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SHORT SALES AND SHAREHOLDERS' UNANIMITY

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Abstract

Short sales collided with Makowski's (1983) view that shareholders' unanimity follows from the competitiveness of the firm in the stock market. However, once short sales are properly modelled, requiring the shares to be borrowed beforehand, unanimity can be restored. Short sales are no longer an externality and have a price: the lending fee. Unanimity prevails under perfectly elastic demands by both shareholders and shareborrowers. This requires the firm's shares not to be on special. Stock prices reflect the evaluation by shareholders who did not entirely encumber their shares or shareborrowers who did not fully re-use the shares that they possess.

1 Introduction

Introducing firms into a model with uncertainty and incomplete markets has been a challenge to the general equilibrium theory at least since the early days when the work by Dreze (1974) and Grossman and Hart (1979) came out. Market incompleteness prevents the firm from having a well defined objective function and shareholders, with different marginal rates of intertemporal and interstate substitution, tend to disagree on what the firm should do. Several solutions have been attempted, and a promising approach relies on assumptions on the competitiveness

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of the firms in the shares market, as in Hart (1979) and Makowski (1983). However, in these papers short sales were ruled out as they seemed to be incompatible with competitiveness. In Makowski's work, short sales are clearly referred to as an externality that cannot be priced in the stock market.

In the current paper we show that short sales and shareholders' unanimity are not incompatible. We abandon the naive modelling of short positions as naked short sales and, instead require agents to borrow the shares before they can be shorted, as actually occurs in reality. Makowski's externality can now be priced in the market where securities are borrowed. Shares that are on high demand for short positions have higher lending fees.

A recent example of the renewed interest in the shareholders' unanimity problem is the paper by Carceles-Poveda and Coen-Pirani (2009), where it is shown that unanimity holds when the technology exhibits constant returns to scale and the bounds on (naked) short sales of shares are non-binding. Our work differs from this interesting paper in that we allow for general technologies for the firms and we do not require non-binding portfolio constraints.

Our approach depends crucially on the correct modelling of short sales. In order to short a security, one needs to obtain it through a Securities Financing Transaction (SFT), either by borrowing it against cash collateral (or another security as collateral), as done in the Securities Lending Market, or by accepting the security as collateral in exchange for a cash loan, as is done in repo markets. In the case of bonds, repo markets are overwhelmingly dominant, but for equity the securities lending markets are still the preferred channel, at least between the beneficial owners and the hedge fund managers that wish to short-sell[2].

Unanimity of shareholders is a convenient property since it simplifies both the definition of the firm's objective and also its agency problem: there is no need for majority voting rules and the firm can simply implement the production plan favoured by all its shareholders, the plan that maximizes the firm's present value.

Although very convenient, unanimity does not often occur when markets are incomplete.

[2] Among dealers or between dealers and the real money agents (money market funds or commercial banks), repo trades using shares tend to be more common (see FSB (2012)).

There are two reasons for disagreement among shareholders. First, ownership of the firm's shares confers access to a stream of dividends that can be used to transfer income across time and across states of nature, that is, the share provides some *hedging* or *spanning* services to the shareholder. Dividends are contingent on the production plan the firm adopts and different shareholders, having different inter-temporal and inter-state marginal rates of substitution, will generally disagree on the production plan that provides the best hedging services.

Apart from these different preferences over the *hedging effect*, there is a second reason for disagreement as even the *income effect* may be perceived differently. Under incomplete markets, there are multiple non-arbitrage deflators and, therefore, shareholders might not be coordinating on how to anticipate the impact of production changes on the share price and on their current incomes. Good surveys describing the problem and the literature can be found in Kreps (1979) and in Magill and Quinzii (1996).

An assumption that guarantees shareholders' unanimity when markets are incomplete is *partial spanning* but it has somehow an *ad-hoc* flavour. The firm's shareholders will be unanimous in their production choices if the production set of the firm is spanned by the equilibrium production plans of all firms. Changes in hedging services are always accompanied by a change of the share's price that exactly compensates the loss in utility suffered by the agents and there is no confusion about what are the correct price of the firm's share and the firm's present value. This assumption appears in Diamond (1967), Ekern and Wilson (1974), and Radner (1974). It is also implied by the stronger (but possibly more tangible) assumption made in Carceles-Poveda and Coen-Pirani (2009) that firms' production functions exhibit constant returns to scale. There are some important drawbacks with partial spanning. First, the firm cannot *innovate* in the sense that it cannot introduce into the market anything which is not already marketed. Second, partial spanning is stronger than it might seem. If technologies are such that there is free disposal, partial spanning implies that markets are complete[3].

Another approach, due to Grossman and Hart (1979), uses the assumption of *competitive*

[3] This is not an issue in Carceles-Poveda and Coen-Pirani (2009), where firms are characterized using production functions instead of production sets.

price perceptions: each individual shareholder expects the change in hedging services that he himself experiences, when the production plan is modified, to be exactly compensated by the variation in the price of the firm's shares. This is not a rational conjecture since the hedging service each agent receives depends on his own preferences and will not coincide with that of other agents. Once this is assumed, the hedging services become irrelevant and all agents agree on selecting a plan that maximizes the present value of the firm. However, since each agent uses his own personal valuation of the stream of dividends, shareholders still cannot agree on a production plan: the second source of disagreement remains.

Given the important shortcomings of partial spanning and of competitive price perceptions, several authors have tried to provide better assumptions that guarantee shareholders' unanimity when markets are incomplete. These assumptions are related to the competitiveness of the firm in the markets where its shares are traded. An example is the model by Hart (1979), where the number of firms is small, the number of agents is large and for each shareholder the hedging effect is much less important than the income effect. Another example is the work by Makowski (1983), who explicitly defines perfect competition by means of the substitutability by other assets, in utility terms, for the firm's shares[4]. In contrast with the competitive price perceptions assumption, Makowski's approach does not suffer from irrational price conjectures. Moreover, competitiveness assumptions do not share the unpleasant features of partial spanning: innovation is not precluded and complete markets are not implied when free disposal is allowed.

Sadly, whatever their advantages are, models that rely on competitiveness assumptions have, until now, shared a very important flaw: short sales were not allowed. This was never by choice as short sales are prevalent in reality and increase efficiency[5] by allowing agents to transfer income from the future into the present and by enlarging their hedging possibilities and, therefore, should not be excluded. But something was missing to make things work with short sales.

[4]Actually, for such adequate substitutability the large number case is not necessary (as we illustrate in Example 1). It can help but may not be sufficient, as we argue in section 3. Having plenty of firms with identical technologies may not be enough for competitiveness if these firms choose different production plans and therefore differentiate their dividends.

[5]See, for example, Chen and Rhee (2010), Bai et al. (2006).

Previous authors have argued that short sales are fundamentally incompatible with firms' competitiveness in stock markets and with shareholders' unanimity. In Makowski (1983)'s model, for example, the source of this incompatibility is the positive externality consisting on the hedging that can only be done by short selling. It seems that this externality could not be priced in the stock market since the stock price reflected just the willingness of agents to take long positions.

In this article we solve the problem: the firm's competitiveness in the stock markets becomes compatible with the way that short sales are actually done in financial markets. Naked short sales (those that were commonly contemplated in the general equilibrium literature) are usually forbidden in reality[6].

Once we have introduced markets where the shares can be borrowed, we provide, in the spirit of Makowski (1983), a definition of what a perfectly competitive firm is, in both the stock and securities lending markets. This is a firm that takes not just the price of its stock as given but also the lending fee at which the firm's shares are borrowed. At a slightly higher price, no one would purchase the firm's shares and, at a slightly higher lending fee, no one would borrow the firm's shares. Short sales do not constitute an externality anymore. The hedging services that agents get when short selling the firm's shares are priced in the securities lending market through the lending fee. When the rebate rate (the money market interest rate minus the lending fee) on a certain security is below the highest available rebate rate[7], the security is said to be *on special*. We prove that specialness is incompatible with perfect competition.

By shareholders we mean, as it is customary, the agents that legally own shares of the firm rather than those that may be in temporary possession of the shares as a result of having borrowed them. Even though it may not be immediate to distinguish the former from the latter, we can assume that the former are those that tend to be in possession of the share when voting takes place. It is true that the latter are entitled to voting rights even if they don't have ownership, but it is commonly observed that shares tend to be returned to the owners before a voting decision

[6]See e.g., S.E.C Exchange Act Rule 10b-21. http://www.sec.gov/divisions/marketreg/tmcompliance/rule10b21-secg.htm.

[7]That is, the *general collateral rate.*

takes place (see Aggarwal et al. (2015) for evidence in the securities lending market).

We prove that if a firm is a perfect competitor, both in the stock and the securities lending markets, its shareholders will unanimously agree on choosing the production plan that maximizes the firm's present value. Makowski had provided a counterexample showing why short sales, in the naive naked form, were incompatible with shareholders unanimity. We reformulate this example (see Example 3), replacing naked short sales by short sales (preceded by borrowing), and show that, in equilibria allowing for short sales, unanimity still depends on whether the firm faces perfectly elastic demand for its shares by both buyers and borrowers.

The assumptions that we use to achieve shareholders' unanimity are less restrictive than others used in the literature. Perfect competition is more general than partial spanning in that it does not place strong restrictions on the firms' technologies or on the number of firms. Perfect competition *can* be a result of the firms' characteristics but can also occur as a consequence of the agents' preferences. To make this point, we give an example where there is only one firm (Example 1). Consumers have indifference curves with linear sections, which facilitates the substitutability between the firm's shares and a one-period real bond. We have a second example, where there are several firms and the condition that ensures competitiveness is closer to the partial spanning assumption (Example 2), since the production set of one firm is generated by the equilibrium production plans of the other firms.

We also address the problem of security pricing when the firm is a perfect competitor in the stock and securities lending markets. The way an agent's operations in these two markets are linked is captured by the *box constraint*. This constraint says that the physical title amount of the share in the agent's possession must be non-negative. If the agent is long in the share he can lend all or part of it. On the other hand, if an agent intends to short-sell a certain amount of shares, he must have borrowed at least that amount. Since the share is in positive net supply, agents' box constraints cannot be all binding. For an agent whose box constraint is not binding, either some part of his long security positions is unencumbered or some part of the shares that he borrowed has not been reused. Such agent has null shadow price for the box constraint and, therefore, the present value of dividends, deflated using his personal marginal rates of substitution, will be exactly equal to the equilibrium price of the share. It does not matter whether the agent is a shareholder or a shareborrower. However, that agent can only be used for pricing as long as he remains with null box shadow prices and for this reason, like in Makowski (1983), we may need to use different reference agents at different moments in time. Moreover, if the firm is perfectly competitive and such *reference agents* are not initial shareholders, then their marginal rates of substitution can be used to evaluate the price of the share *out of the financial equilibrium*. This result is in the spirit of Makowski but it is important to note that in Makowski (1983) such reference agent was always a buyer of the share whereas in our model the agent may be a borrower of the share and, therefore, even a short-seller.

The valuation, at date t, of a stream of dividends associated with the firm's stock at date $t+1$ that is obtained when using reference shareholders' deflators, corresponds to the maximum personal valuation of such stream of dividends among all agents. A pricing function that also corresponds to the maximum valuation among agents has appeared recently in some models[8] (e.g., Bisin et al. (2014) and Acharyaa and Bisin (2014)). As we prove in section 4, this function correctly computes the price of the share as long as the firm is a perfect competitor in the economy and there is at least one reference agent, but these two conditions are sometimes ignored in the literature. In an economy with a finite number of firms, if the firm is not perfectly competitive, or if there are no reference agents, the criterion is correct for the plan that maximizes the firm's present value but will be generally incorrect when evaluating other production plans. We illustrate this point in Example 4.

In the next section we present the model, the equilibrium definitions, and make sure that existence of equilibrium is not an issue. In section 3 we define perfectly elastic demands by shareholders and shareborrowers and provide our unanimity results (including unanimity through time). In section 4 we address the pricing of securities and relate no-specialness to perfect

[8] This is the case in Bisin et al. (2014) only when short sales are not allowed. In the case with short sales, which are modelled by introducing a derivative and not through securities lending, the pricing function is different to the one in our model.

competition. Section 5 concludes.

2 The model

The economy is represented by an event tree D with three dates $t \in \{1, 2, 3\}$[9]. The unique node at date 1 will be denoted by $e = 1$. There is uncertainty at $t = 2$ as $S < \infty$ states of nature may occur. Between date 2 and date 3 there is no uncertainty. So, each node is defined by a date and a state of the world, $e = ts$ and each node $2s$ has only one successor, $3s$. Given two nodes $e, u \in D$ we write $u \geq e$ if $u = t's'$ belongs to the sub-tree with root $e = ts$, and we write $u \in e + 1$ if $u \geq e$ and $t' = t + 1$.

Markets for the C commodities produced in the economy open at each event and we denote the price of commodity c at event e by p_{ec}. The commodity space is $X \equiv \mathbb{R}_+^{C(2S+1)}$. Trading of commodities and securities is done by a set of $I \geq 2$ agents, indexed by i. A bundle of commodities consumed by agent i is denoted by $x^i = (x^i_e)_e \in X$. Production is carried out by F firms, each one being characterized by a production possibility set $Y^f \subset \mathbb{R}^{C(2S+1)}$ $(f = 1, \cdots, F)$. Agents are described by their endowments of commodities, $\omega^i \in X$, by nonnegative endowments, at date 1, of ownership shares of each firm f, ϕ_0^{if}, and by a utility function $U^i : X \to \mathbb{R}$. We assume utility functions of all agents to be locally non-satiated[10]. For each firm f, we normalize the total supply of the firm's shares to one: $\sum_i \phi_0^{if} = 1$.

2.1 Security markets

Securities in this model are the ownership shares of the firms. Owning a firm (or a share of the firm) means owning a stream of dividends across dates and states and the right to lend it. The latter tends to be overlooked in a *cash-flows only* view of finance, but will play a major role in this article. Dividends are contingent on the production plan adopted by the firm and can be

[9]Our model has three dates since we need securities to be traded and have a price at the intermediate date, when the second leg of the securities lending transaction takes place.

