Discussion Papers in Economics

QUALITY GROWTH: FROM PROCESS TO PRODUCT INNOVATION ALONG THE PATH OF DEVELOPMENT

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ISSN: 1749-5075

Quality Growth: From Process to Product Innovation along the Path of Development

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July 2016

Abstract

We propose a demand-driven growth theory where process innovations and product innovations fulfil sequential roles along the growth path. Process innovations must initially set the economy on a positive growth path. However, process innovations alone cannot fuel growth forever, as their benefits display an inherent tendency to wane. Product innovations are therefore also needed for the economy to keep growing in the long run. When the economy fails to switch from a growth regime steered by process innovation to one driven by product innovation, R&D effort and growth will eventually come to a halt. However, when the switch to a product innovation growth regime does take place, a virtuous circle gets ignited. This happens because product innovation effort not only keeps growth alive when incentives to undertake process innovation diminish, but it also regenerates profit prospects from further process innovation effort.

Keywords: Endogeneous Growth, Process and Product Innovation, Nonhomothetic Preferences, Quality Ladders.
JEL Classifications: O30, O31, O41

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1 Introduction

Process and product innovations are two key determinants behind sustained economic growth. Process innovations introduce technological improvements that allow an expansion in the quantity of goods that an economy can deliver. Product innovations foster growth instead by bringing to the market goods of higher quality than those previously available. This paper presents a demand-driven growth theory in which both types of innovations fulfil crucial roles, and where their respective roles display a specific sequential timing. Our theory shows that process innovations must precede product innovations along the path of development. Yet, while process innovations can initially set the economy on a positive growth path, they cannot sustain rising incomes perpetually. Long-lasting growth requires that the economy is also able to start generating product innovations at some point. The reason for this is that the incentives to invest in process innovations tend to wane as physical production continuously expands.

Our model features an economy with a vertically differentiated good, available in a number of (vertically ordered) quality varieties. All the quality varieties are produced with technologies that use labour as their sole input. Both labour productivity and the degree of vertical differentiation are endogenous to the model. Labour productivity increases as a result of process innovations. In particular, process innovations lower the costs of production (in terms of hours of labour), leading to an increase in the physical quantities that may be produced with a given amount of labour. Product innovations instead allow the introduction of better quality varieties of the vertically differentiated good.

Innovations are the outcome of purposeful research and development effort. Hence, investment in process and in product innovations will be the endogenous response to the potential profit associated to each of them. The underlying force leading to their different timings along the growth path stems from our demand side. Individuals exhibit nonhomothetic preferences along the quality dimension. In particular, their willingness to pay for quality upgrading increases as their incomes rise. An implication of this is that product innovations tend to become increasingly profitable along the growth path, since product innovators can charge higher markups when they face richer consumers. However, our nonhomothetic demand structure entails also a flip side. At early stages of development, the economy must rely on process innovations as the source of income growth. This is because the low willingness to pay for quality by consumers with low incomes stifles profit opportunities for product innovators.

Our theory then shows that when incomes are still low process innovation must become the leading actor. Product innovation takes over a more prominent role instead in more mature economies. Furthermore, such transition from process to product innovation effort is essential for sustaining growth in our model. In a context where individuals display decreasing marginal utility on physical consumption, process innovations bring about two opposing dynamic forces. On the one hand, they drive the marginal utility of consumption down. Thus, the prospects of *future* profits from process innovation are endogenously dampened by *current* process innovation efforts. On the other hand, the higher quantity of consumption allowed by process innovations is exactly what spurs profit prospects from product innovation effort. The effect of these two countervailing forces means that it is not guaranteed that an economy will succeed in eventually switching from a growth path steered by process innovation to one steered by product innovation. When it fails to do so, growth will eventually come to a halt due to negative effect of marginal utility on process innovation profits.

The switch to a product innovation growth regime can also ignite a virtuous circle with further process innovation down the road. The incapacity of cost-cutting innovations to spur growth perpetually lies in that a continuous expansion in quantity of production must struggle against the decreasing marginal utility of (physical) consumption that it simultaneously leads to. This struggle makes it increasingly hard to keep profit prospects from process innovation high enough to sustain it forever. Product innovation relaxes this inherent tension, since higher quality goods yield greater utility per unit of physical consumption. In other words, by raising the intrinsic quality of goods offered in the economy, product innovation is able to make the decreasing marginal utility of (physical) consumption less pressing, and thereby regenerate profit prospects from further process innovation effort.

