journal of Economic Theory* **142**, 218-232.

D.Levine (1986). Borrowing Constraints and Expansionary Policy. UCLA unpublished.

D.Levine (1988). Asset Trading Mechanisms and Expansionary Policy. Federal Reserve Bank, Minneapolis, working paper 385.

D.Levine (1989). Efficiency and the Value of Money. *Review of Economic Studies* **56**, 77-88.

M. Marinacci (1998). *An axiomatic approach to complete patience and time invariance. Journal of Economic Theory*, **83**, 105-144.

A.Mas-Colell and W.Zame (2010). Equilibrium theory in infinite dimensional spaces. *Handbook of mathematical economics* **4**, 1835-1898.

M.Pascoa, M.Petrassi and J.Torres-Martinez (2010). Fiat Money and the Value of Binding Portfolio Constraints. *Economic Theory* **46**, 189-209.

E.Phelps (1972). *Inflation Policy and Unemployment Theory.* W.W.Norton, New York.

E.Phelps (1973). Inflation in the Theory of Public Finance. *Swedish Journal of Economics* **75**, 67-82.

L.Raut (1986). Myopic topologies on general commodity spaces. *Journal of Economic Theory* **39**(2), 358-367.

P. Samuelson (1958). An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money. *The Journal of Political Economy* **66** 467-82.

D. Schmeidler (1989). Subjective Probability and Expected Utility without Additivity, *Econometrica* **57**, 571-587.

M Woodford (1990). The Optimum Quantity of Money, in B. Friedman and F. Hahn (eds.), *Handbook of Monetary Economics*, vol. II, ch. 20. North Holland, Amsterdam.

SUPPLEMENT TO: "ON THE EFFICIENCY OF MONETARY EQUILIBRIUM WHEN AGENTS ARE WARY"

Aloisio Araujo

IMPA and EPGE/FGV, Rio de Janeiro, Brazil

aloisio@impa.br

Juan Pablo Gama-Torres

IMPA, Rio de Janeiro, Brazil

jpgamat@impa.br

Rodrigo Novinski

Faculdades Ibmec, Rio de Janeiro, Brazil

rodrigo.novinski@ibmecrj.br

Mario R. Pascoa

University of Surrey, U.K.

Nova School of Business and Economics, Portugal

m.pascoa@surrey.ac.uk

March 7, 2016

A Additional Example

Another example is obtained by taking $n^i \in \mathbb{N}$ and C^i to be the weak* closed convex hull of $\{(\delta^i_m)_{m\in\mathbb{N}} : \delta^i_m(t) = \zeta^i_t + \beta^i/n^i$ for $m+1 \le t \le m+n^i, \delta^i_m(t) = \zeta^i_t$ elsewhere$\}$ to get $U^i(x) := \sum_t \zeta^i_t u^i(x_t) + \beta^i \inf_t \left(\frac{1}{n^i}\sum_{k=1}^{n^i} u^i(x_{t+k})\right)$. The agent is worried about the worst possible cycle of consumption in n^i dates (say one year). The concern is not anymore with the worst consumption or with worst mean losses (worst consumption could be in a cycle in which consumption is large in average). A single low consumption may be negligible at distant dates. Supporting prices are $\pi(c) = \sum_{t=1}^{\infty} \zeta^i_t (u^i)'(x_t)c_t + \beta^i \mathrm{LIM}(\tilde{c})$ where $\tilde{c}_t := \frac{1}{n^i}\sum_{k=0}^{n^i-1} c_{t+k}$.

And when $u^i(y) = \sqrt{y}$, $n^i = 2$, $\eta^i_t = \left(\frac{1}{2}\right)^{t-1}\sqrt{1+1/t}$, $\beta^i = 4$, endowments are defined as $\omega^i_t := 16\frac{1+t}{t} + G^i_t$ where $G^1_t = 6$ if $t = 4k+2$ or $t = 4k+3$ for $k = 0,1,\ldots$ and $G^1_t = -4$ if $t = 4k+1$ or $t = 4k+2$ for $k = 0,1,\ldots$, and $G^2_t = -G^1_t$ for all $t \in \mathbb{N}$, the equilibrium allocation that results from using a Banach limit B in the AD price, that is $x^i_t = 16\frac{1+t}{t}$ and the AD price is $\pi(c) = \sum_{t=1}^{\infty}\left(\frac{1}{2}\right)^{t+2} c_t + \frac{1}{2}B(\bar{c})$ where $\bar{c}_t := c_t + c_{t+1}$. Note that $\bar{\cdot} : \ell^\infty \to \ell^\infty$ is Frechet differentiable and, due to the chain rule, pure charges of each agent are of the form LIM($\bar{\cdot}$). Therefore, the left derivative in the net trade and its lim sup coincide (since $\frac{1}{2}\left(\overline{x^1 - W^1}\right) = (4,-1,-6,-1,4,-1,-6,-1,4,\ldots)$), which implies that $z^i_0 = 5$ (by the implementation argument as in Example 1). For taxes defined as in Example 1, the AD equilibrium is implemented with positive net supply.