[10]For any $x^i \in X$ and any e, there is some $\bar{x}^i \in X^i$ arbitrarily close to x^i that satisfies $\bar{x}^i_{e\prime} = x^i_{e\prime}$ for all $e\prime \neq e$ and $U^i(\bar{x^i}) > U^i(x^i)$.

used to transfer consumption among states.

Shares are traded in stock markets at dates 1 and 2. Each agent decides on an ownership plan $\phi^i \equiv \phi_e^{if} \in \mathbb{R}^{F(S+1)}$, where ϕ_e^{if} represents the position of agent i in firm f's shares at event e. Share prices are denoted by $q \equiv (q_e^f) \in \mathbb{R}^{F(S+1)}$.

Agents can have negative positions of the ownership shares of firms, short sales are permitted. Short selling, however, is not the same as issuing shares. Agents can only trade (perhaps several times over the same settlement date) the finite given amount of shares existing in the economy. Short selling and issuing are different. To understand this difference we need to introduce markets where agents can borrow shares.

2.2 Markets for borrowing and lending shares

Our approach to securities lending has some traits in common with the GEI repo model in Bottazzi, Luque and Páscoa (2012). If an agent wishes to short firm f's shares, he must first go to the securities lending market and *borrow* the desired amount of shares before selling them short. This is the way short-selling is done in reality. Naked short sales have been a useful abstraction in general equilibrium models in the past but are simply not allowed in reality. To better focus on short sales we fix issuance at the initial date, and do not allow for naked shorts.

Securities lending works as follows: The lender of the security transfers the security to the borrower and receives collateral in exchange. The exchange is reversed at a future date which can be left open. In the U.S., up to 98% of security loans are made against cash collateral. In Europe, it is most common to use other securities as collateral. In this article, we will focus on transactions done against cash collateral and for a fixed term. A description of the U.S. market can be found in D'Avolio (2002).

The market practice is that the borrower of the security posts initial margin to cover the risks that the lender incurs in the transaction. Here it is assumed that the value of the collateral posted is H times the market value of the securities being lent, where $H > 1$. The borrower of the security must pay a fee to the lender, which is usually embedded in the *rebate rate*, the interest that the lender of the security pays to the borrower in the second leg of the transaction for the use of the cash collateral. The rebate rate is usually bellow the market rate at which the lender of the security can invest the cash during the term of the transaction, so the fee associated to the loan is this difference between the market rate and the rebate rate. Securities with high fees and low interest rates are said to be “special” and securities with the lowest fees and an interest rate very near the market rate are said to be “general collateral”.

The borrower of a security acquires possession rights associated with the security[11]. However, any coupon or dividend paid to the borrower during the term of the transaction is passed through to the original owner; this is called a *manufactured payment* or a *manufactured dividend*[12].

For simplicity, security lending takes place only at date 1. Agent i's position on firm f's shares in the securities lending market is represented by the variable z^{if}. If $z^{if} > 0$ the agent is borrowing the security and if $z^{if} < 0$ he is lending it. Given the margin that is applied, the amount of funds that must be pledged as collateral when borrowing one unit of firm f's shares in the securities lending market is given by $H^f q_1^f$. The rebate rate that the lender of firm f's shares pays to the borrower for the use of the cash collateral is denoted by ρ^f.

An important feature of a borrowing transaction is that the borrower has the right to re-lend the security or to sell it in the stock market. This process may occur many times over for the same settlement period. The re-usability of the securities implies that, although issuance is fixed at the initial date, this is not sufficient to guarantee that short positions on the securities are bounded. This is a new instance of Hart (1975)'s problem: when short sales of real assets are not bounded, equilibrium may fail to exist. To circumvent this problem we assume that there is an exogenous bound on the positions agents hold in the securities lending market, $|z^{if}| \leq M$, for all i and all f. This, together with the box constraint is enough to bound short positions. We adopt this exogenous bounds approach since the focus of this article is not on existence but on shareholder's unanimity and perfect competition in the presence of short sales. However, it

[11]Together with voting rights although these tend not to be exercised since shares are usually returned to the original owners for the voting date.

[12]See (Maxime Bianconi et al., 2010, section I)

is important to mention that in reality there are several legal and institutional constraints that bound agents' short positions.

The present model of securities lending is formally quite close to the repo model by Bottazzi, Luque and Páscoa (2012). The key difference is that here cash is used as collateral and it is the securities borrower that pledges the collateral to the securities lender, while in repo there is a cash loan against a security that is pledged as collateral by the cash borrower (security lender). In both cases the value of the collateral usually exceeds that of the loan: actually in securities lending an initial margin is always posted ($H > 1$) while in repo a haircut is many times collected by the cash lender ($H \leq 1$). See for example Bottazzi, Luque and Páscoa (2012) for detailed models of repo where the bounds on short sales, rather than being exogenous, are driven by different institutional arrangements (segregated haircuts or a combination of constrained dealers with non-dealers that do not collect haircut).

2.3 Feasible market plans and the box constraint

Given a set of production decisions $y \equiv (y^f)$, market prices and rebate rates, (p, q, ρ), agents decide on a plan (x^i, ϕ^i, z^i), consisting of consumption and portfolios in the stock and securities lending markets. Let us define the budget constraints that these plans must satisfy.

At date 1, the securities lending market opens. Borrowing and lending positions are bounded exogenously as follows:

$$(1) \qquad -M \leq z^{if} \leq M$$

Agents' budget constraint at date 1 is:

$$(2) \qquad p_1(x_1^i - \omega_1^i) + \sum_f q_1^f(\phi_1^{if} - \phi_0^{if}) + \sum_f q_1^f H^f z^{if} \leq \sum_f \phi_0^{if} p_1 y_1^f$$

Date 1 budget constraints include initial endowments of goods and securities and dividends paid to initial shareholders.

The fact that, to short-sell a security, an agent must first borrow it, is reflected in a restriction we will refer to as the *box constraint*. This restriction states that the agent must hold a nonnegative amount of the security in his possession. That is, the sum of his original position (from a previous period), plus security transactions in the current stock market, plus current transactions in the securities lending market (if open) must be nonnegative. At date 1 the box constraint is:

$$\phi_1^{if} + z^{if} \geq 0 \ , \ \forall f \tag{3}$$

Securities lending markets do not open at date 2. Agents trade securities only in stock markets, and receive dividends from the firms according to their shares positions at the end of date 1. Borrowing and lending transactions that occurred at date 1 are settled at date 2: for transactions where shares of firm f were borrowed, lenders repay the cash they have received as collateral at the rate ρ^f. The budget constraint at node $e = 2s$ for agent i is given by:

$$p_{2s}(x_{2s}^i - \omega_{2s}^i) + \sum_f q_{2s}^f(\phi_{2s}^{if} - \phi_1^{if}) \leq \sum_f \phi_1^{if} p_{2s} y_{2s}^f + \sum_f (1+\rho^f) q_1^f H^f z^{if} \tag{4}$$

Since securities lending markets do not open at date 2, the box constraint at node $e = 2s$ reduces to a no short sales constraint:

$$\phi_{2s}^{if} \geq 0 \ , \ \forall f \tag{5}$$

At date 3 there is no trading of securities[13], agents consume from their endowments and dividends corresponding to them according to their securities positions at the end of date 2:

$$p_{3s}(x_{3s}^i - \omega_{3s}^i) \leq \sum_f \phi_{2s}^{if} p_{3s} y_{3s}^f \tag{6}$$

[13]The third date serves for guaranteeing that the shares of the firm retain value at the second date, when borrowing and lending transactions are settled.

Finally, the following non-negativity restrictions must hold: $x^i_e \geq 0$, $\forall e$.

Given parameters (y, p, q, ρ), an agent's plan of consumption, of securities, and of borrowing positions (x^i, ϕ^i, z^i), will be called *feasible* if it satisfies equations (1), (2), (3), (4), (5) and (6). We will represent the set of all feasible plans for agent i by $B^i(y, p, q, \rho)$, and $B^i_\star(y, p, q, \rho)$ will represent the subset of utility maximizing plans in $B^i(y, p, q, \rho)$.

2.4 Equilibrium

In this section we present various concepts of equilibrium that will be needed to study the production decisions taken by the firms. We start by defining an *exchange equilibrium*, where production decisions are taken as given.

Definition 1 (**Exchange equilibrium**). *Given firms' production decisions* $y = (y^1, \cdots, y^F)$*,* $(x, \phi, \theta, \psi, p, q, \rho)(y)$ *is an exchange equilibrium parametrized by* y*, if*

(a) for each agent i*,* $(x^i, \phi^i, z^i) \in B^i_\star(y, p, q, \rho)$*,*

(b) commodity markets clear: $\sum_i \left(x^i - \omega^i\right) = \sum_f y^f$*,*

(c) stock markets clear: $\sum_i \phi^{if}_e = 1$ *at each event* e*,*

(d) securities lending markets: $\sum_i z^i = 0$*.*

$EE(y)$ *will denote the set of all exchange equilibria parametrized by* y*.*

Each firm has conjectures $\upsilon^f = (\upsilon^f_e) : Y^f \to \mathbb{R}^{S+1}$ about how the price of its shares will vary with its production plan. Firm f also conjectures about the rebate rate paid by lenders of its stock, which is also conditional on the production plan the firm selects, and is given by $\tau^f : Y^f \to \mathbb{R}$.

The net market value of the firm equals dividends paid at node e plus the equity value[14]. Firm f's conjectured net market value at node e, given production decisions y^f, is given by

[14]Note that in some nodes, where $p_e y^f_e < 0$, this corresponds to investment in the firm which shareholders must bear. At other nodes, where $p_e y^f_e > 0$, it constitutes positive dividends that the firm distributes among its shareholders.

$\pi_e^f(y^f) \equiv p_e y_e^f + \upsilon_e^f(y^f)$.

Next we define a *financial equilibrium*, which is an exchange equilibrium with the added requirement that firms choose the production plans that maximize their respective net market values at date 1.

Definition 2 (Financial equilibrium). *$(x, \phi, z, y, p, q, \rho)$ is a financial equilibrium if*

(a) $(x, \phi, z, p, q, \rho)(y) \in EE(y)$,

(b) for each firm f, $y^f \in \text{argmax}_{\bar{y}^f \in Y^f} \pi_1^f(\bar{y}^f)$, and, in each event e, $\upsilon_e^f(y^f) = q_e^f$ and $\tau^f(y^f) = \rho^f$.

FE will denote the set of financial equilibria in the economy.

Finally, we present the concept of *quasi-equilibrium*, which is an exchange equilibrium in which all firms, with the possible exception of one, choose the production plans that maximizes their respective net market values at date 1. This concept will help us study whether or not present value maximization is a good objective for the firm. By "good" we mean a course of action that maximizes shareholders' utilities.

Definition 3 (Quasi-equilibrium). *We say that $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho})$ is a quasi-equilibrium given firm f's production plan $\bar{y}^f$ if*

(a) $(\bar{x}, \bar{\phi}, \bar{z}, p, \bar{q}, \bar{\rho})(\bar{y}) \in EE(\bar{y})$,

(b) for each firm $g \neq f$, $\bar{y}^g \in \text{argmax}_{y^g \in Y^g} \pi_1^g(y^g)$,

(c) for each firm g, in each event e, $\upsilon_e^g(y^g) = \bar{q}_e^g$ and $\tau^g(y^g) = \bar{\rho}^g$.

The set of quasi-equilibria when firm f's production plan is y^f will be denoted by $QE(\bar{y}^f)$.

2.5 Existence of exchange equilibrium

It is well understood that the main difficulty when proving the existence of equilibrium arises when short sales are unbounded, as shown by Hart (1975). This is not a problem in our model, where the exogenous upper bound on borrowing of securities, together with the box constraint, bound agents' short positions.

We now introduce some assumptions that will be used in some of the results that follow.

Assumption 1. *For every agent i:*

1. *commodity endowments are strictly positive in every node e,*
2. *stock endowments are such that $\sum_f \phi_0^{if} > 0$,*
3. *the agent's preferences satisfy:*
   (a) *U^i is concave and twice continuously differentiable,*
   (b) *$DU^i(x) \in \mathbb{R}_{++}^{(1+2S)C}$, $\forall x \in \mathbb{R}_{++}^{(1+2S)C}$,*
   (c) *$\forall c \in \mathbb{R}$, the set $[U^i]^{-1}(c)$ is closed in $\mathbb{R}_{++}^{(1+2S)C}$,*
   (d) *for every (y, p, q, ρ): $(x^i, \phi^i, z^i) \in B_\star^i(y, p, q, \rho)$ implies that for each event e, there is some commodity c such that $x_{ec}^i > 0$ and $\partial U^i(x^i)/\partial x_{ec}^i > 0$.*

Part (3d) of assumption 1 ensures that marginal utilities of income (the Lagrange multipliers of the budget constraints) are positive.

The next assumption will be considered in some of our results:

Assumption 2. *$y = (y^1, \cdots, y^F)$ is such that:*

1. *for each good, there is at least one firm producing that good at dates 2 and 3,*
2. *for every agent i, the initial endowment of goods exceeds the agent's initial investment in the several firms he has ownership shares[15]: $\omega_1^i + \sum_f \phi_0^{if} y_1^f \gg 0$.*

Given that short sales are bounded, the existence of exchange equilibria follows as a natural extension of Bottazzi, Luque and Páscoa (2012) in our framework:

Proposition 1. *If assumption 1 holds and the vector of firms production plans $y = (y^1, \cdots, y^F)$ satisfies assumption 2, then there exists an exchange equilibrium with production plans y.*

[15] This is equivalent to assuming that the value of initial endowments $(p_1\omega_1^i)$ dominates the cost of the initial investment ($p_1 \sum_f \phi_0^{if} y_1^f$) regardless of what the relative commodity prices (p_1) might be.

3 Unanimity

Before presenting the main unanimity results, we introduce two important definitions that will help us characterize equilibria and the firms for which the *shareholders' unanimity* property holds.

3.1 Perfectly elastic demand by shareholders and by shareborrowers

The following definitions describe a perfectly competitive firm. A perfectly competitive firm has a limited ability to influence the price of its shares, both the price at which the shares are traded and also, the rebate rate paid to borrowers of the share in exchange for the cash collateral that they post.