Despite the large number of articles dealing with either process or product innovation effort, it is hard to find models in the literature where both are explicitly involved together in steering the economy along the growth path, while at the same time playing a distinctive role as growth engines.[1] The only exception that we are aware of is Foellmi, Wuergler and Zweimüller (2014). They build a growth model where firms must choose between product innovations to introduce new luxury goods that will be consumed only by the rich, or process innovations that turn luxury goods into mass consumption goods also available to the poor. Their model depicts

[1] Models where growth is the result of the of new technologies that allow an increase in physical production (i.e., process innovations) can be found in: Shleifer (1986), Romer (1990), Aghion and Howitt (1992), Jones (1995), Kortum (1997). Examples of models where growth is driven by the introduction of final goods of higher quality than before (i.e., product innovations) are: Segerstrom et al. (1990), Grossman and Helpman (1991a, 1991b), Stokey (1991), Segerstrom (1998). A third type of innovation, which is neglected by our model, is that one that leads to a horizontal expansion in the variety of goods, as in Judd (1985), Romer (1990), Grossman and Helpman (1991c, Ch. 3). We relegate to the concluding section a brief discussion on the possible effects of introducing variety-expanding innovations within the context of our model.

situations where this type of product cycle takes place as the optimal behaviour by firms, and use it to explain how some new goods first introduced during the 20th century have later on become available as mass consumption goods (e.g., automobiles, refrigerators, etc.).[2] The main focus of our model is somewhat different, as it studies how the interplay between process and product innovation efforts can sustain a continuous increase in incomes in the long run, and how the preeminence of each type of innovation tends to change along the growth path. In that sense, our model is mostly concerned with how an economy may keep growing *beyond* a mass consumption economy, in a context where rising incomes increasingly tilt consumer preferences towards quality expansion (and away from quantity expansion).[3]

A key aspect behind our demand-driven growth model is therefore the nonhomotheticity of preferences along the quality dimension; in particular, the notion that willingness to pay for quality upgrading rises as physical consumption increases. This is in fact a property of the preference structure that has been previously incorporated in several trade models: Flam and Helpman (1987), Murphy and Shleifer (1997), Fajgelbaum, Grossman and Helpman (2011, 2015), Jaimovich and Merella (2012, 2015). This paper introduces such type of preference structure into an endogenous growth model that ties together process and product innovations, leading to dynamics where growth is primarily driven by cost-cutting innovations at early stages of development, and by quality-upgrading innovations as economies mature.

The notion that firms' actively invest in quality upgrading in order to cater to richer consumers or markets has recently received support by a growing strand of empirical papers; e.g. Verhoogen (2008), Brambilla, Lederman and Porto (2012), Manova and Zhang (2012), Bas and Strauss-Kahn (2015), Flach (2016). Our theory not only predicts such quality-upgrading effort by firms in response to rising consumer incomes, but more generally a gradual shift from process innovation to product innovation effort along the growth path. This specific sequence of growth phases seems also to be in line with the R&D behaviour exhibited by firms across countries with different income levels.

[2] Matsuyama (2002) also studies an endogenous growth model where goods initially affordable to the rich become gradually mass consumption goods affordable to all individuals. In his model, however, technological change is not the result of purposeful R&D effort, but it arises because of industry-specific learning-by-doing.

[3] Foellmi and Zweimüller (2006) also present a demand-driven endogenous growth model where individuals display non-homothetic preferences. Their model differs from ours substantially, especially on two fundamental aspects of our story. First, in their model there is no quality differentiation, and their nonhomotheticities are the result of hierarchical preferences with a horizontal continuum of goods, similar to Matsuyama (2000). Second, their model features only cost-cutting innovations, which is combined with a setup cost that must be incurred to open new sectors/product lines.

Table 1 displays a series of simple correlations between the innovation intensity by type of innovation at the firm level, and the income per head of the country where the firm is located. The data on innovation intensity by type of innovation (process innovation vs. product innovation) is taken from the Eurostat *Community Innovation Survey* (CIS), which collects information about the innovation activity by enterprises.[4] The CIS asks the surveyed firms if they have introduced innovations during the previous two years, and whether these innovations pertain to process innovation or product innovation (or both).[5]

The first column in Table 1 regresses the share of surveyed firms in each country that have introduced either process innovations or product innovations (or both) during the surveyed period on the income per head of each country. The correlation between the two variables is positive and highly significant, suggesting that firms in richer economies tend to introduce more innovations (of any type) than those in poorer economies.

More revealing of how the nature of the innovation activities undertaken by firms changes over the growth path are the results in the second and third column of Table 1. The second column considers as the dependent variable *only* those firms that have introduced *process innovations*. Conversely, the third column considers *only* those firms that have introduced *product innovations*. In both cases, we see again positive and significant correlations between the share of firms who generated innovations and the income per head of the country where the firms are located. However, the magnitude of the correlation between the share of firms doing product innovation and income per head is stronger than that one obtained when we consider instead the share of firms doing process innovation. Another way to see how the relevance of the different types of innovation effort shifts along the growth path is portrayed in column four. This column uses as dependent variable the ratio of firms that introduced product innovations

[4]In Table 1, we are using the CIS that covers the innovation activities of firms during the period 2004-06. This survey was conducted in the 25 EU members states (as of 2006), plus Norway, Bulgaria, Romania, Croatia and Turkey. The CIS covers firms in sectors coded by NACE Rev. 1.1 as C (mining and muarrying), D (manufacturing), E (electricity, gas and water supply), I (transport, storage and communication), J (financial intermediation), G51 (wholesale trade and commission trade), K72 (computer and related activities), K74.2 (architectural and engineering activities), and K74.3 (technical testing and analysis).