B On Example 1 and the Marginal Utility in the Direction of Net Trades

We show here that for a utility function U of the form given by (4), if z^* is an optimal portfolio plan in $B^A(q, y^i_0, \omega^i)$ (defined in Subsection 8.2.1) such that, at $x^* := x(z^*) \ggg 0$, we have $\inf x^*$ not attained and $\lim_s x^*_s = \inf_s x^*_s$, then

$$\delta^- U(x^*)(x^*; x^* - \omega^i) = \mu(x^* - \omega^i) + \alpha \limsup(x^* - \omega^i)$$

for $\alpha > 0$ equal to the norm of the pure charge component of a supergradient of U at x^*, where μ is given by $\mu_t = \zeta_t u'(x_t^*)$.

We will estimate $\lim_{r\to 0} \frac{1}{r}[U \circ x(z^* + rz^*) - U \circ x(z^*)]$. Consider the direction $\Delta \in \ell^\infty$ given by $\Delta_t = q_t z_{t-1}^* - q_t z_t^*$. Notice that $\lim_{r\to 0} \frac{1}{r} \sum_{t\geq 1} \zeta_t [u(x_t^* + r\Delta_t) - u(x_t^*)] = \sum_{t\geq 1} \zeta_t \lim_{r\to 0} \frac{1}{r}[u(x_t^* + r\Delta_t) - u(x_t^*)] = \sum_{t\geq 1} \zeta_t u'(x_t^*)\Delta_t$. So, what we still need to do is to estimate $\lim_{r\uparrow 0} \frac{1}{r}\beta[\inf_t u(x_t^* + r\Delta_t) - \inf_s u(x_s^*)]$, which is $\delta^- \inf_t u(x^*, \Delta)$, the left-derivative of the function $\inf_t u(.)$ along the direction Δ evaluated at x^*.

Observe that there exists $\chi > 0$ such that $\forall r \in (-\chi, 0)$ the following holds: $(1+r)z^* > 0$ is a non-negative plan, $x(z^* + rz^*)$ satisfies (16) and $x(z^* + rz^*) = x^* + r(x^* - \omega) \ggg 0$.

Claim: $\lim_{r\uparrow 0} \frac{1}{r}[\inf_t u(x_t^* + r\Delta_t) - \inf_t u(x_t^*)] = u'(\underline{x}^*) \limsup_t \Delta_t$

Proof. Let us suppose that x_t^* converges to $\underline{x}^* = \inf x^*$, then $\lim_{r\uparrow 0} \frac{1}{r}[\inf_t u(x_t^* + r\Delta_t) - u(\underline{x}^*)]$ since $\inf(.) : \ell^\infty \to I\!R$ is a concave function.

Fixed $r \in (-\chi, 0)$ and given $\epsilon > 0$, it is valid for all τ large enough that $(1/r)[\inf_t u(x_t^* + r\Delta_t) - u(\underline{x}^*)] + \epsilon = (-1/r)[u(\underline{x}^*) - \epsilon r - \inf_t u(x_t^* + r\Delta_t)] \geq (-1/r)[u(x_\tau^*) - u(x_\tau^* + r\Delta_\tau)] \geq u'(x_\tau^*)\Delta_\tau$. Making $\tau \to \infty$ we get $(1/r)[\inf_t u(x_t^* + r\Delta_t) - u(\underline{x}^*)] + \epsilon \geq \limsup_t u'(x_t^*)\Delta_t = u'(\underline{x}^*) \limsup_t \Delta_t$, for an arbitrary $\epsilon > 0$.