Definition 4 (**Perfectly elastic demand**). *Let* $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho}) \in QE(\bar{y}^f)$ *be a quasi-equilibrium, and let* $E_T \equiv \{1, 21, \cdots, 2S\}$ *be the set of nodes where there is trading of securities. The demand for firm f's shares is perfectly elastic both in the stock and the securities lending markets, in the quasi-equilibrium, when for every agent i, it is true that:*

If the agent's plan in the quasi-equilibrium $(\bar{x}^i, \bar{\phi}^i, \bar{z}^i)$ *is such that* $\bar{\phi}_e^{if} > 0$ *in some event* $e \in E_T$ *(agent is a buyer of the share in the stock market), or* $\bar{z}^{if} > 0$ *(agent is a borrower of the share), then there is another (optimal) plan* $(\bar{x}_*^i, \bar{\phi}_*^i, \bar{z}_*^i,) \in B_\star^i(\bar{y}, p, \bar{q}, \bar{\rho})$*, in which* $\bar{\phi}_{*e}^{if} \leq 0$ *in all events* $e \in E_T$*, and* $\bar{z}_*^{if} \leq 0$ *(agent is neither a buyer nor a borrower).*

If the demand for firm f's shares is perfectly elastic according to definition 4, whoever is trying to sell the share in the stock market is a *price taker* in the sense that if he tried to sell the firm's shares at a higher price, he would loose all of his buyers. Every potential buyer could do just as well (in utility terms) with plans that did not include long positions on the shares of the firm. For the same reason, if the shares of the firm were to be lent at a slightly lower rebate rate, no one would borrow the share any more.

The general collateral rate is the highest rebate rate that lenders of securities pay to borrowers. This highest rebate rate is associated to the lowest lending fee paid by borrowers, which in turn corresponds to the borrowing of securities for which there is no special demand (hence the term *general collateral*). In other words, a security classified as general collateral has no particular *possession value*. Of the sample of securities studied in D'Avolio (2002), 91% were general collateral, lent for an average fee of 17 basis points. Transactions done with general collateral are more likely to be driven by the supply and demand for cash than the supply and demand for individual securities. The lending fee associated with the firm's shares is the price that a borrower of shares is willing to pay to be in possession of these particular shares in order to reuse them, either by short-selling or by re-lending them. It is the presence of this price that makes the short sales no longer an externality as they were in Makowski (1983)'s model with naked short sales.

If borrowers of the firm's shares are not willing to accept being paid a rebate rate below the general collateral rate, and if this rate coincides with money market interest rate, then the lending fee is zero, and one can infer that there is no *possession value* for the firm's shares and, therefore, the firm faces a perfectly elastic demand in the securities lending market. Examples 1 and 2 will illustrate such absence of possession value. A precise relation between perfectly elastic demands and no-specialness will be established in Theorem 4, in section 4.1.

Definition 5 (Perfectly competitive firm). *Let* $(x, \phi, z, y, p, q, \rho) \in FE$ *be a Financial Equilibrium. We say that firm* f *is a* perfect competitor *in the equilibrium if for any production plan* $\bar{y}^f \in Y^f$ *that firm* f *chooses to implement, and for every quasi-equilibrium* $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho}) \in QE(\bar{y}^f)$*, it is true that:*

1. *Firm* f*'s production decision does not affect prices or dividends of other firms:* $p_e \bar{y}^g_e = p_e y^g_e$*,* $\bar{q}^g_e = q^g_e$*,* $\bar{\rho}^g = \rho^g$*, at all events* e*, and for all firms* $g \neq f$*.*

2. *There is a perfectly elastic demand for firm* f*'s shares in the quasi-equilibrium.*

The first part of definition 5 states that firm f's choice of production plan does not affect other firms' prices. The second part says that f is limited in how it can affect its own prices: it can choose its production plan $\bar{y}^f$ which induces a quasi-equilibrium in $QE(\bar{y}^f)$, but the way in which the choice of the production plan $\bar{y}^f$ induces quasi-equilibrium price and rebate rates for the firm's shares is constrained by the fact that there is a perfectly elastic demand for its shares.

That is, the quasi-equilibrium price of the share can not be influenced by either shareholders or by share-borrowers (as these are the two types of agents that sell or short-sell the shares to any buyer). Any attempt to raise the price above the quasi-equilibrium level, would fail since the buyers can always find good substitutes and maintain their utility levels.

Similarly, an attempt by lenders of the shares to charge a lending fee (pay a rebate rate) above (below) the quasi-equilibrium level would also fail due to the adequate substitutability that borrowers of shares enjoy.

The above notion of a firm being perfectly competitive in the markets for its shares is not related in a straightforward manner to having many firms. Actually, the large numbers case is not necessary, as we illustrate in the following examples, it may help (as we discuss in example 3) but is not sufficient either. In fact, if firms do not have the same technology or even when a same technology set allows them to pick different production plans, firms end up offering, through the dividends, hedging services that are not perfect substitutes, and therefore, demands would still be non-perfectly elastic in spite of the large numbers involved.

In the literature some other notions of competitiveness are used to obtain unanimity. One important example, the *spanning* condition, implies that firms cannot innovate. Essentially, shareholders of a firm will be unanimous in their preferred production plan if the firm is competitive in the sense that it cannot introduce new goods into the market or produce the same goods in a way that differs much from that of the other firms'. Whatever production plan the firm decides to implement, it is a linear combination of the equilibrium production plans in the economy.

The notion of competition in definition 5 is less restrictive than the spanning condition because firms are allowed to innovate by selecting a production plan that is not spanned by the equilibrium plans of the firms in the economy. However, in any quasi-equilibrium, investors manage to find an alternative optimal portfolio where positions on the shares of this firm are absent. In Makowski

(1983) a firm is competitive when investors manage to do without long positions in the firm's shares, but as the hedging done through short selling could not be priced, short selling was an externality that had to be ruled out. In the present article, a firm is competitive when investors manage to find good substitutes for both long and short positions in the firm's shares. In the spirit of Ostroy (1980), trading the shares of a competitive firm, being either long or short, contributes no surplus to the economy.

Let us now give two examples of economies in which one firm is a perfect competitor according to our definition. These examples will show that, in our definition of perfect competition, what matters is not just what the production plans of the firms are, but also the way in which the technology of a firm affects the utility of the agents. Competitiveness also depends on the agents preferences.

To illustrate this last point, we present an example of an economy with only one firm. There are no restrictions on the firm's technology but we assume linear utilities and this allows for an easy substitution of the firm's shares by positions in a riskless one-period real bond.

Example 1. Suppose there is only one firm. There is only one commodity produced by the firm and traded at each node. This commodity will be the numeraire (that is, $p_e = 1$, $\forall e$). We assume that apart from hedging in the firm's shares, agents have at their disposal a one-period real bond whose real returns (in the single consumption good) are $1 + \rho_b$. We denote positions in such bond by b^i (measured in the equivalent first-date consumption units). Agents' budget constraints for nodes 1 and $2s$ become:

$$(x_1^i - \omega_1^i) + q_1^f(\phi_1^{if} - \phi_0^{if}) + b^i \le \phi_0^{if} y_1^f - q_1^f H^f z^{if}$$

$$(x_{2s}^i - \omega_{2s}^i) + q_{2s}^f(\phi_{2s}^{if} - \phi_1^{if}) - (1 + \rho_b)b^i \le \phi_1^{if} y_{2s}^f + (1 + \rho^f) q_1^f H^f z^{if}$$

Utilities of all the agents are linear: $U^i(x^i) = \sum_{ts} \beta^{t-1} \sigma_{ts}^i x_{ts}^i$, where $\beta \in (0, 1]$ is the discount factor and σ_{ts}^i is the probability agent i assigns to the occurrence of date-state ts. Since the unique state of the world is known at date 1, and there is no uncertainty from date 2 to date 3, we must have $\sigma^i_1 = 1\ \forall i$, and $\sigma^i_{2s} = \sigma^i_{3s}\ \forall i, \forall s$. Also, $\sum_s \sigma^i_{2s} = 1\ \forall i$. We will show that demand for the firm's shares is perfectly elastic at the quasi-equilibria where the condition $\rho = \rho_b$ holds.

Consider a quasi-equilibrium in which the solutions to agents' problems are interior, that is, there is positive consumption at all date-states. Let λ^i_e denote the multipliers for the budget constraints at node e. From the agents first order conditions we have that, for interior solutions, $\partial U(x^i)/\partial x^i_e = \lambda^i_e$. Computing $\partial U(x^i)/\partial x^i_e$, we have $\lambda^i_{ts} = \beta^{t-1}\sigma^i_{ts}$. In particular, $\lambda^i_1 = 1$, and for every agent i, it must be true that $(1+\rho_b) = 1/\sum_s(\beta\sigma^i_{2s}) = 1/\beta$ (see appendix).

Suppose $\rho = \rho_b$. To prove that the demand for the firm's shares is perfectly elastic, we must show that whenever an agent has positive long positions, at any date, in the stock or in the securities lending market, it is possible to substitute the agent's plan with some other optimal plan that does not include the firm's shares. We will start by showing that this is true for long positions purchased at date 1.

The first order conditions of the agent i's problem (see appendix) imply that in an equilibrium with $z^{if} < M$ the following equation holds:

$$\sum_s \beta\sigma^i_{2s}\left(\left(1+\rho^f\right)q^f_1 - \left(y^f_{2s} + q^f_{2s}\right)\right) = 0 \tag{7}$$

Suppose the agent's portfolio was initially (ϕ^i, z^i, b^i). There is a portfolio $(\tilde{\phi}^i, \tilde{z}^i, \tilde{b}^i)$ which doesn't include shares of the firm, and doesn't change the level of utility attained by the agent. Simply set $\tilde{\phi}^{if}_1 = 0$, $\tilde{z}^{if} = 0$ and $\tilde{b}^i = b^i + q^f_1(\phi^{if} + H^f z^{if})$.

With the new portfolio, consumption at date 1 will remain unaltered. At date 2, state s, the change in consumption (and in income) will be $\phi^{if}_1\left((1+\rho^f)q^f_1 - (q^f_{2s} + p_{2s}y^f_{2s})\right)$. After the change, the agent will consume more in some states and less in others. Suppose that consumption at every state remains positive after the change[16]. Equation (7) implies that the overall change in utility after the change in portfolio will be zero. An agent can always do as well by not having

[16]Consumption after the change of portfolio will remain positive if the cost of the original portfolio of firm f's shares was a small enough fraction of the agent's total income. This in turn will be more likely when there are more firms and shares of firm f constitute only a small portion of the agent's portfolio.

long positions on firm f's shares on the securities lending and stock markets. The linearity of the preferences guarantees that the same substitution works in a quasi-equilibrium where $z^{if} = M$.

This proves that the demand for the firm's shares is perfectly elastic at date 1. The same argument could be done for the portfolio at the second date if a one period real bond was traded at that date too. Alternatively, since there is no uncertainty between dates 2 and date 3, the firm would also face a perfectly elastic demand for its shares if there were no inter-temporal discounting, that is, if $\beta = 1$. In this case an agent which initially had a long position $\phi^{if}_{2s} > 0$, can simply choose another portfolio in which $\tilde{\phi}^{if}_{2s} = 0$ instead. The change in portfolio will increase the agent's income at node $2s$ by $q^f_{2s}\phi^{if}_{2s}$ and decrease his income at node $3s$ in the amount $\phi^{if}_{2s}y_{3s}$. In terms of utility, the agent will gain $\beta q^f_{2s}\phi^{if}_{2s}$ and loose $\beta^2\phi^{if}_{2s}y_{3s}$. From the first order conditions the share's price at date 2 satisfies $q^f_{2s} = (\lambda^i_{3s}/\lambda^i_{2s})y^f_{3s}$ and $\lambda^i_{ts} = \beta^{t-1}\sigma^i_{ts}$. Since $\sigma^i_{2s} = \sigma^i_{3s}$, when $\beta = 1$ the increment in expected utility from the change in income at node $2s$ is exactly compensated by the decrease in expected utility from the change at node $3s$.

We have seen that, in this example, if the condition $\rho^f = \rho_b$ holds at all quasi-equilibria, the firm will be a perfect competitor in the economy. A stronger result which is valid in general and not only in this example is given in Theorem 4, which states that if firm f is a perfect competitor in the financial equilibrium, then $\rho^f = \rho_b$ in all quasi-equilibria.

This example imposes restrictions on agents' preferences rather than on firms' technologies. What is required from the agents is that they all have linear utilities, so that the null first order effect given by equation (7) coincides with the actual utility variation experienced by the agents when they substitute their long positions on the shares of the firm by positions in the one-period bond.

We made no assumptions on spanning. The payments of the risk-free loan will, in general, not be equal to the dividends paid by the firm, but when the interest rates coincide ($\rho^f = \rho_b$) the equality holds in terms of the average across states, weighted by marginal utilities of income. Hence, for linear utilities, the proposed change in portfolio has no effect on agents' expected utility.

The example also highlights another important feature of equilibria and economies in which firms are perfectly competitive in the sense of Definition 5: some linearity of preferences is needed when there are few firms or few agents. To be more precise, plans $(\bar{x}^i, \bar{\phi}^i, \bar{z}^i)$ and $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*,)$ in Definition 4 must be such that $U^i(\alpha\bar{x}^i + (1-\alpha)\bar{x}^i_*) = U^i(\bar{x}^i) = U^i(\bar{x}^i_*)$ for all $\alpha \in [0,1]$.

In contrast with example 1, the following example depends strongly on the structure of the firms' technologies but has no restrictive assumptions on preferences.

Example 2. Suppose there are three firms and only one commodity in the economy. The margin applied in borrowing transactions is the same for the three firms $H = H^1 = H^2 = H^3$. Firm 1 will be a perfect competitor in any financial equilibrium if its shares can be substituted by some portfolio consisting of shares of firms 2 and 3 in such a way that income at every node stays the same. In that case, agents optimal consumption remains attainable and agents utilities do not decrease as result of the substitution.