[5]The CIS specifies a *process innovation* as follows 'A process innovation is the implementation of a new or significantly improved production process, distribution method, or support activity for your goods or services. The innovation must be new to the enterprise, but it does not need to be new to your sector [...] [It] excludes purely organisational innovations'. Regarding a *product innovation*, the CIS defines those as 'A product innovation is the market introduction of a new good or service or a significantly improved good or service with respect to its capabilities, such as improved software, user friendliness, components or sub-systems. The innovation must be new to your enterprise, but it does not need to be new to your sector [...].'.

over those reporting the introduction of process innovations. Regressing this ratio on the income per head of the country where those firms are located we obtain again a correlation coefficient that is positive and significant.

Taken together, the set of correlations displayed in Table 1 point out towards a clear pattern: while R&D activities are larger in richer countries than in poorer ones, this behaviour is more pronounced for product innovations than for process innovations. In particular, the cross-country data seems to suggest that the prominence of product innovation effort relative to process innovation effort is greater in countries with higher income per capita. Our paper offers a demand-driven endogenous growth model that can rationalise these correlations.[6]

The rest of the paper is organised as follows. Section 2 presents the setup and main assumptions of the model. Section 3 studies a simplified version of the model with two quality varieties and where the only source of technical change is due to process innovations, showing that growth is eventually bound to come to a halt. Section 4 allows also the introduction of product innovations into the two-quality-variety model, and shows that product innovations may help sustaining growth for longer. Section 5 extends the model to an environment with an infinite number of quality varieties, which may allow economies to sustain positive growth in the long run. Section 6 provides some concluding remarks. All relevant proofs are relegated to the appendices.

2 Setup of the Model

Life evolves in discrete time over an infinite horizon, starting in $t = 1$. In each period t a unit mass of individuals is spontaneously born. Individuals live for one period only. All individuals are endowed with two units of time: one unit of *labour* time and one unit of *spare* time. Labour time is supplied inelastically to firms. The unit of spare time may instead be used as leisure time or, alternatively, to undertake R&D effort.

The economy's output consists of a final consumption good. The consumption good is

[6] Although there exists a small empirical literature that investigates the behaviour of firms in terms of R&D investment, differentiating between process and product innovation [e.g., Cohen and Klepper (1996), Huergo and Jaumandreu (2004), Parisi et. al. (2006), Harrison et. al. (2014)], none of these papers uses data collected from a large and diverse sample of different countries. These studies either use firm-level data from one single country (like Cohen and Klepper (1996) for US, Huergo and Jaumandreu (2004) for Spain, or Parisi et. al. (2006) for Italy), or from a small number of countries with similar levels of income (like Harrison et. al. (2014) for France, Germany, Spain and UK). As a result, this literature is silent about correlations between income per capita and intensity of investment in process innovation relative to product innovation.

Table 1: Process Innovation and Product Innovation Intensity at Different Income Levels

	Dependent Variable			
	ratio of firms doing process and/or product innovation	ratio of firms doing process innovation	ratio of firms doing product innovation	ratio of firms doing product to firms doing process innov.
GDP per head (PPP)	0.049***	0.020*	0.040***	0.074***
	(0.012)	(0.010)	(0.009)	(0.025)
R-squared	0.37	0.12	0.42	0.24
Number countries	30	30	30	30

Standard errors reported in parentheses. All data corresponds to the *Community Innovation Survey* (2004-06). GDP per head data corresponds to year 2006, and is taken from Penn Tables and measured in PPP (divided by 10,000). The ratios in the first three columns are computed using as the numerator the number of firms in a given country answering 'Yes' to the relevant innovation question over the total number of surveyed firms in that country as the denominator.
* significant 10%; ** significant 5%; *** significant 1%.

potentially available in a discrete number of vertically ordered quality levels: $q \in Q$. We normalise the lowest value in the set Q to unity. Henceforth, we will refer to the different quality levels as *varieties*, and to the lowest quality level ($q = 1$) as the *baseline* quality variety.

2.1 Technologies

Production takes place within firms. Firms are risk neutral and are active for one period only (that is, a firm that operates in period t will close down at the end of t). There is free entry to the final good sector and opening a new firm entails no setup cost.

In $t = 1$ the economy inherits a technology from the pre-historic period $t = 0$. The inherited technology allows transforming one unit of labour into one unit of the consumption good, but only in its baseline quality variety $q = 1$. All firms active in $t = 1$ have free access to the technology inherited from $t = 0$. In addition to this, individuals may undertake innovation effort in order to create blueprints for generating new technologies. In what follows, we explain in further detail the different set of technologies available in each period t and how they originate.