To prove the reverse inequality, notice that, under the hypothesis, $\delta U(x^*; \mathbb{1}(n)) = \sum_{t>n} \zeta^t u'(x_t^*) + \beta u'(\underline{x}^*)$ and, therefore, any supergradient has a pure charge component with norm $\beta u'(\underline{x}^*)$ by Lemma 1. Hence, for any supergradient T of U at x^* we have $T(\Delta) = \sum_{t\geq 1} \zeta_t u'(x_t^*)\Delta_t + \beta u'(\underline{x}^*)\mathrm{LIM}(\Delta)$, for some generalized limit LIM. So, $\delta^- \inf_t u(x^*, \Delta) \leq u'(\underline{x}^*) \limsup_t \Delta_t$.

Now, if there is a subsequence S such that $\Delta_t \geq 0$, $\inf_S x_t^i = \underline{x}^*$ and $\limsup_S \Delta_t = \limsup \Delta_t$, the left derivative on the direction $\{\Delta_t\}_t$ is $u'(\underline{x}^*) \limsup_t \Delta_t$, which concludes the proof. □

C Proof of Theorem 1

Proof. $(x_t)_{t\in\mathbb{N}} \ggg 0$ implies that the marginal utility in x_t of the function u is uniformly bounded from above and below, that is, $0 < m \leq u'(x_t) \leq M < \infty$ implying that the condition of uniform convegence can be written as $\lim_t \sup_{\delta \in C}\left\{\sum_{s\geq t}\delta_s\right\} = 0$. Therefore,

$$\begin{aligned}
0 &\geq \lim_{n\to\infty}\lim_{h\to 0^-}\frac{-1}{h}\left(U(x+h\mathbb{1}_{E_n})-U(x)\right)\\
&= \lim_{n\to\infty}\lim_{h\to 0^-}\frac{-1}{h}\left(\inf_{m\in C}\left\{\sum_{t=1}^{n-1}\delta_t u(x_t)+\sum_{t=n}^{\infty}\delta_t u(x_t+h)\right\}-\inf_{m\in C}\left\{\sum_{t=1}^{\infty}\delta_t u(x_t)\right\}\right)\\
&\geq \lim_{n\to\infty}\lim_{h\to 0^-}\frac{-1}{h}\left(\inf_{m\in C}\left\{\sum_{t=n}^{\infty}\delta_t\left(u(x_t+h)-u(x_t)\right)\right\}\right)\\
&\geq \lim_{n\to\infty}\lim_{h\to 0^-}\frac{-1}{h}\left(\inf_{m\in C}\left\{\sum_{t=n}^{\infty}\delta_t\left(Mh+o(h)\right)\right\}\right) = -M\lim_{n\to\infty}\sup_{m\in C}\left\{\sum_{t=n}^{\infty}\delta_t\right\}=0,
\end{aligned}$$

which concludes proves that the left derivative is 0, for $h \to 0^+$ is analogous. □

D Proofs for Section 6.

Proof of Proposition 3. We start by implementing the efficient allocation in an auxiliary economy without taxes but with portfolio constraints. Consider $B^A\left(Q, y_0^i, \omega^i\right)$ the set of plans (x, z) satisfying

$$x_t - \omega_t^i \leq Q_t\left(z_{t-1}-z_t\right) + R_t z_{t-1}\ \forall t \in \mathbb{N} \tag{1}$$

Lemma 7 can be adapted to an asset paying dividends if we replace (15) by (1) and (16) by $\mu_t Q_t = \mu_{t+1}(Q_{t+1}+R_{t+1})$. We now denote $x_t(z) \equiv Q_t(z_{t-1}-z_t)+R_t z_{t-1}$. Then, by (A4), $\rho^i \lim p_t Q_t z_t^i = \nu^{iL}(x^i - W^i)$ and we use the portfolio constraint $\lim p_t Q_t z_t \geq \limsup\left(x(z)-\omega^i\right)$, where $x_t(z) = \omega_t^i + Q_t\left(z_{t-1}-z_t\right)+R_t z_{t-1}$. Now $x^i(z)$ satisfies the AD budget equation if and only if $\nu\left(x^i(z)-\omega^i\right) - \lim_t p_t Q_t z_t^i = z_0^i\left(\nu(R) - \lim p_t Q_t\right)$, equivalently,

$$(\rho^i)^{-1}\tilde{\nu}^i\left(x^i-\omega^i\right) - \nu\left(x^i-\omega^i\right) = z_0^i\left(p_1 Q_1 - \sum_{t=1}^{\infty}p_t R_t - \nu(R)\right) \tag{2}$$