Suppose $(x, \phi, z, \bar{y}, p, q, \rho) \in FE$ is a financial equilibrium, that $\bar{y}^2$ and $\bar{y}^3$ are the production plans that maximize the present value of firms 2 and 3, respectively, and that firm 1's production set satisfies the condition

(8) $$Y^1 \subset \left\{ y \in \mathbb{R}^{1+2S} | y = \alpha_2\bar{y}^2 + \alpha_3\bar{y}^3, (\alpha_2, \alpha_3) \in \mathbb{R}^2_+ \right\}$$

Then, if $y^1 \in Y^1$ and $y^1 = \alpha_2^{y^1}\bar{y}^2 + \alpha_3^{y^1}\bar{y}^3$, a stock portfolio consisting of one unit of firm 1's shares $(\phi^{i1}, \phi^{i2}, \phi^{i3}) = (1, 0, 0)$, pays the same dividends, in all nodes, as a portfolio consisting of $\alpha_2^{y^1}$ units of firm 2's shares plus $\alpha_3^{y^1}$ units of firm 3's shares $(\phi^{i1}, \phi^{i2}, \phi^{i3}) = (0, \alpha_2^{y^1}, \alpha_3^{y^1})$. More generally, the stock portfolios $(\phi^{i1}, \phi^{i2}, \phi^{i3})$ and $(0, \phi^{i2} + \alpha_2^{y^1}\phi^{i1}, \phi^{i3} + \alpha_3^{y^1}\phi^{i1})$, pay exactly the same dividends in all nodes of the event tree. This substitution leaves the returns from the whole portfolio unchanged but this still does not guarantee that agents will be able to consume exactly the same as before the substitution, due to the change in portfolio costs. Previous consumption is still attainable as long as the condition $y^1 = \alpha_2^{y^1}\bar{y}^2 + \alpha_3^{y^1}\bar{y}^3$, which guaranteed unchanged returns implies also that security and borrowing costs remain the same, given, respectively, by:

$$q^1 = \alpha_2^{y^1} q^2 + \alpha_3^{y^1} q^3 \tag{9}$$

$$\rho^1 q_1^1 = \alpha_2^{y^1} \rho^2 q_1^2 + \alpha_3^{y^1} \rho^3 q_1^3 \tag{10}$$

When conditions (9) and (10) hold, an agent that substitutes the whole securities and borrowing portfolio $\big((\phi^{i1}, z^{i1}), (\phi^{i2}, z^{i2}), (\phi^{i3}, z^{i3})\big)$ by the portfolio $\big((0,0), (\phi^{i2}, z^{i2}) + \alpha_2^{y^1}(\phi^{i1}, z^{i1}), (\phi^{i3}, z^{i3}) + \alpha_3^{y^1}(\phi^{i1}, z^{i1})\big)$ will experience no change in income at every date-state and, therefore, no loss in utility.

For $y^1 = \alpha_2^{y^1}\bar{y}^2 + \alpha_3^{y^1}\bar{y}^3$ to imply (9) and (10) it is enough that shadow prices of the agents' budget constraints are zero, and in particular, that the shares of the three firms are not on special. Null shadow prices imply that there is no possession value for any of the securities in the economy.

3.2 Unanimity results

Let us now present our main unanimity result. The following theorem shows that perfect competition is enough to guarantee that initial shareholders agree on the best production plan for the firm.

Theorem 1 (Shareholders' Unanimity). *Let $(x, \phi, z, y, p, q, \rho) \in FE$ be a financial equilibrium and suppose firm f is a perfect competitor in this equilibrium. Given any production plan $\bar{y} \in Y^f$ let $(\bar{x}, \bar{\phi}, \bar{z}, p, \bar{q}, \bar{\rho})(\bar{y}) \in QE(\bar{y}^f)$ be a quasi-equilibrium. Then:*

(a) Each initial shareholder (i such that $\phi_0^{if} > 0$) is not worse off in the financial equilibrium than in the quasi-equilibrium: $U^i(x^i) \geq U^i(\bar{x}^i)$.
Also, if $\pi_1^f(y^f) > \pi_1^f(\bar{y}^f)$, the inequality is strict: $U^i(x^i) > U^i(\bar{x}^i)$.

(b) An agent with $\phi_0^{if} = 0$ is as well off in the financial equilibrium as in the quasi-equilibrium: $U^j(x^j) = U^j(\bar{x}^j)$.

Remark 1. Remember that, by definition, the difference between y^f and $\bar{y}^f$ is that y^f maximizes the present value of the firm at date 1: $\pi_1^f(y^f) \geq \pi_1^f(\bar{y}^f)$.

The intuition behind the proof is that if the firm is a perfect competitor in the financial equilibrium, we can find new consumption plans for the agents that are utility-equivalent to the equilibrium and quasi-equilibrium production plans, respectively (the resulting allocations might not be market clearing). These new plans will be comparable, in terms of utility, because they also have the characteristic of being utility-maximizers in their respective budget sets and that one of them or both will belong in the other's budget set.

Proof. First, note that given any quasi-equilibrium $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho})$, and for every agent i, there exists some alternative plan $(\bar{x}_*^i, \bar{\phi}_*^i, \bar{z}_*^i) \in B_\star^i(\bar{y}, p, \bar{q}, \bar{\rho})$, with $(\bar{\phi}_{*1}^{if}, \bar{z}_*^{if}, (\bar{\phi}_{*2s}^{if})_s) = (0, 0, (0, \cdots, 0))$.

To see this, note that the firm is a perfect competitor in the economy, so we know that agent i could choose a feasible plan $(\bar{x}_*^i, \bar{\phi}_*^i, \bar{z}_*^i) \in B_\star^i(\bar{y}, p, \bar{q}, \bar{\rho})$, such that $(\bar{\phi}_*^{if}, z_*^{if}) \leq 0$.

This alternative plan is feasible, so it satisfies the box constraint, $\bar{\phi}_{*1}^{if} + \bar{z}_*^{if} \geq 0$. This, together with $\bar{\phi}_{*1}^{if} \leq 0$ and $\bar{z}_*^{if} \leq 0$, imply $\bar{\phi}_{*1}^{if} = \bar{z}_*^{if} = 0$. The same for the second date when, again by feasibility, $\bar{\phi}_{*2s}^{if}$ satisfies the box constraint (5) and satisfies $\bar{\phi}_{*2s}^{if} \leq 0$, so that we actually have $\bar{\phi}_{*2s}^{if} = 0$.

That is, the box constraints imply that if a firm is a perfect competitor in the financial equilibrium, then every agent can dispense with not only long positions in the shares of the firm, but with all positions altogether. In other words, every agent can ignore the presence of the firm in the economy and not loose any utility by doing that.

Next, given the equilibrium production plan $y^f \in Y^f$, and an arbitrary production plan $\bar{y}^f \in Y^f$, from what we have just proved, for every agent there are plans (x_*^i, ϕ_*^i, z_*^i) and $(\bar{x}_*^i, \bar{\phi}_*^i, \bar{z}_*^i)$ such that $U^i(x_*^i) = U^i(x^i)$ and $U^i(\bar{x}_*^i) = U^i(\bar{x}^i)$.

If $\phi_0^{if} = 0$ we have that $(x_*^i, \phi_*^i, z_*^i) \in B^i(\bar{y}, p, \bar{q}, \bar{\rho})$ and that $(\bar{x}_*^i, \bar{\phi}_*^i, \bar{z}_*^i) \in B^i(y, p, q, \rho)$, that is, both plans are feasible at the prices corresponding to the other (quasi)equilibrium. This is true because prices and dividends associated to other firms do not change in the quasi-equilibrium, and because both plans exclude firm f from the agent's portfolio. This implies $U^i(\bar{x}_*^i) \geq U^i(x_*^i)$

and $U^i(\bar{x}^i_*) \leq U^i(x^i_*)$, that is, $U^i(\bar{x}^i) = U^i(\bar{x}^i_*) = U^i(x^i_*) = U^i(x^i)$. This proves part (b) of the theorem.

If an agent is an initial shareholder, we have that, from the definitions of financial equilibrium and quasi-equilibrium we know that $\pi_1^f(y^f) = p_1 y_1^f + q_1^f \geq p_1 \bar{y}_1^f + \bar{q}_1^f = \pi_1(\bar{y}^f)$. This implies that the agent's plan in the quasi-equilibrium is feasible at equilibrium prices: $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i(y, p, q, \rho)$. This in turn implies $U^i(\bar{x}^i) = U^i(\bar{x}^i_*) \leq U^i(x^i_*) = U^i(x^i)$. If $\pi_1^f(y^f) < \pi_1(\bar{y}^f)$ the strict preference of x^i over $\bar{x}^i$ comes from the monotonicity of preferences that we have assumed.

□

Shareholders are the agents with a positive long position on the firm's stock. As we mentioned before, an agent that has borrowed the share has, while the share is in his possession all voting rights associated with it. However, the usual practice is that the share will be returned to its original owner before a vote. See for example Aggarwal et al. (2015)[17].

From the proof of Theorem 1, it is clear that, if an agent held an initial short position ($\phi_0^{if} < 0$) he would, in fact, prefer if the firm implemented the plan that minimizes its present value. By construction, this is not possible in our model with three dates because all initial positions are nonnegative, but it will be possible when we extend the model to accommodate more dates and consider the firms' decisions in the future, when agents' initial positions at date $t+1$ will be the positions they have chosen at the end of t.

An interesting question in an economy with $T > 3$ dates, is whether, at dates $t > 1$, *new shareholders would agree with initial shareholders or would prefer that the firm implemented a different production plan.* To answer this question we need to introduce the following assumption:

[17] Notice that if not all shares borrowed were being returned on time to the original owners before a vote was taken, it would be possible to have more security long positions at that time than the fixed net supply (the 100% ownership) but the voters would be those in possession of the latter (possibly borrowers doing a short sale and, therefore, more interested in minimizing the firm's present value instead, which shows why it is important and a prevailing practise to return shares before voting takes place). Notice that our results address the unanimity of initial shareholders, assumed to have positive endowments of the shares.

Assumption 3. *If* $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho})$ *is a quasi-equilibrium given firm* f*'s production plan* $\bar{y}^f$*, then at every node* e *in the event tree there is some agent* i *who:*

1. *is not an initial shareholder:* $\bar{\phi}_0^{if} = 0$*,*

2. *has a non-binding box constraint at node* e*.*

We will refer to agents that satisfy assumption 3 as *reference agents*. Note that there are always agents satisfying the second part of the assumption. This follows from aggregation of the box constraints and the market clearing conditions[18]. The first part of the assumption is also likely to be satisfied in reality because, although this is not imposed in our model, the securities lending market's supply and demand sides tend to be segmented. On the supply side we have *beneficial owners*. These include pension funds, insurance companies, mutual funds, endowments and foundations and sovereign wealth funds. On the other hand, the demand side is mainly comprised of hedge funds and proprietary traders[19]. Agents on the demand side are unlikely to be initial shareholders.

Assumption 3 only imposes the extra requirement that at least one of the agents with non-binding box constraint is also not an initial shareholder.

As we will see in Theorem 3, reference agents play a fundamental role in our model. These agents' marginal rates of substitution can be used to compute the price of the firm's shares both in and out of equilibrium (that is, in any other quasi-equilibrium).

Consider an economy with $T > 3$ dates in which securities lending markets open in the first $T-2$ dates. There is uncertainty between dates t and $t+1$ for $t = 1, 2, \cdots, T-2$ and there is no uncertainty between dates $T-1$ and T. All definitions and assumptions are to be extended in the natural way.

Let $(x, \phi, z, y, p, q, \rho) \in FE$ be a financial equilibrium and let $D(e) = \{u \geq e\}$ be the sub-tree of events with root e. Define the economy $\mathcal{E}(x, \phi, z, y)(e)$ as an economy in which the

[18] Take for example node $e = 1$, we have $\sum_i(\phi_1^{if} + \theta^{if} - \psi^{if}) = \sum_i \phi_1^{if} + \left(\sum_i \theta^{if} - \sum_i \psi^{if}\right) = 1 + 0 > 0$, so we must have $\phi_1^{if} + \theta^{if} - \psi^{if} > 0$ for at least one agent i.

[19] Proprietary trading occurs when a financial firm trades with its own money in search of profit, rather than trading with clients money and only making profit from commissions.

consumption sets of the agents are $X^i(e) = \{\bar{x}^i \in X^i | \bar{x}^i_u = x^i_u,\ \forall u \notin D(e)\}$, preferences of the agents remain the same as in the original economy, production sets of the firms satisfy $Y^f(e) = \{\bar{y}^f \in Y^f | \bar{y}^f_u = y^f_u,\ \forall u \notin D(e)\}$ for all firms, and the available portfolios for each agent i are in the set $\{(\bar{\phi}^i, \bar{z}^i) | \bar{\phi}^i_u = \phi^i_u, \bar{z}^i_u = z^i_u,\ \forall u \notin D(e)\}$.

Theorem 2 (**Unanimity through time**). *Suppose that preferences satisfy assumption 1. Suppose also that $(x, \phi, z, y, p, q, \rho)$ is a Financial Equilibrium in an economy with $T > 3$ dates, and that firm f satisfies assumption 3 and is a perfect competitor in the truncated economy $\mathcal{E}(x, \phi, z, y)(e)$. Then*

1. *agents that immediately prior to node e had positive positions ($\phi^{if}_{e-1} > 0$), will unanimously agree on the production plan $y^f(e)$ that the firm should implement,*

2. *and $y^f(e)$ coincides with $y^f(1)$, the plan chosen by initial shareholders (those with $\phi^{if}_0 > 0$).*

Proof. The proof of Theorem 2 is given in the Appendix, after the proof of Theorem 3. □

3.3 Is competition compatible with Short Sales?

One important shortcoming in other models where shareholders' unanimity is obtained via some assumption regarding the competitiveness of firms, is that in those models there is a fundamental incompatibility between perfect competition and the short selling of the firm's shares. We have proven that this need not be the case when the services agents get by short selling the security are priced correctly in the securities lending market.