2.1.1 New and Inherited Technologies I: the effects of process innovation

All agents born in period t inherit at birth all the technological know-how that has been generated before t by prior events of process innovation. We broadly refer to all the technologies that were already available before period t as *inherited technologies*. Inherited technologies may be further improved upon through *current* process innovation effort. When an individual alive in t exerts process innovation effort, he will create a blueprint that may lead to a *new technology* with higher labour productivity than that one available in $t - 1$.

We assume that, in each period t, there is a unit mass of process innovation blueprints that may possibly be created; we denote this set of potential blueprints by $B_t^p = [0,1]$. We also assume that only one of these blueprints will turn out to be successful (ex post) in raising labour productivity; we label this successful blueprint by s_t^p, where $s_t^p \in B_t^p$. Neither individuals nor firms know (ex-ante) which element of B_t^p will turn out to be s_t^p. They all know, however, that all elements in B_t^p carry the same probability to become s_t^p, and that there is always one element that will become s_t^p. Both firms and individuals find out which element of B_t^p turns out to be s_t^p after market exchanges of blueprints take place. (We explain the within-period timing of the model in further detail in Section 2.3.)

Note that not all potential blueprints will be necessarily created in any particular period t. It proves then convenient to let $b_t^p \subseteq B_t^p$ denote the subset of blueprints that were actually created as a result of process innovation effort in t. A process innovation will be generated in a generic period τ if and only if $s_\tau^p \in b_\tau^p$. We can thus observe that the number of process innovations generated *before* period t can be written as follows:

$$R_{t-1} = \begin{cases} \sum_{\tau=1}^{t-1} 1\{s_\tau^p \in b_\tau^p\} & \text{if} \quad t \geq 2 \\ 0 & \text{if} \quad t = 1 \end{cases}, \tag{1}$$

where $1\{s_\tau^p \in b_\tau^p\}$ represents an index function that is equal to 1 when $s_\tau^p \in b_\tau^p$, and 0 otherwise.

We can now describe formally the set of technologies available to firms active in period t resulting from process innovation effort.

Assumption 1 *Consider some generic period $t \geq 1$. The technological options resulting from process innovation effort that are available to firms active in period t are the following ones:*

(i) Inherited technology: *If a firm active in t does not purchase any process innovation blueprints, the firm will be able to produce*

$$1 + \sigma R_{t-1}$$

units of the baseline quality variety, $q = 1$, with each unit of labour it hires, where $\sigma > 0$ and R_{t-1} is given by (1).

(ii) New technologies: *If a firm active in t purchases a bundle of process innovation blueprints, and the bundle contains s_t^p, the firm will be able to produce*

$$1 + \sigma R_{t-1} + \sigma$$

units of the baseline quality variety, $q = 1$, with each unit of labour it hires.

The first part of Assumption 1 stipulates that the effects of process innovation on labour productivity accumulate over time, and are transferred to the next generations as inherited technologies. The second part describes the effects of current process innovation: successful blueprints boost labour productivity by $\sigma > 0$ units relative to the inherited technology.[7]

2.1.2 New and Inherited Technologies II: the effects of product innovation

The economy may also have available improved technologies that originate from product innovation effort. Product innovation effort leaves labour productivity unchanged, but it allows the production of higher quality varieties than before.

Analogously as done for process innovations, we now assume that in each period t there is a unit mass of product innovation blueprints that may possibly be created, and denote this set by $B_t^q = [0, 1]$. Only one of these blueprints, denoted by $s_t^q \in B_t^q$, will be successful (ex post) in generating quality upgrading. No one knows ex-ante which element will turn out to be s_t^q, but everyone knows that there is always one s_t^q and that all elements of B_t^q are equally likely to become s_t^q. Again, both firms and individuals find out s_t^q after market exchanges of (product innovation) blueprints take place.

Assumption 2 below summarises in detail the quality upgrading effect of product innovation effort. It also states the (physical) productivity of labour at the different levels of quality in which the final good is available. In order to do this, we label by q_t the *highest* quality variety that is available at the end of period t. It proves convenient again to specifically label the subset of blueprints that were actually created as a result of product innovation effort in t, which we do by $b_t^q \subseteq B_t^p$.

Assumption 2 *Consider some generic period $t \geq 1$. The technological options resulting from product innovation effort that are available to firms active in period t are the following ones:*

(i) Inherited technology: *If a firm active in t does not purchase any new blueprints, the firm will be able to produce*

$$\frac{1 + \sigma R_{t-1}}{q_{t-1}} \tag{2}$$

units of the consumption good in the quality level q_{t-1}, where

$$q_{t-1} = \begin{cases} 1 + \rho \sum_{\tau=1}^{t-1} 1\{s_\tau^q \in b_\tau^q\} & \text{if} \quad t \geq 2 \\ 1 & \text{if} \quad t = 1 \end{cases},$$

[7]Note that, since $R_0 = 0$, Assumption 1 implicitly states that in $t = 1$ the inherited technology is the pre-historic technology, which allows any firm to produce one unit of the baseline quality variety with each unit of labour it hires.

with $\rho > 0$.