If $x^i - \omega^i$ converges for every agent, we choose Q_1 such that $p_1 Q_1 = \nu(R) + \sum_{t=1}^{\infty} p_t R_t$ and $z_0^i \geq 0$ such that $M z_0^i < \underline{W}^i$ where $\underline{W}^i = \inf_t\left\{W_t^i\right\}$. If $x^i - \omega^i$ does not converge for some i, we choose Q_1 big enough so that $p_1Q_1 > \nu(R) + \sum_{t=1}^{\infty} p_tR_t$ and z_0^i satisfying the condition above and $Mz_0^i < \underline{W}^i$. Let us define a constant sequence of taxes by

$$\tau_t(y) := \left(\frac{1}{1+\left\|\tilde{Q}\right\|_1\gamma}\right)\left(\left(\underline{\beta}^{-1}\limsup(Q_t(y_{t-1}-y_t)+R_ty_{t-1})-\lim_t y_t\right)\vee 0\right)$$

where $\gamma := \prod_{t=1}^{\infty}\left(\frac{R_t}{Q_t}+1\right)$, $\left(\tilde{Q}_t\right)_t = (1/Q_t)_t \in \ell^1$ and $\underline{\beta} := p_1Q_1 - \sum_{t=1}^{\infty} p_t$.

The relationship between y and z is given by:

$$z_t - y_t = \sum_{r=1}^{t} \frac{\prod_{s=0}^{t-r-1}(Q_{t-s}+R_{t-s})}{\prod_{j=0}^{t-s}(Q_{t-s})}\tau(y) = \sum_{r=1}^{t}\frac{\tau(y)}{Q_r}\left(\prod_{s=0}^{t-r-1}\left(1+\frac{R_{t-s}}{Q_{t-s}}\right)\right)$$

Making the proper substitutions we have $\sum_{r=1}^{\infty}\frac{\underline{\beta}\tau(y)}{Q_r}\gamma \geq \limsup\big(x(y)-\omega^i\big) - \lim_t p_tQ_ty_t$ with equality for $y = y^i$, where y^i is the asset portfolio, which implements the AD allocation with taxes τ. □

Let us prove next Theorem 5 and leave the proof of Theorem 4 to the end.

Proof of Theorem 5. Let us assume that $p_1q_1 = 1$ and $||\nu^{iL}|| = 1$, where ν^{iL} is the pure charge of the supergradient of U^i that takes the highest value on $x^i - W^i$. Now, let us present sufficient conditions for individual optimality:

Lemma 1. *Let $(\tilde{y}^*, \tilde{z}^*)$ be a feasible portfolio and let $x^* = x(\tilde{y}^*, \tilde{z}^*)$. Suppose there is $T \in \partial U(x^*)$ with $T = \mu + \nu$, $\mu \in \ell^1_+$ and $\nu \in pch_+$ such that for every node s_t and both promises $j = 1, 2$, $\mu_{s_t}Q_{s_t}(j) = \sum_{s_{t+1}^- = s_t}\mu_{s_{t+1}}(R_{s_t}(j) + Q_{s_t}(j))$ and $\mu_{s_t}q_{s_t} = \sum_{s_{t+1}^- = s_t}\mu_{s_{t+1}}q_{s_t}$ and $\lim_t\left(\sum_{s_t\in\mathcal{S}_t}\left[\mu_{s_t}Q_{s_t}\tilde{y}^*_{s_t} + \mu_{s_t}q_{s_t}\tilde{z}^*_{s_t}\right]\right) = \nu(x^*-\omega)$. Suppose also that every feasible portfolio $(\tilde{y}, \tilde{z})$ satisfies the portfolio constraint*

$$\lim_t\left(\sum_{s_t\in\mathcal{S}_t}\left[\mu_{s_t}Q_{s_t}\tilde{y}_{s_t} + \mu_{s_t}q_{s_t}\tilde{z}_{s_t}\right]\right) \geq \nu\left(x(\tilde{y},\tilde{z})-\omega\right).$$

Then $(\tilde{y}^*, \tilde{z}^*)$ is an optimal solution for the consumption problem with sequential budget constraints.