An agent wishing to short sell must be willing to pay a lending fee, that is, for the cash he posted as collateral, this agent must accept an interest rate (the rebate rate) which will sometimes differ from what his personal marginal rates of intertemporal substitution would predict. To see what this difference may be, let us look at the agent's first order condition on security borrowing:

$$(1 + \rho^f) = \frac{\lambda^i_1}{\sum_s \lambda^i_{2s}} - \frac{\mu^{if}_1 - \nu^{if}_+ + \nu^{if}_-}{H^f q^f_1 \sum_s \lambda^i_{2s}},$$

where λ_e^i and μ_e^{if} are the multipliers of node e's budget and box constraint, respectively and ν_+^{if}, ν_-^{if} are the multipliers of the upper and lower bounds on z^{if}, respectively. The first term on the right hand side is what the agent would normally expect to receive when lending cash. The second term, measures the possession value of the security. Possession has value because it enables the agent to short-sell the security. The result is that if the agent is interested in short-selling, he potentially accepts an interest rate lower than his valuation of the cash flow corresponding to the loan and the difference is the cost the agent pays for the benefits of short-selling (see Section 4.1).

To illustrate that there is no incompatibility between unanimity of shareholders and short sales, we present an example adapted from Makowski (1983). In his economy, naked short sales were the reason why the thesis of Theorem 1 did not hold. Once we introduce securities lending markets, this thesis holds even when short sales are allowed, provided that firms are competitive (in both the stock and securities lending market).

Example 3 (On Makowski's counterexample).

3.1 A counterexample with naked short sales

This example is adapted from Makowski (1983). Consider an economy in which there are only two dates ($T = 2$) and no uncertainty ($S = 1$). There are 2 agents. One of the agents, denoted by j, has initial endowments $\phi_0^j = 0$, $\omega^j = (9, 2)$, the other agent, denoted by k, has endowments $\phi_0^k = 1$, $\omega^k = (3, 2)$. The production set is given by $Y = \{(0, 0), (-2, 1)\}$. Agents preferences are $U^j(x_1^j, x_2^j) = x_1^j + x_2^j$ and $U^k(x_1^k, x_2^k) = x_1^k + \xi x_2^k$, where $0 < \xi < 1/2$. Utilities are linear for convenience. As discussed in example 1, some linearity in agents' utilities is necessary if firms are to be perfect competitors[20]. Given the restriction on ξ, agent k will prefer to transfer as much consumption as possible from date 2 to date 1.

For the moment we will assume that naked short sales are allowed, by which we mean that

[20]There are two differences between this example and Example 2 , in Makowski (1983). First, Makowski does not specify a utility function, just shows some indifference curves. Second, in Makowski (1983) there are J agents of type j while in ours, $J = 1$. In Makowski (1983) the section of indifference curves that must be linear for the firm to be a perfect competitor, shrinks as J increases. In the limit, linearity is reduced to differentiability.

agents do not need to borrow in the securities lending market before short selling, that is, there are no securities lending markets and there is no box constraint.

In this economy, a quasi-equilibrium in which each agent consumes his initial endowment and there is no production, is supported by prices $p = (1, 1)$ and $q_1 = 0$. In this quasi-equilibrium, the present value of the firm is 0.

When we have production and the plan adopted by the firm is $y = (-2, 1)$, a quasi-equilibrium emerges where $x^k = (4, 0)$, $x^j = (6, 5)$, $\phi_1^k = -2$, $\phi^j = 3$. This quasi-equilibrium is supported by prices $p = (1, 1)$, $q_1 = 1$. Note that in this case, the present value of the firm is $\pi(y) = y_2 + q_1 = -2 + 1 = -1$.

Now that we know the present value of the firm in every quasi-equilibrium we can label the one with no production as the unique financial equilibrium for this economy because it is in this quasi-equilibrium that the firm's present value is maximized. Note that agent k, the initial shareholder, is strictly better off in the quasi-equilibrium with production while agent j attains the same level of utility in both quasi-equilibria. Also note that the firm is a perfect competitor in this economy[21]. This shows that the result in Theorem 1 cannot be extended to apply to naked short sales: The initial shareholder of a perfectly competitive firm is better off in the quasi-equilibrium where the present value of the firm is the lowest.

3.2 The example redone with securities lending markets.

We now consider the same economy with the addition of securities lending markets. When the production plan is $(0, 0)$ the unique equilibrium is, as before, one in which each agent consumes his initial endowment and $q_1 = 0$ so the value of the firm is 0.

More interesting, when the production plan is $(-2, 1)$ it is possible to transfer consumption from one date to another, something that benefits agent k because he prefers to transfer as much consumption as possible from date 2 to date 1.

Although there may be multiple quasi-equilibria when $y = (-2, 1)$, we will restrict ourselves

[21]To obtain the correct definition of perfectly competitive firm in an economy without securities lending markets, the definition for perfectly elastic demand should be modified to refer only to long positions in the stock market, as there are no lending markets.

to studying only those in which agent j has a perfectly elastic demand for the firm's shares both in the stock and the securities lending market. That is, only to quasi-equilibria in which, if the agent holds a long position in the securities lending market, he does no worse and no better than when he consumes his initial endowment. This is the condition for perfectly elastic demand in this special case in which there is only one firm and there are no other securities with which the agent can substitute the firm's shares. We restrict our attention to this case because, otherwise, we already know that the hypotheses of Theorem 1 are not satisfied.

If agent k transfers as much consumption as possible from the second to the first date, and agent j has a perfectly elastic demand for the firm's shares both in the stock and securities lending markets, portfolios must be of the form $(\phi_1^k, z^k) = (\eta, -\eta)$ and $(\phi_1^j, \theta^j, \psi^j) = (1-\eta, \eta)$, where $\eta \geq 0$ satisfies the condition:

$$0 < \eta = \min\left\{\frac{2}{H-1}, M\right\}. \tag{11}$$

This quasi-equilibrium is supported by prices $q_1 = 1 + \rho = 1$ and consumption is $(x_1^k, x_2^k) = (2 + \eta(H-1), 2 - \eta(H-1))$ and $(x_1^j, x_2^j) = (8 - \eta(H-1), 3 + \eta(H-1))$.

Table 1 summarizes some variables of interest in the two quasi-equilibria we have found:

QE	y	$U^k(x^k)$	$U^j(x^j)$	$\pi(y)$
1	$(0,0)$	$3+2\xi$	11	0
2	$(-2,1)$	$2(1+\xi)+\eta(H-1)(1-\xi)$	11	-1

Table 1: Quasi-equilibria with securities lending markets

Suppose $M > 2/(H-1)$. Here are the important facts:

- From equation (11) we have that $\eta = 2/(H-1)$.
- The initial shareholder's (agent k) utilities are $3 + 2\xi$ and 4 in quasi-equilibria 1 and 2, respectively. Since $\xi < 1/2$, the initial shareholder is better off in quasi-equilibrium 2.
- The firm's present value is greater in quasi-equilibrium 1 than in quasi-equilibrium 2.

According to Theorem 1, if the firm is a perfect competitor, initial shareholders should be better off in the quasi-equilibrium where the value of the firm is maximized. This is not the case in this economy and the reason why this happens is that the firm is not a perfect competitor.

If the firm were a perfect competitor, every agent, and in particular agent k should be able to do just as well with a plan that does not include borrowing of the firm's shares ($z^k = 0$) as with plans that do include borrowing ($z^k > 0$). Since in this economy there are no other firms, whose shares could serve as substitutes, the firm is a perfect competitor only if every agent is indifferent between having the firm's shares in his portfolio or not.

When we add the securities lending market to Makowski's economy we end up with a counterexample but of a different kind. While Makowski's example showed the incompatibility of unanimity and short sales, ours shows how crucial the competitiveness assumption is to get the unanimity result of Theorem 1, which allows for short sales.

In this example, the initial shareholder (agent k) does not have a short position on the shares of the firm. If instead of borrowing the share in the securities lending market, agent k borrowed it in the repo market, he would actually have a short position but, again, the thesis in Theorem 1 would not hold due to the firm's lack of competitiveness (See Example 5 in the appendix).

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4 Pricing of securities

Our definitions of equilibrium and quasi-equilibrium require that agents have conjectures on how the price of the stock of a firm varies in response to a change in its production plan. Pricing securities is straightforward when markets are complete. In that case, marginal rates of substitution between future and present consumption are equalized across agents, and these rates can be used by any agent as deflators when computing the price of a security.

In the economy we are considering, where markets can be incomplete, marginal rates of substitution will generally differ across agents and so, their personal valuations of a given security may disagree. We will show that, under certain assumptions, some agents (the reference agents) will have the correct valuation of the firm's stock price in equilibrium, and the price computed using these agents deflators in any quasi-equilibrium will also be correct.

The following theorem characterizes the price of the stock of the firm, contingent on the firm's production decision.

Theorem 3 (Pricing of stock and of the rebate rates). *Let $(x, \phi, z, y, p, q, \rho)$ be a financial equilibrium. Suppose assumption 1 holds and that firm f satisfies assumption 3 and is a perfect competitor. Then, the following functions for date 1 and date-state $2s$, correctly compute the price of firm f's stock for any given production plan $\bar{y}^f \in Y^f$ that the firm decides to implement:*

$$\upsilon_1^f(\bar{y}^f) = \max_{i \in I} \sum_s \frac{\lambda_{2s}^i}{\lambda_1^i} \left(p_{2s}\bar{y}_{2s}^f + \upsilon_{2s}^f(\bar{y}^f) \right) \tag{12}$$

$$\upsilon_{2s}^f(\bar{y}^f) = \max_{i \in I} \frac{\lambda_{3s}^i}{\lambda_{2s}^i} \left(p_{3s}\bar{y}_{3s}^f \right) \tag{13}$$

Moreover, function $\tau^f(\bar{y}^f)$ correctly computes the rebate rate:

$$\tau^f(\bar{y}^f) = \max_{i \in I} \sum_s \frac{\lambda_{2s}^i}{\lambda_1^i} - 1 \tag{14}$$

These functions use agent i's marginal utilities of income (λ_e^i), in the financial equilibrium.

Lets study closely what the theorem says. For any FE, there are agents i_1 and i_2 for which the maxima on the right hand side of (12) and (13) are attained. The expressions in Theorem 3 give us the maximum personal valuation, at date t, of the stream of dividends paid by the security at date $t+1$. Given assumption 3, this maximum valuation will always correspond to reference agents as these agents have null shadow prices for the box constraint. Most remarkably, the theorem says that we can use the deflators obtained in equilibrium, when the firm's production plan is y^f, to compute the price of the firm's stock in any other quasi-equilibrium in which the firm chooses to produce $\bar{y}^f$ instead.

Another remarkable feature of this pricing function that differentiates it from others in the literature is that, sometimes, the correct valuation of the share is obtained by using as deflators the marginal rates of substitutions of agents with short positions on the share. The reason for this is that reference agents can be both shareholders or shareborrowers (possibly holding short positions on the share). In other price functions appearing in the literature, the correct price of the shares is computed using a weighted average of *shareholders'* valuations but never considering short-sellers' valuations. In Grossman and Hart (1979), the correct price of the stock is given by a weighted average of initial shareholders' personal valuations in which the weights, are given by the proportion of total shares each agent holds. Diamond (1967) and Dreze (1974) propose a weighted average of new shareholders instead. A version of the pricing function given in Theorem 3 in which only shareholders can act as reference agents appears in Makowski (1983) and, more recently, when short sales are not allowed, in Bisin et al. (2014) and in Acharyaa and Bisin (2014).

We should point out that there is another important difference between our work and that in Bisin et al. (2014), where, when short sales are not allowed, the pricing function in Theorem 3 is assumed to be correct[22] and, from that, unanimity of shareholders follows. However, unanimity is not the main focus of their paper. We prove the converse implication: we characterize perfect competition and, *from that*, both unanimity and the pricing function in Theorem 3 follow.

To see that this implication is not at all vacuous we give a counterexample. In general, it is not *rational* to expect the pricing function defined in Theorem 3 to be correct (not even in the simpler case when short sales are not allowed). Given assumption 3, this pricing function will always be correct for the financial equilibrium production plans but, if the firm is not a perfect competitor or, if there are not reference shareholders in the economy, it may be incorrect for other production plans, as the following example shows.

Example 4. Consider an economy with one good ($C = 1$), one firm ($F = 1$) and three agents ($I = 3$) which are denoted j, k, l. There are two dates and two states at the second date. The firm chooses one of two possible production plans: $Y = \{y^1, y^2\} = \{(-1, 1, 1/6), (-1, 1/6, 1)\}$. Agent l is the only initial owner of the firm at date one $\phi_0^l = 1$, $\phi_0^j = \phi_0^k = 0$. Initial endowments of the commodity are as follow: $(\omega_1^l, \omega_{21}^l, \omega_{22}^l) = (0, 1, 1)$, $(\omega_1^j, \omega_{21}^j, \omega_{22}^j) = (3, 2, 1)$, $(\omega_1^k, \omega_{21}^k, \omega_{22}^k) =$

[22]In Bisin et al. (2014), short sales are not modelled using securities lending markets. Instead, financial intermediaries who can issue claims corresponding to short positions on the firm's stock is introduced and, a pricing function different from the one in Theorem 3 is used by firms in this environment.

$(3,1,2)$. For simplicity, suppose there are no markets for borrowing shares[23].

The utility functions of the agents are $U^l(x_1^l, x_{21}^l, x_{22}^l) = 10{x_1^l}^{\frac{9}{10}} + {x_{21}^l}^{\frac{9}{10}} + {x_{22}^l}^{\frac{9}{10}}$, $U^j(x_1^j, x_{21}^j, x_{22}^j) = {x_1^j}^{\frac{9}{10}} + 2{x_{21}^j}^{\frac{9}{10}} + \varepsilon^j {x_{22}^j}^{\frac{9}{10}}$ and $U^k(x_1^k, x_{21}^k, x_{22}^k) = {x_1^k}^{\frac{9}{10}} + \varepsilon^k {x_{21}^k}^{\frac{9}{10}} + 2{x_{22}^k}^{\frac{9}{10}}$, where $\varepsilon^j = 12(3^{\frac{1}{10}} - 1)(7/18)^{\frac{1}{10}}$ and $\varepsilon^k = 6\left[(399/200)/(201/200)^{\frac{1}{10}} - 2/3^{\frac{1}{10}}\right](7/6)^{\frac{1}{10}}$.

All utilities are increasing in consumption in every date-state so agents will consume whatever they have available for consumption in that date-state. The only way to transfer consumption from date one to date two is by buying the share of the firm. Since agent l values consumption at date 1 much more than consumption at date 2, we will suppose (correctly) that he will sell his share and use the proceeding for consumption. Considering this, agent l's consumption will be $(x_1^l, x_{21}^l, x_{22}^l) = ((q-1), 1, 1)$.