(ii) New technologies: *If a firm active in t purchases a bundle of product innovation blueprints, and the bundle contains s_t^q, the firm will be able to produce*

$$\frac{1 + \sigma R_{t-1}}{q_t} \tag{3}$$

units of the consumption good in the quality level q_t, where $q_t = q_{t-1} + \rho$.

Assumption 2 describes how a successful product innovation blueprint in t, s_t^q, allows the production of a variety whose quality level is $\rho > 0$ units higher than the one that was available at the end of $t - 1$ (namely, $q_t = q_{t-1} + \rho$). It also states how past product innovations alter future production possibilities: the variety q_t will also be readily available from $t + 1$ onwards as an inherited technology, without the need of any further product innovation effort.

Two additional remarks are worth stressing here. Firstly, Assumption 1 and 2 taken together imply that past process innovations generate productivity improvements that are *not* quality-specific. More precisely, the numerators in (2) and (3) entail that improvements in labour productivity owing to *prior* process innovations apply identically to *all* the existing quality varieties of the consumption good. Secondly, the denominators in (2) and (3) entail that the unit labour requirements are greater for higher quality varieties of the consumption good than for lower quality ones.

2.2 Preferences

The utility function of an individual alive in period t is given by:

$$U_t = \ln \left[\sum\nolimits_{q \in Q_t} \max \left\{ q\,x(q), [q\,x(q)]^q \right\} \right] + \eta(1 - \varepsilon). \tag{4}$$

In (4), $x(q)$ denotes the quantity of the quality variety $q \in Q_t$ consumed by the individual, and $Q_t \subseteq Q$ is the subset of quality varieties available in period t.[8] Next, ε is an index function that takes the value of 1 if the individual decided to undertake R&D effort (either in the form of process or product innovation) and 0 if he instead chose to use his time endowment as leisure, with $\eta > 0$ being the utility of leisure.

Two important mathematical properties of (4) are worth mentioning now. Firstly, since the lowest value of the set Q_t is $q = 1$, the term $\sum_{q \in Q_t} \max \{q\,x(q), [q\,x(q)]^q\}$ turns out to be

[8]Naturally, before any product innovation takes place, $Q_t = \{1\}$; that is, the only element of Q_t is the baseline quality variety, $q = 1$.

a sum of convex functions in $x(q)$. As a consequence of this, in the optimum, individuals will select a *corner* solution for their consumption plan; that is, a solution characterised by $x(q) > 0$ for some $q \in Q_t$ and zero for all other quality varieties of the good. Secondly, the expression $\max\{q\,x(q), [q\,x(q)]^q\}$ in (4) means that higher quality varieties magnify the level of utility that an individual obtains from a given physical amount of the consumption good. Moreover, this magnifying effect becomes stronger the larger the value of physical consumption $x(q)$. This is a crucial feature of our model, as it will lead to a non-homothetic behaviour in the demand for quality. In particular, the exponential effect of q on physical consumption $x(q)$ leads to demand functions where the willingness to pay for higher quality varieties of the final good is increasing in the level of spending of the consumer.[9]

2.3 Timing of Events

In each period t, actions take place with the following within-period timing:

1. Agents are born and inherit all the technologies that were available at the end of $t-1$.
2. Each individual i decides whether to use his spare time endowment (ε_i) as leisure or innovation effort. In case of the latter, they must also choose between process or product innovation effort. When choosing process innovation effort (resp. product innovation effort), they must also select which specific blueprint to create from the set B_t^p (resp. from the set B_t^q).
3. Individuals sell their innovation blueprints to firms.
4. Nature reveals the identities of the successful blueprints: $s_t^p \in B_t^p$ and $s_t^q \in B_t^q$.
5. Individuals sell their labour endowments to firms. Production and consumption take place.

[9] The expression $\max\{q\,x(q), [q\,x(q)]^q\}$ in (4) ensures that the quality dimension q is always a desirable feature to consumers. As it will become apparent in the next sections, given the assumptions of our model, in equilibrium it will always be the case that $q\,x(q) \leq [q\,x(q)]^q$ will hold for all individuals. All the following results in our model will thus be based on the part of (4) that exhibits the exponential functional form $[q\,x(q)]^q$. (A formal proof of this fact is provided in Appendix C.)

3 Endogenous Growth via Process Innovation

In this section, we study the dynamic behaviour of the economy under the assumption that only process innovation is feasible. Doing this allows a cleaner description of the conditions under which process innovation arises in equilibrium, and how it endogenously generates its own tendency to eventually stop. In addition, for most part of the paper, we further simplify the analysis by considering an environment where there exist only two potential quality varieties of the consumption good; that is, $Q=\{1, 1+\rho\}$. Nevertheless, in Section 5 we show how the model easily extends to a more general environment with an infinite number of quality varieties, and the dynamic implications of this. Finally, we restrict all our analysis to symmetric equilibria under pure strategies.[10]

3.1 Process Innovation Effort in $t=1$

Consider a generic individual born in $t = 1$. This agent may use his spare time endowment as process innovation effort or, else, simply enjoy it as leisure. If the individual chooses the former, he must also select the specific process innovation blueprint to create from among B_1^p. Since there is a unit mass of individuals and the set B_1^p has also unit mass, in equilibrium, no individual will choose to create a blueprint that is also being created by another individual (in other words, there will be no duplication of blueprints in equilibrium).[11]

We take the baseline quality variety as the *numeraire*. Recall from Assumption 1 that in $t = 1$ any firm can produce one unit of the baseline quality variety simply by using the inherited technology. Given free entry, this implies that in equilibrium each individual will receive a wage equal to one during $t = 1$. We label this wage by $w_1 = 1$.