As usual, implementation follows by imposing the portfolio constraint and choosing $(z_0^i)_i$ such that the AD budget equation holds, that is, $(\rho^i)^{-1}\tilde{\nu}^i\left(x^i-\omega^i\right) - \nu\left(x^i-\omega^i\right) = \tilde{y}_0^i\big(\lim_t\sum_{s_t}\mu_{s_t}Q_{s_t} - \nu(R)\big) + \tilde{z}_0^i\lim_t\sum_{s_t}{\mu^i}_{s_t}q_{s_t}$.

As in Theorem 2, the no short sale constraints of money and the Lucas trees are satisfied by adding an extra money holding A at $t=0$ to all agents[1].

Similarly to the proof of Theorem 2, we first implement the efficient allocation with personal taxes, $\tau^i_{s_t}(y,z)$, given by the pure charge $\tilde{\nu}(x^i(y,z)-\omega)$, and then we define additional personal taxes, $\gamma^i_{s_t}((\tilde{y},\tilde{z}),(y,z))$, such that the sum $\tau_{s_t}(\tilde{y},\tilde{z}) = \tau^i_{s_t}(y,z) + \gamma^i_{s_t}((\tilde{y},\tilde{z}),(y,z))$ have the following form

$$\begin{aligned}\tau_{s_t}(\tilde{y},\tilde{z}) =& \tilde{p}_t \max\Bigg\{0, \limsup_{s_r}\left(Q_{s_r}\left(\tilde{y}_{s_{r-1}}-\tilde{y}_{r_t}\right) + R_{s_r}\tilde{y}_{s_{r-1}} + q_{s_r}\left(\tilde{z}_{s_{r-1}}-\tilde{z}_{s_t}\right)\right)\Bigg] \\ & - \lim_t\left(\sum_{s_t\in\mathcal{S}_t}\left[p_{s_t}Q_{s_t}\tilde{y}_{s_t} + p_{s_t}q_{s_t}\tilde{z}_{s_t}\right]\right) + A\Bigg\}\end{aligned}$$

where $\{\tilde{p}_t\}_t \in \ell^1_{++}$, $||\tilde{p}||_1 = 1$ and $\tilde{p}_t \max_{s_t} Q_{s_t} \to 0$, $\tilde{p}_t \max_{s_t} q_{s_t} \to 0$. Therefore there is an initial money holding $\tilde{z}^i_0 + A + \gamma^i$ such that the efficient allocation can be implemented with impersonal taxes that in equilibrium are not necessarily zero where γ^i is defined as in the proof of Theorem 2.

□

Proof of Theorem 4. We start by characterizing the supergradients of the utility (13).

Lemma 2. *Consider a consumption plan $x^* \ggg 0$ such that $\inf_{s\geq 1}\mathbb{E}_s\left[u\left(x^*_s\right)\right] > \mathbb{E}_t\left[u\left(x^*_t\right)\right]$ for all $t\geq 0$ and $\mathbb{E}_t\left[u\left(x^*_s\right)\right] \to \inf_{s\geq 1}\mathbb{E}_s\left[u\left(x^*_s\right)\right]$.*

$\pi \in \partial U\left(x^*\right)$ if and only if it is given by

$$\pi(x) = \sum_{t\geq 0}\zeta^t\mathbb{E}_t\left[u'(x^*_t)\cdot x_t\right] + \beta\,\nu\left(\left(\mathbb{E}_t\left[u'(x^*_t)\cdot x_t\right]\right)_{t\geq 0}\right)$$

where $\nu \in pch(\ell^\infty)$ such that $\|\nu\| = 1$[2].

Proof. It is enough to show that, given $x \in \ell^\infty(\mathcal{S})$,

$$\inf_t \mathbb{E}_t[u(x_t)] - \inf_t \mathbb{E}_t[u(x^*_t)] \;\leq\; \nu\left((\mathbb{E}_t[u'(x^*_t)\cdot(x_t - x^*_t)])_{t\geq 0}\right).$$

[1] Due to the indeterminacy produced by the three assets in the economy, large money holding at $t=0$ can prevent short sales also for the Lucas tree.

[2] The right part of the π is a bounded functional on ℓ^∞ due to $x \ggg 0$.