Agents $i \in \{j, k\}$ will consume according to how much of the share they buy: $(x_1^i, x_{21}^i, x_{22}^i) = (\omega_1^i - q\phi_1^i, \omega_{21}^i + \phi_1^i y_{21}, \omega_{22}^i + \phi_1^i y_{22})$.

A quasi-equilibrium for this economy, when $y = y^1 = (-1, 1, 1/6)$ is supported by the price $q = 2$. Agents' positions on the firm's share are $\phi_1^l = \phi_1^k = 0$, $\phi_1^j = 1$. The corresponding allocations are: $(x_1^l, x_{21}^l, x_{22}^l) = (1,1,1)$, $(x_1^j, x_{21}^j, x_{22}^j) = (1, 3, 7/6)$ and $(x_1^k, x_{21}^k, x_{22}^k) = (3,1,2)$. With these allocations and prices, markets clear and each agent maximizes his utility. The present value of the firm at $t = 1$ is $y_1 + q = 1$.

When the production plan adopted by the firm is $y = y^2 = (-1, 1/6, 1)$, a quasi-equilibrium is supported by the price $q = 399/200$. Agents' positions on the firm's share are $\phi_1^l = \phi_1^j = 0$, $\phi_1^k = 1$. Agents' consumption is $(x_1^l, x_{21}^l, x_{22}^l) = (1,1,1)$, $(x_1^j, x_{21}^j, x_{22}^j) = (3,2,1)$ and $(x_1^k, x_{21}^k, x_{22}^k) = (201/200, 7/6, 3)$. With these allocations, markets clear and each agent maximizes his utility. In this case, the present value of the firm at $t = 1$ is $y_1 + q = 199/200 < 1$.

Now that we know that the quasi-equilibrium with production plan $y = y^1$ maximizes present value of the firm, we can label it as the (unique) financial equilibrium. It is easy to see that pricing function $\upsilon(\cdot)$, defined in Theorem 3, correctly computes the price of the share in equilibrium.

[23] The definition for perfectly elastic demand should be modified to refer only to long positions in the security market and exclude long positions in the securities lending market. In this example, a firm will be a perfect competitor only if it is so in the sense of Makowski (1983).

First, we must compute the deflators derived from the multipliers of the agents' problem *in the financial equilibrium*: $\lambda^l_{21}/\lambda^l_1 = (1/10)$, $\lambda^l_{22}/\lambda^l_1 = (1/10)$, $\lambda^j_{21}/\lambda^j_1 \approx 1.7919$, $\lambda^j_{22}/\lambda^j_1 \approx 1.2484$, $\lambda^k_{21}/\lambda^k_1 \approx 1.3743$, $\lambda^k_{22}/\lambda^k_1 \approx 2.0827$.

With these deflators in hand, we can compute the share's price from the Theorem 3's function: $\upsilon(y^1) = \max_{i \in I} \left\{(\lambda^i_{21}/\lambda^i_1)y^1_{21} + (\lambda^i_{22}/\lambda^i_1)y^1_{22}\right\} = \max\{(1/10)(1) + (1/10)(1/6), (1.7919)(1)$ $+ (1.2484)(1/6), (1.3743)(1) + (2.0827)(1/6)\} = \max\{0.11\overline{66}, 2, 1.7214\} = 2$.

This is not surprising at all since the pricing function is derived from the first order conditions of the agents' problem *in equilibrium.* Note that the correct price corresponds to the valuation of the reference agent[24], which in this case is agent j: $\upsilon(y^1) = (\lambda^j_{21}/\lambda^j_1)y^1_{21} + (\lambda^j_{22}/\lambda^j_1)y^1_{22} = 2$.

However, if we were to use $\upsilon(\cdot)$ to compute the price of the share in the quasi-equilibrium we would get: $\upsilon(y^2) = \max_{i \in I} \left\{(\lambda^i_{21}/\lambda^i_1)y^2_{21} + (\lambda^i_{22}/\lambda^i_1)y^2_{22}\right\} = \max\{(1/10)(1/6) + (1/10)(1),$ $(1.7919)(1/6)+(1.2484)(1), (1.3743)(1/6)+(2.0827)(1) = \max\{0.11\overline{66}, 1.5471, 2.3118\} \approx 2.3118 \neq 399/200$.

So the function $\upsilon(\cdot)$ is incorrect in the quasi-equilibrium. Note that the assumptions of Theorem 3 are not satisfied. In particular, the firm is not a perfect competitor in the financial equilibrium.

4.1 Perfect competition, possession value and specialness

In our discussion of examples 1 and 2 we pointed out that, in both examples, null shadow prices for the agents' constraints involving firm f's shares, are enough to guarantee that the firm is a perfect competitor in the economy. Null shadow prices imply, in particular, an absence of *possession value* for the firm's shares. Let us be more precise.

Suppose first that there were no bounds on securities borrowing and lending. A positive shadow price μ^{if}_e for the box constraint, of firm f's shares at node e, can be interpreted either as the utility a shares borrower would gain if he were allowed to short-sell more than the amount

[24] In a model without short sales, a reference agent is simply a buyer of the share who is not an initial shareholder.

he has borrowed, or as the utility that a shares lender would gain if he were allowed to lend more than what he has in his possession. In both of these interpretations, the agent is limited in his short-selling or in his lending by his (limited) possession of the share. This makes the possession of the share valuable for the agent.

If in addition there is a bound on shares borrowing, the looser is this bound, the lower will be the value attached to the possession of the borrowed shares. On the other hand, in the presence on a bound on shares lending, the looser is the bound, the higher will be the value attached to the possession of shares that can be lent. That is, the possession value for firm f's shares is defined as $\mu_1^{if} - \nu_+^{if} + \nu_-^{if}$, where ν_+^{if} and ν_-^{if} are the multipliers for the restrictions $z^{if} \leq M$ and $z^{if} \geq -M$, respectively.

Another context where null shadow prices are related to unanimity of shareholders can be found in Carceles-Poveda and Coen-Pirani (2009). In that article, short sales are naked (there are no securities lending markets) and bounded exogenously. Null shadow prices (but for different constraints) together with a specific technology are assumed by these authors in order to derive their unanimity result.

In example 1 the *no possession value* follows from the fact that the rebate rate is equal to the yield rate of a one-period real bond $\rho = \rho_b$. This makes the shares of the firm substitutable by trading the price-indexed bond. In example 2, when all shadow prices are null, the shares of firm 1 can be replaced by a portfolio consisting of shares of firms 2 and 3.

While in the above examples absence of possession value implied that the firm is a perfect competitor, the converse will be established in Theorem 4: possession value for the firm's shares must be null when the firm is perfectly competitive.

When a share is *General Collateral* (GC) there is no particular demand to borrow it or special desire to lend it more than any other share. General collateral securities are substitutable among them and are traded at the common rate ρ^{GC}, the *General Collateral Rate.* General collateral securities are high quality and liquid. The GC rate can be thought of as being driven by the supply and demand for cash and not by the supply and demand for individual assets. It is therefore reasonable to assume that, at least for some agents, the GC has no possession value, as follows:

Assumption 4. *For any quasi-equilibrium* $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho})$ *induced by production plan* $\bar{y}^f$ *of firm* f*, there exist at least one non-initial shareholder* $\tilde{i}$ *(*$\bar{\phi}_0^{\tilde{i}f} = 0$*) for whom*

$$(1 + \bar{\rho}^{\mathrm{GC}}) = \frac{\bar{\lambda}_1^{\tilde{i}}}{\sum_s \bar{\lambda}_{2s}^{\tilde{i}}},$$

where $\bar{\lambda}_e^{\tilde{i}}$ *are the multipliers corresponding to node* e*'s budget constraint in agent* $\tilde{i}$*'s problem.*

A security is said to be trading *on special* on the securities lending market, whenever the rebate rate associated with the security is below the corresponding GC rate. In other words, a security trades on special if agents pay a higher lending fee when borrowing it than when they borrow general collateral.

We will first note that a perfectly competitive firm can never influence the general collateral rate, that is, ρ^{GC} will be the same regardless of the production plan implemented by the firm (see Lemma 3 in the Appendix).

Second, we note that the actions by a perfectly competitive firm can never determine whether its shares trade on special or not.

Lemma 1 (**A competitive firm cannot induce specialness**)**.** *Suppose assumptions 1 and 4 hold, and that firm* f *is a perfect competitor. Then the firm's production plans cannot affect whether the firm's shares are on special or not.*

The next result takes a step forward. It says that it is enough to have one agent not interested in re-using the share in some quasi-equilibrium, to infer that the share is not on special (in any quasi-equilibrium and in particular in the financial equilibrium).

Lemma 2. *Under assumption 1, given a quasi-equilibrium* $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho})$ *for some production plan* $\bar{y}^f$*, and agent* $\tilde{i}$ *as in assumption 4, we can say that* ***for any quasi-equilibrium*** $(\tilde{x}, \tilde{\phi}, \tilde{z}, p, \tilde{q}, \tilde{\rho})(\tilde{y}) \in QE(\tilde{y}^f)$ *we have: If* $0 < \bar{z}^{\tilde{i}f} < M$ *and* $\bar{\phi}_1^{\tilde{i}f} = 0$ *, then* $\tilde{\rho}^f = \rho^{\mathrm{GC}}$.

We are now ready to state an important necessity result. Assuming that the null vector can always be chosen as a production plan we can construct the corresponding quasi-equilibrium (with $\tilde{q}_1^f = 0$) as in Lemma 2 and infer that the shares of a perfectly competitive firm are never on special.

Theorem 4 (**Absence of specialness is necessary for perfect competition**). *Under assumptions 1 and 4, if* $0 \in Y^f$*, and firm* f *is perfectly competitive, then in equilibrium (and in any quasi-equilibrium) the firm's shares are not on special (*$\tilde{\rho}^f = \rho^{\mathrm{GC}}$*).*

(The proof follows from Lemma 2, as already explained).

5 Concluding remarks

This paper contributes to two strands of the literature on general equilibrium with incomplete markets. The first is an old strand concerned with the relation between shareholders' unanimity and perfect competition in the markets where the shares of the firm are traded. The second is a recent attempt to capture in a GEI framework the increasing role of Securities Financing Transactions (SFT), which are transactions where securities are used to borrow cash (as in repo markets), or cash is used to borrow securities (as in securities lending markets).

We have proven that, if a firm is a perfect competitor in both the stock and the securities lending markets, its shareholders will be unanimous in their selection of the production plan. Previously, it was believed that short sales and shareholders' unanimity where incompatible, in models where unanimity was driven by perfect competition. We have shown that this is not the case once short sales are modelled the way they are actually done in reality (by borrowing first the shares), and not in the naive way of previous models. Under such reformulation of short sales, what was a counterexample in Makowski (1983) no longer illustrates that unanimity driven by perfect competition is incompatible with allowing for short sales (see Example 3).

The concept of a perfectly competitive firm may appear to be abstract but we relate it to the simpler concept of *no specialness*, which refers to observable variables. The shares of the firm are said to be *on special* if the rebate rate at which they are borrowed is below the general collateral rate. Our results on the relation between possession value, specialness and perfect competition provide important insights that were absent in the previous literature. First, a perfectly competitive firm can never influence whether or not the firm's shares trade on special. Second, assuming that inaction is a possible production choice for the firm, *no specialness* becomes a necessary condition for the firm to be a perfect competitor.

Conversely, we illustrate in Examples 1 and 2 that when the shadow values of all constraints involving the firm's shares are null (which in particular implies no possession value) the firm becomes a perfect competitor.

Finally, Example 4 serves as an important reminder that Makowski's criterion for pricing securities should not be used if the firm is not a perfect competitor. Otherwise, the pricing function shown in Theorem 3 will be correct when the firm selects the plan that maximizes its present value but will otherwise be incorrect.

A Appendix

This appendix contains proofs and computations for some of the results and examples.

Proof of Proposition 1. Agents' borrowing positions are bounded by (1). This, together with the box constraint implies $\phi_1^{if} \geq -M$ for every i and f. Also, market clearing allocations must satisfy $\phi_1^{if} \leq 1 + IM$. Since, at date 2 there is no borrowing of securities, attainable allocations satisfy $\phi_{2s}^{if} \leq 1$. Finally, agents' consumption is bounded from bellow by 0 and from above by $x_e^i \leq \omega_e^i + \sum_f y_e^f$, for all i and all e. This proof follows that of Proposition 1 in Bottazzi, Luque and Páscoa (2012) with some minor adjustments. We consider a sequence of auxiliary economies in which the box constraint and the lower bound on securities lending positions are relaxed. Step 1 of the proof must be modified as follows.

Rebate rates are decided at date 1, when borrowing contracts are negotiated. Let $R^f \equiv 1/(1+\rho^f)$. Let $\tilde{z}^{if} \equiv (1+\rho^f)q_1^f z^{if}$. The agent's (relaxed) budget constraints can be rewritten as:

$$p_1 x_1^i + \sum_f q_1^f \phi_1^{if} + \sum_f R^f H^f z^{if} \leq p_1 \omega_1^i + \sum_f \phi_0^{if}(q_1^f + p_1 y_1^f) \tag{15}$$

$$q_1^f \phi_1^{if} + R^f \tilde{z}^{if} \geq -1/n \ , \forall f \tag{16}$$

$$R^f \tilde{z}^{if} \leq q_1^f M + 1/n \ , \forall f \tag{17}$$

$$R^f \tilde{z}^{if} \geq -(q_1^f M + 1/n) \ , \forall f \tag{18}$$

$$p_{2s} x_{2s}^i + \sum_f q_{2s}^f \phi_{2s}^{if} \leq p_{2s}\omega_{2s}^i + \sum_f \phi_1^{if}(q_{2s}^f + p_{2s} y_{2s}^f) + \sum_f H^f \tilde{z}^{if} \tag{19}$$

$$\phi_{2s}^{if} \geq -1/n \ , \forall f \tag{20}$$

$$p_{3s} x_{3s}^i \leq p_{3s}\omega_{3s}^i + \sum_f \phi_{2s}^{if} p_{3s} y_{3s}^f \tag{21}$$

Equations (15), (19) and (21) are the budget constraints with modified variables at nodes 1, $2s$ and $3s$, respectively. Equations (16) and (20) are the new box constraints, after the change of variables, and being relaxed. With the change of variables, the relaxed bounds on borrowing positions are given by (17) and (18). Finally, we must have $x_e^i \geq 0$, $\forall e$.