Individuals may also receive earnings from selling the blueprints they created via process innovation effort to firms. Free entry and firm competition imply that the equilibrium price of a blueprint must equal the *expected* return it generates. Consider thus a hypothetical firm that purchased the blueprint that (ex post) turned out to be s_1^p. This firm will be able to produce $1 + \sigma$ units of the baseline quality variety of the final good by hiring all the available labour supply (which has unit mass). Since each unit of labour costs $w_1 = 1$, the blueprint s_1^p will thus

[10] There exist some parametric configurations of the model under which mixed-strategy equilibria would arise alongside symmetric pure-strategy equilibria. However, ruling out those cases does not seem to a major source of concern to our results, as when mixed-strategy equilibria do exist, they turn out to be locally unstable (see footnote 17 in Section 4.2 for some further discussion on this issue).

[11] The same non-duplication result will hold in every period t, given that the mass of agents is equal to one for all generations and all the sets B_t^p have unit mass as well.

yield a profit equal to σ. Recall now that firms do not know (ex ante) which blueprint will turn out to be s_1^p, and all blueprints in B_1^p carry the same probability to become s_1^p. As a result, an individual alive in $t = 1$ will be able to sell a process innovation blueprint to a firm for a price equal to σ (which is the expected return it will generate).[12]

Let $Y_{i,1}$ denote the lifetime earnings of a generic individual i at the end of $t = 1$. There are, in principle, two components in $Y_{i,1}$. The first is the earnings received as a wage, $w_1 = 1$. The second, in case individual i decided to exert process innovation effort, is the payment received for selling the resulting blueprint to some firm (which is equal to σ). Thus, when individual i invests his spare time endowment in process innovation,

$$Y_{i,1}\left(\varepsilon_{i,1} = 1\right) = 1 + \sigma. \tag{5}$$

Instead, if i chooses to consume his spare time endowment as leisure,

$$Y_{i,1}\left(\varepsilon_{i,1} = 0\right) = 1. \tag{6}$$

The optimal decision for i will naturally depend on which of the two actions (i.e., $\varepsilon_{i,1} = 1$ or $\varepsilon_{i,1} = 0$) leads to a higher utility level. Using (4), it follows that a generic individual i alive in $t = 1$ will set $\varepsilon_{i,1} = 1$ if and only if: $\ln(Y_{i,1}\left(\varepsilon_{i,1} = 1\right)) > \ln(Y_{i,1}\left(\varepsilon_{i,1} = 0\right)) + \eta$. (For the rest of the paper we will assume that in case of indifference the agent chooses $\varepsilon_i = 0$.) Hence, using (5) and (6), and noting that $Y_{i,1}$ depends only on the action by individual i (i.e., $\varepsilon_{i,1}$), we can finally obtain the condition that ensures that, in equilibrium, individuals alive in $t = 1$ will exert process innovation effort. Namely:

$$\ln(1 + \sigma) > \eta. \tag{7}$$

Otherwise, they will all set $\varepsilon_1 = 0$.

Condition (7) shows (quite intuitively) that a larger value of σ is instrumental to sustaining an equilibrium with positive process innovation effort. An equilibrium where $\varepsilon_1 = 1$ requires that the *additional utility* obtained from $Y_{i,1}(\varepsilon_{i,1} = 1)$, relative to that obtained from $Y_{i,1}(\varepsilon_{i,1} = 0)$, more than compensates the disutility of effort incurred when $\varepsilon_{i,1} = 1$, which leads (7).

[12] Rigorously speaking, since we have a continuum of individuals i such that $\int_0^1 di = 1$, each blueprint carries a probability di to become the successful blueprint, and hence it generates an expected return equal to $\left[\int_0^1 (1+\sigma)dj - \int_0^1 1dj\right] di = \sigma di$ when the firm that purchased it hires also the *entire* available labour supply. From now on, to keep a lighter notation, we disregard this measure issue and simply write σ as the expected return of any process innovation blueprint.

3.2 Process Innovation Effort in a Generic t

Assumption 1 states that any firm active in period t can freely use the inherited technology from $t-1$. Hence, without needing to buy any new blueprints, any firm active in t will be able to produce $1+\sigma R_{t-1}$ units of baseline quality variety with each unit of labour it hires. Competition among firms for workers will thus lead to an equilibrium wage:

$$w_t = 1 + \sigma R_{t-1}. \tag{8}$$

Any new process innovation blueprint created in period t will boost labour productivity, in expectation, by σ units. Therefore, in any period t, the price at which individuals will be able to sell their newly created blueprints will be equal to σ.