Given $\varepsilon > 0$, we have, for $t_1 > 0$ large enough,

$$\inf_t \mathbb{E}_t[u(x_t)] - \inf_t \mathbb{E}_t[u(x_t^*)] - \varepsilon \;<\; \mathbb{E}_{t_1}[u(x_{t_1})] - \mathbb{E}_{t_1}[u(x_{t_1}^*)] \;\leq\; \mathbb{E}_{t_1}[u'(x_{t_1}^*) \cdot (x_{t_1} - x_{t_1}^*)].$$

Making $t_1 \to \infty$, we get $\inf_t \mathbb{E}_t[u(x_t)] - \inf_t \mathbb{E}_t[u(x_t^*)] - \varepsilon \;\leq\; \liminf_t \mathbb{E}_t[u'(x_t^*) \cdot (x_t - x_t^*)]$. Now $\|\nu\| = 1$ implies $\nu(z) \geq \liminf z \; \forall z \in \ell^\infty$ and ε is arbitrary.

To prove the other part of the lemma, let us use some results of nonsmooth analysis (see Clarke [1990]) for $U(x) = V \circ \phi(x)$ where $\phi : \ell^\infty(\mathcal{S}) \to \ell^\infty$ and $V : \ell^\infty$ are given by $x \mapsto (\phi_t(x))_{t\in\mathbb{N}} := (\mathbb{E}_t[u(x_t)])_{t\in\mathbb{N}}$ and $y \mapsto V(y) := \sum_{t\geq 1} \delta_t y_t + \beta \inf_t y_t$.

Since U is concave and Lipschitz[3] close to x^* (since $x^* \ggg 0$), we have that $\partial_c U(x^*) = \partial U(x^*)$ (see page 36 proposition 2.2.7), where $\partial_c F(y)$ is the Clarke subdifferential, see page 10. Notice that V have also the same property.

And since ϕ is Lipschitz close to x^*, we have that ϕ is strictly differentiable (see page 30 proposition 2.2.4). And as a consequence of the Chain Rule (see page 45 proposition 2.3.10), we have that $\partial U(x) \subseteq \partial V(\phi(x^*)) \circ \phi'(x^*)$ which concludes the proof. □

To proceed with the proof of Theorem 4, let us look first at the case where the zero-net-supply promises are available. As a preliminary step, we obtain the following personal taxes:

$$\tau^i_{s_t}(y) = \frac{1}{\alpha^i \underline{\beta}^i + \gamma^i} \max\Bigg\{0, \limsup_{\{\{t_r\}_r \,:\, \lim_r \mathbb{E}_{t_r}[u^i(x_{s_{t_r}})] = \inf_t \mathbb{E}_t[u^i(x_{s_t})]\}} \mathbb{E}_t\Big[u_i'(x_t^i)\Big(Q_{s_t}\big(y_{s_{t-1}} - y_{s_t}\big) + R_{s_t} y_{s_{t-1}}\Big)\Big] - \lim_t \Big(\sum_{s_t} \mu^i_{s_t} Q_{s_t} y_{s_t}\Big)\Bigg\}$$

where $\alpha^i = \limsup_{\{\{t_r\}_r \,:\, \lim_r \mathbb{E}_{t_r}[u^i(x_{s_{t_r}})] = \inf_t \mathbb{E}_t[u^i(x_{s_t})]\}} \mathbb{E}_t\left[u^{i'}\left(x^i_{s_t}\right)\right]$, $\underline{\beta}^i = \mu^i_{1,1} Q_{1,1} - \sum_{s_t} \mu^i_{s_t} R_{1,s_t}$, and $\gamma^i = \sum_{s_j} \left[\frac{\prod_{i=j+1}^\infty \left(1 + \frac{R_{1,s_i}}{Q_{1,s_i}}\right)}{Q_{1,s_j}} \left(\lim_{t\to\infty} \sum_{\tilde{s} \,:\, \tilde{s}_t^{-(t-j)} = s_j} \mu^i_{1,s_t} Q_{1,s_t}\right)\right]$.

Under assumption H'2, these taxes can be rewritten in an impersonal way, by an argument similar to the one used in the proof of Theorem 5.

[3] In the sup-norm.

If I.O.U.s were added in order to complete markets, we can define optimality conditions, similarly to Lemma 1, which allow us to define a fiscal policy τ to avoid the *long-run improvement opportunities*, under a no short sales constraint on the Lucas trees. □

References

A.Araujo, R.Novinski and M.Pascoa (2011). General Equilibrium, Wariness and Efficient Bubbles. *Journal of Economic Theory* **53** 785-811 (2011).

F. H. Clarke (1990). *Optimization and Nonsmooth Analysis.* Society for Industrial and Applied Mathematics, Philadelphia.