Auctioneer at date 1 selects $(p_1, q_1, R) \in \Delta^{L+2F}$ to maximize:

$$p_1 \left(\sum_i (x_1^i - \omega_1^i) - \sum_f y_1^f \right) + q_1 \sum_i (\phi_1^i - \phi_0^i) + \sum_f R^f H^f \sum_i \tilde{z}^{if}$$

Auctioneer at node $2s$ selects $(p_{2s}, q_{2s}) \in \Delta^{L+F}$ to maximize:

$$p_{2s} \left(\sum_i (x_{2s}^i - \omega_{2s}^i) - \sum_f y_{2s}^f \right) + q_{2s} \sum_i (\phi_{2s}^i - \phi_1^i)$$

Finally, auctioneer at node $3s$ selects $(p_{3s}) \in \Delta^L$ to maximize

$$p_{3s} \left(\sum_i (x_{3s}^i - \omega_{3s}^i) - \sum_f y_{3s}^f \right) + q_{3s} \sum_i (\phi_{3s}^i - \phi_{2s}^i)$$

In step 2 of the proof we need to modify the argument showing the lower semicontinuity of the consumers' constraint correspondences. This is achieved by finding an interior point in the correspondence for any given set of prices. Let

$$d_{ts} \equiv \begin{cases} 1, \text{ if } p_{ts} y_{ts}^f \geq 0 \ \forall f \\ -\min_f \{p_{ts} y_{ts}^f\}, \text{ otherwise} \end{cases} \quad \text{and} \quad g_{ts} \equiv \begin{cases} 1, \text{ if } p_{ts} y_{ts}^f \leq 0 \ \forall f \\ \max_f \left\{ p_{ts} y_{ts}^f \right\}, \text{ otherwise} \end{cases}$$

If $p_1 \neq 0$. Let $x^i = 0$, $\tilde{z}^{if} = 0$, $\phi_{2s}^{if} = 0$, and

$$\phi_1^{if} = \frac{1}{2F} \min \left\{ p_1 \left(\omega_1^i + \sum_f \phi_0^{if} y_1^f \right), \left(\frac{p_{2s} \omega_{2s}^i}{d_{2s}} \right)_s \right\}$$

If $p_1 = 0$, $q_1 \neq 0$, then there is some firm g such that $q_1^g \neq 0$. Let $x^i = 0$; $\tilde{z}^{if} = 0$, $\forall f$; and let

$$\phi_{2s}^{if} = -\frac{1}{2F} \min \left\{ \frac{p_{3s} \omega_{3s}^i}{g_{3s}}, \frac{F}{n} \right\} \quad \text{and} \quad \phi_1^{if} = -\frac{1}{2F} \min \left\{ \left(\frac{p_{2s} \omega_{2s}^i}{g_{2s}} \right)_s, \left(-F \phi_{2s}^{if} \right)_s \right\}$$

If $p_1 = q_1 = 0$, $R^f \neq 0$. Let $x^i = 0$, $\phi_{2s}^i = 0$. Also, let

$$\phi_1^{if} = \frac{1}{2F} \min\left\{ \left(\frac{p_{2s}\omega_{2s}^i}{d_{2s}} \right)_s \right\} \text{ and } H^f \tilde{z}^{if} = -\frac{1}{2F} \min\left\{ \left(p_{2s}\omega_{2s}^i + \sum_f \phi_1^{if}(p_{2s}y_{2s}^f + q_{2s}^f) \right)_s, \frac{F}{n} \right\}$$

The last modification that we must do to the proof of Proposition 1 in Bottazzi, Luque and Páscoa (2012) is the argument that bounds from below ${}^n\hat{R}^f$, the renormalized rebate rates corresponding to the n-th economy in the sequence (the one in which the budget constraints have been relaxed by subtracting or adding $1/n$). The agent's first order conditions, with respect to $\tilde{z}^{if}$, in the n-th economy in the sequence give us:

$$(22) \qquad {}^n\hat{R}^f = \sum_s \frac{{}^n\lambda_{2s}^i}{{}^n\lambda_1^i} + \frac{({}^n\mu_1^{if} - {}^n\nu_+^{if} + {}^n\nu_-^{if})\,{}^n\hat{R}^f}{{}^n\lambda_1^i H^f}$$

Where ${}^n\lambda_e^i$ are the Lagrange multipliers corresponding to node e's budget constraint, ${}^n\mu_1^{if}$ the multipliers corresponding to security f's box constraint at date 1, ${}^n\nu_+^{if}$ and ${}^n\nu_-^{if}$ the multipliers corresponding to the upper and lower bounds on borrowing positions. In equilibrium there must always be an agent i such that ${}^n\nu_+^{if} = 0$, so (22) implies that ${}^n\hat{R}^f \geq \sum_s \frac{{}^n\lambda_{2s}^i}{{}^n\lambda_1^i}$.

□

Computational details for Example 1. Let $\lambda_e^i \geq 0$ denote the multiplier for the budget constraints at node e, $\mu_e^{if} \geq 0$ the multipliers for the corresponding box constraint, $\underline{\nu}^{if} \geq 0$ and $\overline{\nu}^{if} \geq 0$ the multipliers corresponding to the lower and upper bound on z^{if}, respectively. In equilibrium, the following equations, (obtained from the first order conditions with respect to the variables ϕ_{2s}^{if}, ϕ_1^{if}, z^{if}, and b^i), must hold:

$$(23) \qquad q_{2s}^f = \frac{\lambda_{3s}^i}{\lambda_{2s}^i}\left(y_{3s}^f\right) + \frac{\mu_{2s}^{if}}{\lambda_{2s}^i}, \forall i$$

$$(24) \qquad q_1^f = \sum_s \frac{\lambda_{2s}^i}{\lambda_1^i}\left(y_{2s}^f + q_{2s}^f\right) + \frac{\mu_1^{if}}{\lambda_1^i}, \forall i$$

$$(25) \qquad (1+\rho^f) = (1+\rho_b) - \frac{\mu_1^{if}}{q_1^f H^f \sum_s \lambda_{2s}^i} + \frac{(\overline{\nu}^{if} - \underline{\nu}^{if})}{q_1^f H^f \sum_s \lambda_{2s}^i}, \forall i$$

$$(26) \qquad (1+\rho_b) = \frac{\lambda_1^i}{\sum_s \lambda_{2s}^i}, \forall i$$

If $\rho^f = \rho_b$, we see that equation (25) implies $\mu_1^{if} + \underline{\nu}^{if} = \overline{\nu}^{if}$. If $z^{if} < M$ then $\overline{\nu}^{if} = 0$, which implies $\mu_1^{if} = 0$ and $\underline{\nu}^{if} = 0$. In this case, equations (24) and (25) imply:

$$\sum_s \frac{\lambda_{2s}^i}{\lambda_1^i} \left((1 + \rho^f) q_1^f - \left(y_{2s}^f + q_{2s}^f \right) \right) = 0 \tag{27}$$

Substituting $\lambda_{ts}^i = \beta^{t-1} \sigma_{ts}^i$ in equation (27), we have:

$$\sum_s \beta \sigma_{2s}^i \left(\left(1 + \rho^f\right) q_1^f - \left(y_{2s}^f + q_{2s}^f \right) \right) = 0 \tag{7}$$

□

Example 5 (Makowski's economy with repo markets). We only consider the case analogous to Example 3.2 when the production plan is $(-2, 1)$. Since we are dealing with repo markets and are interested only in the case when agent k is able to transfer consumption from date 2 to date 1, we must have $H < 1$. Again, we restrict ourselves to studying only those quasi-equilibria where agent j has a perfectly elastic demand for the firm's shares both in the stock and repo market. If agent k transfers as much consumption as possible from the second to the first date, and agent j has a perfectly elastic demand for the firm's shares both in the stock and repo markets, portfolios must be of the form $(\phi_1^k, z^k) = (-\eta, \eta)$ and $(\phi_1^j, \theta^j, \psi^j) = (1 + \eta, -\eta)$, where $\eta \geq 0$ satisfies the condition $0 < \eta = \min \left\{ \frac{2}{1-H}, M \right\}$. This quasi-equilibrium is supported by prices $q_1 = 1 + \rho = 1$ and consumption is $(x_1^k, x_2^k) = (2 + \eta(1 - H), 2 - \eta(1 - H))$ and $(x_1^j, x_2^j) = (8 - \eta(1 - H), 3 + \eta(1 - H))$.

The following table summarizes the two quasi-equilibria:

QE	y	$U^k(x^k)$	$U^j(x^j)$	$\pi(y)$
1	$(0, 0)$	$3 + 2\xi$	11	0
2	$(-2, 1)$	$2(1 + \xi) + \eta(1 - H)(1 - \xi)$	11	-1

Table 2: Quasi-equilibria with repo markets

Here, we arrive at the exact same conclusions as in Example 3.2, with the only difference that the initial shareholder is short in the firm's shares in quasi-equilibrium 2. Recall that this agent is transferring income from date 2 to date 1. In both cases (repo and securities lending) this is done through the presence of $q(\phi^i + Hz^i) < 0$ in the budget equation, where $\phi^i = -z^i$ from the box constraint. In repo $H < 1$ and we should have $\phi^i < 0$ and $z^i > 0$. In securities lending $H > 1$ and we should have $\phi^i > 0$ and $z^i < 0$.

Proof of Theorem 3. Let $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho}) \in QE(\bar{y}^f)$ and let $(\bar{\lambda}^i, \bar{\mu}^{if})$ be the multipliers corresponding to the budget and box constraints, when the optimal consumption plan chosen by the agent is $(\bar{x}^i, \bar{\phi}^i, \bar{z}^i)$. Since firm f is a perfect competitor in the financial equilibrium, there is some plan $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i_\star(\bar{y}, p, \bar{q}, \bar{\rho})$ with $\bar{\phi}^{if}_{*e} = 0 \ \forall e$ and $\bar{z}^{if}_* = 0$ (as we saw in the proof of Theorem 1). Since $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$ is utility maximizing, and given the concavity assumptions that we have made for the agents' utilities, $(\bar{\lambda}^i, \bar{\mu}^{if})$ also satisfy KT first order conditions at the point $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$. From assumption 1 and the first order conditions we have $\bar{\lambda}^i_e = (\partial U^i(\bar{x}^i_*)/\partial x^i_{ec})(1/p_{ec})$. Also, **in equilibrium**, the following relations between share prices and dividends are satisfied:

$$\text{(p.1)} \qquad q^f_{2s} = \frac{\lambda^i_{3s}}{\lambda^i_{2s}} \left(p_{3s} y^f_{3s}\right) + \frac{\mu^{if}_{2s}}{\lambda^i_{2s}} \ , \forall f$$

$$\text{(p.2)} \qquad q^f_1 = \sum_s \frac{\lambda^i_{2s}}{\lambda^i_1} \left(p_{2s} y^f_{2s} + q^f_{2s}\right) + \frac{\mu^{if}_1}{\lambda^i_1} \ , \forall f$$

Here (λ^i, μ^{if}) are the **equilibrium** multipliers corresponding to agent i's budget and box constraints in the financial equilibrium $(x, \phi, z, y, p, q, \rho)$. Note that the maximum valuation of the dividends, $\sum_s \frac{\lambda^i_{2s}}{\lambda^i_1} \left(p_{2s} y^f_{2s} + q^f_{2s}\right)$ and $\frac{\lambda^i_{3s}}{\lambda^i_{2s}} \left(p_{3s} y^f_{3s}\right)$, is realized for agents whose box constraint multipliers satisfy $\mu^{if} = 0$. As we saw when we proved Theorem 1, if $\phi^{if}_0 = 0$ then $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i_\star(y, p, q, \rho)$ and so, if $\phi^{if}_0 = 0$, (λ^i, μ^{if}) are also multipliers for $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$ and $\lambda^i_e = (\partial U^i(\bar{x}^i_*)/\partial x^i_{ec})(1/p_{ec}) = \bar{\lambda}^i_e$.

From assumption 3 there is some agent i such that $\bar{\phi}^{if}_0 = 0$ and such that $\bar{\phi}^{if}_{2s} > 0$. For this agent, $\bar{\mu}_{2s}^{if} = 0$ and

$$\bar{q}_{2s}^{f} = \frac{\bar{\lambda}_{3s}^{i}}{\bar{\lambda}_{2s}^{i}} \left(p_{3s} \bar{y}_{3s}^{f} \right)$$

But by our previous discussion, $\bar{\lambda}_{3s}^{i} = \lambda_{3s}^{i}$ and $\bar{\lambda}_{2s}^{i} = \lambda_{2s}^{i}$, so actually

$$\bar{q}_{2s}^{f} = \frac{\bar{\lambda}_{3s}^{i}}{\bar{\lambda}_{2s}^{i}} \left(p_{3s} y_{3s}^{f} \right) = \frac{\lambda_{3s}^{i}}{\lambda_{2s}^{i}} \left(p_{3s} y_{3s}^{f} \right) = \upsilon_{2s}(\bar{y})$$

Again, by assumption 3 there is some agent j such that $\bar{\phi}_0^{jf} = 0$ and such that $\bar{\phi}_1^{jf} + \bar{z}^{jf} > 0$. For this agent, $\bar{\mu}_1^{jf} = 0$ and

$$\bar{q}_1^{f} = \sum_s \frac{\bar{\lambda}_{2s}^{j}}{\bar{\lambda}_1^{j}} \left(p_{2s} \bar{y}_{2s}^{f} + \bar{q}_{2s}^{f} \right)$$

But by our previous discussion, $\bar{\lambda}_{2s}^{j} = \lambda_{2s}^{j}$ and $\bar{\lambda}_1^{j} = \lambda_1^{j}$, so actually

$$\bar{q}_1^{f} = \sum_s \frac{\bar{\lambda}_{2s}^{j}}{\bar{\lambda}_1^{j}} \left(p_{2s} \bar{y}_{2s}^{f} + \bar{q}_{2s}^{f} \right) = \sum_s \frac{\lambda_{2s}^{j}}{\lambda_1^{j}} \left(p_{2s} \bar{y}_{2s}^{f} + \bar{q}_{2s}^{f} \right) = \upsilon_1(\bar{y})$$

□

Proof of Theorem 2. If an agent i is an initial shareholder in node e, that is, if $\phi_{e-1}^{if} > 0$, and if firm f is a perfect competitor in the truncated economy $\mathcal{E}(x, \phi, z, y)(e)$, then the same arguments as in the proof of Theorem 1 apply. In that case, if $\pi_e^f(y^f) \equiv p_e y_e^f + q_e^f \geq p_e \bar{y}_e^f + \bar{q}_e^f = \pi_e^f(\bar{y}^f)$, for all $\bar{y}^f \in Y^f(e)$, then $U^i(x^i) \geq U^i(\bar{x}^i)$.