Using this result, together with (8), we can now generalise (5) and (6) for a generic economy whose inherited technology embodies R_{t-1} past process innovations. In this case, the lifetime earnings of an individual alive in t as a function of ε_t will be given by:

$$Y_t(\varepsilon_t = 1) = 1 + \sigma R_{t-1} + \sigma, \tag{9}$$

$$Y_t(\varepsilon_t = 0) = 1 + \sigma R_{t-1}. \tag{10}$$

From (9) and (10), it follows that a necessary and sufficient condition for an equilibrium with $\varepsilon_t^* = 1$ to hold, is that $\ln(1+\sigma R_{t-1}+\sigma) > \ln(1+\sigma R_{t-1}) + \eta$. This condition leads to the following result.

Lemma 1 *Consider an economy in period t which up until $t-1$ has gone through R_{t-1} periods where, in equilibrium, individuals used their spare time endowment as process innovation effort. Then, the equilibrium in period t will feature $\varepsilon_t^* = 1$ if and only if:*

$$\ln\left(1 + \frac{\sigma}{1+\sigma R_{t-1}}\right) > \eta. \tag{11}$$

The result in Lemma 1 generalises condition (7) to any period t.[13] This simple generalisation yields also some additional insights. First, notice that $\sigma/(1+\sigma R_{t-1})$ is strictly decreasing in R_{t-1}, and it converges to zero when $R_{t-1} \to \infty$. Hence, there will always be a value of R_{t-1} that is large enough such that (11) will fail to hold, and period t will thus feature an equilibrium with $\varepsilon_t^* = 0$ (that is, where individuals in t do *not* exert process innovation effort). Moreover, the ratio $\sigma/(1+\sigma R_{t-1})$ is strictly increasing in σ. As a result, the threshold value of R_{t-1} beyond which process innovation will necessarily stop tends to be greater for economies with

[13] It is straightforward to observe that conditions (7) and (11) coincide when $R_{t-1} = 0$.

a larger σ. The exact value of this threshold is pinned down by the level of R_{t-1} that makes left-hand side of (11) equal to η, namely:

$$\overline{R}(\sigma) \equiv \frac{1+\sigma-e^{\eta}}{\sigma(e^{\eta}-1)}. \tag{12}$$

Using (12), we can now observe that Lemma 1 implies that there is a maximum number of (subsequent) periods such that the economy will be able to exhibit an equilibrium with positive process innovation effort. Furthermore, the fact that $\overline{R}'(\sigma) > 0$ also means that economies with a larger σ are able to sustain an equilibrium with positive process innovation effort for longer.[14]

To conclude this section, we can finally describe the full dynamic behaviour of R_t in this version of the model where only process innovation effort is allowed. In order to deal with integer issues, we introduce one additional threshold:

Definition 1 *Let* $\bar{t}(\sigma) \equiv \text{integer}\{\overline{R}(\sigma)+1\}$*, where* $\overline{R}(\sigma)$ *is defined by (12).*

Proposition 1 *When condition (7) holds, implying that* $\overline{R}(\sigma) > 0$*, the economy will experience an equilibrium with positive process innovation effort in all periods from* $t = 1$ *until* $t = \bar{t}(\sigma) \geq 1$*. In any period* $t > \bar{t}(\sigma)$*, the economy will experience an equilibrium without process innovation effort. This, in turn, implies that* $R_t = t$ *whenever* $t \leq \bar{t}(\sigma)$*, while* $R_t = \bar{t}(\sigma)$ *for all* $t > \bar{t}(\sigma)$*.*

Proposition 1 shows that, when condition (7) holds, the economy will keep investing in process innovation in each period $t \geq 1$ until reaching $t = \bar{t}$ (notice, however, that it may well be the case that $\bar{t} = 1$).[15] While this happens productivity will accordingly grow, which is reflected in the fact that $R_t = t$ whenever $t \leq \bar{t}$. Once the economy goes past period $\bar{t}$, process innovation effort stops forever, and R_t remains thereafter constant and equal to $\bar{t}$.

The intuition behind the fact that process innovation eventually comes to halt hinges on the decreasing marginal utility of consumption. When an individual is contemplating whether or not to use his time endowment for process innovation, he faces a trade-off between higher consumption versus higher leisure. Setting $\varepsilon_t = 1$ allows a level of consumption equal to $Y_t(\varepsilon_t = 1) = 1 + \sigma R_{t-1} + \sigma$. On the other hand, $\varepsilon_t = 0$ leads a lower level of consumption, $Y_t(\varepsilon_t = 0) = 1 + \sigma R_{t-1}$, but it yields an additional non-pecuniary benefit, η. The difference between $Y_t(\varepsilon_t = 1)$ and $Y_t(\varepsilon_t = 0)$ is always equal to σ. However, as the economy goes

[14] Notice also that when $1+\sigma \leq e^{\eta}$, we have that $\overline{R}(\sigma) \leq 0$, and no process innovation will ever take place. In fact, $1+\sigma > e^{\eta}$ is just the same condition as (7) for $t = 1$.