What needs to be proven then, is that $y^f \in \operatorname{argmax}_{\tilde{y}^f \in Y^f(e)} \pi_e^f(\tilde{y}^f)$, where y^f is the plan chosen by initial shareholders at date 1. The only difficulty in proving this lies in the way in which share prices might change in the truncated economies. It will be enough to prove that if e is a node at time t and e' is a successor of e at time $t+1$, then $\pi_e^f(y^f) \geq \pi_e^f(\bar{y}^f)$ implies $\pi_{e'}^f(y^f) \geq \pi_{e'}^f(\bar{y}^f)$, for all $\bar{y}^f \in Y^f(e')$. This result will imply $y^f \in \operatorname{argmax}_{\tilde{y}^f \in Y^f(e)} \pi_e^f(\tilde{y}^f)$, just start at $e = 0$ and apply the result in successive nodes of the relevant branches of the events tree until you reach the root of the sub-tree that defines the truncated economy. The next argument is inspired by Makowski (1983).

Now, let $\lambda^{\hat{i}}$ be the multipliers of a node e reference agent in the quasi-equilibrium where production is y^f, and let $\lambda^{\tilde{i}}$ be the multipliers of a node e reference agent in the quasi-equilibrium when production is $\bar{y}^f$. In particular, we have that (see the proof of Theorem 3):

$$\sum_{u \in e+1} \frac{\lambda_u^{\tilde{i}}}{\lambda_e^{\tilde{i}}} (p_u \bar{y}_u^f + \upsilon_u^f(\bar{y}^f)) \geq \sum_{u \in e+1} \frac{\lambda_u^j}{\lambda_e^j} (p_u \bar{y}_u^f + \upsilon_u^f(\bar{y}^f)) \ , \forall j \in I, \tag{28}$$

where $e+1$ is the set of successor nodes of e.

Then we have that:

$$\begin{aligned}
p_e y_e^f + \upsilon_e^f(y^f) &= p_e y_e^f + \sum_{u \in e+1} \frac{\lambda_u^{\hat{i}}}{\lambda_e^{\hat{i}}} (p_u y_u^f + \upsilon_u^f(y^f)) \\
&\overset{(a)}{\geq} p_e \bar{y}_e^f + \sum_{u \in e+1} \frac{\lambda_u^{\tilde{i}}}{\lambda_e^{\tilde{i}}} (p_u \bar{y}_u^f + \upsilon_u^f(\bar{y}^f)) \\
&\overset{(b)}{\geq} p_e \bar{y}_e^f + \sum_{u \in e+1} \frac{\lambda_u^{\hat{i}}}{\lambda_e^{\hat{i}}} (p_u \bar{y}_u^f + \upsilon_u^f(\bar{y}^f)) \\
&\overset{(c)}{=} p_e y_e^f + \sum_{\substack{u \in e+1 \\ u \neq e'}} \frac{\lambda_u^{\hat{i}}}{\lambda_e^{\hat{i}}} (p_u y_u^f + \upsilon_u^f(y^f)) + \frac{\lambda_{e'}^{\hat{i}}}{\lambda_e^{\hat{i}}} (p_{e'} \bar{y}_{e'}^f + \upsilon_{e'}^f(\bar{y}^f))
\end{aligned}$$

A brief explanation of these inequalities:

(a) follows from our initial hypothesis: $\pi_e^f(y^f) = p_e y_e^f + \upsilon_e^f(y^f) \geq p_e \bar{y}_e^f + \upsilon_e^f(\bar{y}^f) = \pi_e^f(\bar{y}^f)$.

(b) follows from (28), as we saw in the proof of Theorem 3, the valuation of reference agents is always greater or equal to the valuation of all other agents.

(c) follows from the fact that both y^f and $\bar{y}^f$ belong in $Y^f(e')$ so we actually have that $y_e^f = \bar{y}_e^f$ and $y_u^f = \bar{y}_u^f$ for $u \in e+1$, $u \neq e'$.

These inequalities imply that $\pi_{e'}^f(y) = p_{e'} y_{e'}^f + \upsilon_{e'}^f(y^f) \geq p_{e'} \bar{y}_{e'}^f + \upsilon_{e'}^f(\bar{y}^f) = \pi_{e'}^f(\bar{y})$, as we wanted to prove.

□

Lemma 3 (The GC rate is unique across quasi-equilibria). *Suppose assumptions 1 and 3 hold, and that firm f is a perfect competitor. Let $(x, \phi, z, y, p, q, \rho)$ be a financial equilibrium.*

Given any production plan $\bar{y} \in Y^f$ let $(\bar{x}, \bar{\phi}, \bar{z}, p, \bar{q}, \bar{\rho})(\bar{y}) \in QE(\bar{y}^f)$ be a quasi-equilibrium. Then $\bar{\rho}^{\mathrm{GC}} = \rho^{\mathrm{GC}}$.

Proof of Lemmas 1 and 3. Let $(\tilde{x}, \tilde{\phi}, \tilde{z}, p, \tilde{q}, \tilde{\rho})(\tilde{y}) \in QE(\tilde{y}^f)$ and $(\bar{x}, \bar{\phi}, \bar{z}, p, \bar{q}, \bar{\rho})(\bar{y}) \in QE(\bar{y}^f)$ be two different quasi-equilibria, each of which corresponds to a different selection of production plan by firm f. We will prove that $\bar{\rho}^{\mathrm{GC}} = \tilde{\rho}^{\mathrm{GC}}$, and that $\tilde{\rho}^f < \tilde{\rho}^{\mathrm{GC}} \Leftrightarrow \bar{\rho}^f < \bar{\rho}^{\mathrm{GC}}$ and $\tilde{\rho}^f = \tilde{\rho}^{\mathrm{GC}}$ $\Leftrightarrow \bar{\rho}^f = \bar{\rho}^{\mathrm{GC}}$.

Suppose agent i is agent $\tilde{i}$ in assumption 4. In particular, $1 + \tilde{\rho}^{\mathrm{GC}} = \tilde{\lambda}^i_1 / \sum_s \tilde{\lambda}^i_{2s}$. Let $(\bar{\lambda}^i, \bar{\mu}^{if})$ be the multipliers corresponding to the budget and box constraints, and $(\bar{\nu}^{if}_+, \bar{\nu}^{if}_-)$ be the multipliers corresponding to upper and lower bound restrictions on borrowing positions, respectively, when the optimal consumption plan chosen by the agent is $(\bar{x}^i, \bar{\phi}^i, \bar{z}^i)$. Since firm f is a perfect competitor in the financial equilibrium, there is some plan $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i_\star(\bar{y}, p, \bar{q}, \bar{\rho})$ with $\bar{\phi}^{if}_{*e} = 0\ \forall e$ and $\bar{z}^{if}_* = 0$ (as we saw in the proof of Theorem 1). The same applies to quasi-equilibrium $(\tilde{x}, \tilde{\phi}, \tilde{z}, p, \tilde{q}, \tilde{\rho})(\tilde{y})$, where $(\tilde{\lambda}^i, \tilde{\mu}^{if}, \tilde{\nu}^{if}_+, \tilde{\nu}^{if}_-)$ are the multipliers for the agent's problem when he chooses consumption plan $(\tilde{x}^i, \tilde{\phi}^i, \tilde{z}^i)$. There is also an optimal plan $(\tilde{x}^i_*, \tilde{\phi}^i_*, \tilde{z}^i_*) \in B^i_\star(\tilde{y}, p, \tilde{q}, \tilde{\rho})$ with $\tilde{\phi}^{if}_{*e} = 0\ \forall e$ and $\tilde{z}^{if}_* = 0$.

This implies that if a firm f is perfectly competitive, then $\bar{\rho}^f \leq \bar{\rho}^{\mathrm{GC}}$. This is because the multipliers $(\bar{\lambda}^i_*, \bar{\mu}^{if}_*, \bar{\nu}^{if}_{*+}, \bar{\nu}^{if}_{*-})$ associated to the plan $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$ are such that $\bar{\nu}^{if}_{*+} = \bar{\nu}^{if}_{*-} = 0$ because $\bar{z}^i_* = 0$ and this implies that:

$$\begin{aligned} 1 + \bar{\rho}^f &= \frac{\bar{\lambda}^i_*}{\sum_s \bar{\lambda}^i_{*2s}} + \frac{-\bar{\mu}^{if}_{*1} + \bar{\nu}^{if}_{*+} - \bar{\nu}^{if}_{*-}}{H^f q^f_1 \sum_s \bar{\lambda}^i_{*2s}} = \frac{\bar{\lambda}^i_*}{\sum_s \bar{\lambda}^i_{*2s}} - \frac{\bar{\mu}^{if}_{*1}}{H^f q^f_1 \sum_s \bar{\lambda}^i_{*2s}} \\ &\leq \frac{\bar{\lambda}^i_*}{\sum_s \bar{\lambda}^i_{*2s}} = 1 + \bar{\rho}^{\mathrm{GC}} \end{aligned}$$

Given the concavity assumptions that we have made for the agents' utilities, $(\bar{\lambda}^i, \bar{\mu}^{if}, \bar{\nu}^{if}_+, \bar{\nu}^{if}_-)$ and consumption plan $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$ satisfy the KT first order conditions, as do multipliers $(\tilde{\lambda}^i, \tilde{\mu}^{if}, \tilde{\nu}^{if}_+, \tilde{\nu}^{if}_-)$ and consumption plan $(\tilde{x}^i_*, \tilde{\phi}^i_*, \tilde{z}^i_*)$. As we saw in the proof of Theorem 3, when $\phi^{if}_0 = 0$, as is the case for agent i given assumption 4, we have that $(\tilde{x}^i_*, \tilde{\phi}^i_*, \tilde{z}^i_*) \in B^i_\star(\bar{y}, p, \bar{q}, \bar{\rho})$, and that $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i_\star(\tilde{y}, p, \tilde{q}, \tilde{\rho})$. Again, we will have that $(\bar{\lambda}^i, \bar{\mu}^{if}, \bar{\nu}^{if}_+, \bar{\nu}^{if}_-)$ and consumption plan

$(\tilde{x}^i_*, \tilde{\phi}^i_*, \tilde{z}^i_*)$ satisfy the KT first order conditions, as do multipliers $(\tilde{\lambda}^i, \tilde{\mu}^{if}, \tilde{\nu}^{if}_+, \tilde{\nu}^{if}_-)$ and consumption plan $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$, and, as in the proof of Theorem 3, $\bar{\lambda}^i = \tilde{\lambda}^i$. This implies, from the first order conditions, $\tilde{\rho}^{\mathrm{GC}} = \bar{\rho}^{\mathrm{GC}} = \rho^{\mathrm{GC}}$. The first order conditions also imply that

$$\begin{aligned} 1 + \tilde{\rho}^f &= \frac{\tilde{\lambda}^i_1}{\sum_s \tilde{\lambda}^i_{2s}} + \frac{1}{H^f \tilde{q}^f_1} \left(\frac{-\tilde{\mu}^{if} + \tilde{\nu}^{if}_+ - \tilde{\nu}^{if}_-}{\sum_s \tilde{\lambda}^i_{2s}} \right) = 1 + \rho^{\mathrm{GC}} + \frac{1}{H^f \tilde{q}^f_1} \left(\frac{-\bar{\mu}^{if} + \bar{\nu}^{if}_+ - \bar{\nu}^{if}_-}{\sum_s \bar{\lambda}^i_{2s}} \right) \\ &= 1 + \rho^{\mathrm{GC}} + \frac{1}{H^f \tilde{q}^f_1} \left(\frac{-\bar{\mu}^{if} + \bar{\nu}^{if}_+ - \bar{\nu}^{if}_-}{\sum_s \tilde{\lambda}^i_{2s}} \right) \end{aligned}$$

and

$$\begin{aligned} 1 + \bar{\rho}^f &= \frac{\bar{\lambda}^i_1}{\sum_s \bar{\lambda}^i_{2s}} + \frac{1}{H^f \bar{q}^f_1} \left(\frac{-\bar{\mu}^{if} + \bar{\nu}^{if}_+ - \bar{\nu}^{if}_-}{\sum_s \bar{\lambda}^i_{2s}} \right) = 1 + \rho^{\mathrm{GC}} + \frac{1}{H^f \bar{q}^f_1} \left(\frac{-\tilde{\mu}^{if} + \tilde{\nu}^{if}_+ - \tilde{\nu}^{if}_-}{\sum_s \tilde{\lambda}^i_{2s}} \right) \\ &= 1 + \rho^{\mathrm{GC}} + \frac{1}{H^f \bar{q}^f_1} \left(\frac{-\tilde{\mu}^{if} + \tilde{\nu}^{if}_+ - \tilde{\nu}^{if}_-}{\sum_s \bar{\lambda}^i_{2s}} \right) \end{aligned}$$

So we have $\bar{\rho}^f \, (\overset{=}{<}) \, \rho^{\mathrm{GC}} \Leftrightarrow -\tilde{\mu}^{if} + \tilde{\nu}^{if}_+ - \tilde{\nu}^{if}_- \, (\overset{=}{<}) \, 0 \Leftrightarrow -\bar{\mu}^{if} + \bar{\nu}^{if}_+ - \bar{\nu}^{if}_- \, (\overset{=}{<}) \, 0 \Leftrightarrow \tilde{\rho}^f \, (\overset{=}{<}) \, \rho^{\mathrm{GC}}$.

□

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