[15] From now on, to lighten up notation, we will often write down the period-threshold function $\bar{t}(\sigma)$ simply as $\bar{t}$, and avoiding thus making its dependence on σ explicit when this creates no confusion.

through subsequent rounds of process innovation and R_{t-1} accordingly rises, both $Y_t(\varepsilon_t = 1)$ and $Y_t(\varepsilon_t = 0)$ will increase. In a context with decreasing marginal utility of consumption, higher values of $Y_t(\varepsilon_t = 1)$ and $Y_t(\varepsilon_t = 0)$ in turn imply that the consumption gap between them tends to become less and less appealing relative to the required leisure sacrifice η.

4 Introducing Product Innovation

We proceed now to introduce product innovation effort into the model. We will consider that the only quality variety that was available at the end of period $t-1$ was the baseline quality; namely, $q_{t-1} = 1$. In this circumstance, when an individual alive in t exerts product innovation effort, he will design a blueprint from the set B_t^q which may eventually allow the production of the quality variety $q_t = 1 + \rho$.

Before moving on to the full equilibrium analysis of the model, it proves convenient to first address the following two questions: *i*) how much must a firm pay to an individual for a product innovation blueprint?; *ii*) what is the price that a firm producing the quality variety $q_t = 1 + \rho$ will charge for this commodity?

Lemma 2 *Consider a generic individual i alive in period t who uses his spare time endowment as product innovation effort. If a firm wishes to purchase the product innovation blueprint designed by i, it will have to pay him:*

$$\pi_{i,t}^q = P_t \frac{1 + \sigma R_{t-1}}{1 + \rho} - (1 + \sigma R_{t-1}), \tag{13}$$

where P_t in (13) denotes the price the firm will charge for each unit of the quality variety $q = 1 + \rho$, should the acquired blueprint turn out to be the successful one (i.e., s_t^q), and which in equilibrium will be given by:

$$P_t = (1 + \rho) Y_t^{\rho/(1+\rho)}. \tag{14}$$

The first result in Lemma 2 stipulates the amount to be paid to a product innovator in t ($\pi_{i,t}^q$). This equals the (expected) earnings derived from selling the higher-quality variety of the good, after paying the (equilibrium) wage $w_t = 1 + \sigma R_{t-1}$ for each unit of labour hired. Next, Lemma 2 shows in (14) that the price charged for the variety of the good of quality $1 + \rho$ rises with the lifetime earnings of the cohort alive in t. The responsiveness of P_t to Y_t is a direct implication of our preference structure in (4), where the quality index q magnifies the utility derived from the physical quantity of consumption, $x(q)$. Such preference structure leads to a nonhomothetic behaviour where the willingness to pay for the higher-quality variety (i.e., $q = 1 + \rho$) rises with Y_t.[16]

The equilibrium value of P_t thus depends on the level of Y_t, which is itself also an equilibrium object. In particular, since $Y_{i,t} = w_t + \pi_{i,t}$, the level of Y_t will be ultimately a function of the innovation effort decisions by all individuals alive in period t. Equation (9) stated how Y_t rises with subsequent rounds of process innovation effort. The following lemma complements those results when we allow for product innovation as well.

Lemma 3 *Consider an economy in a generic period t. Suppose that up until period $t-1$ there were R_{t-1} periods in which individuals used their spare time endowment as process innovation effort. In addition, suppose that $q_{t-1} = 1$ (i.e., by the end of period $t-1$ only the baseline quality variety was available). Then, if the cohort alive in period t use their spare time endowment to undertake product innovation effort, the level of Y_t will be given by:*

$$Y_t = (1 + \sigma R_{t-1})^{1+\rho}. \tag{15}$$

Lemma 3 shows that the larger the value of R_{t-1} the greater Y_t will be. This is due to the fact that productivity improvements resulting from *prior* process innovations are cumulative across the different varieties of the consumption good. Furthermore, it is interesting to notice that when $R_{t-1} = 0$, the expression (15) yields $Y_t = 1$. In other words, in the absence of *any* previous rounds of process innovations, product innovation effort cannot induce by itself a rise in incomes. Therefore, some initial rounds of process innovation effort are required in order to ignite income growth. In turn, as consumers' incomes grow owing to productivity improvements stemming from process innovation, this may endogenously generate sufficient profit incentives to start investing in product innovation at some point along the growth path. We next study how this particular growth sequence may arise as an equilibrium outcome of the model.

Before moving on to the next section, we need to add some additional notation to distinguish the type of innovation effort that individuals choose to do. We use henceforth $\varepsilon_{t,p} = 1$ for process innovation effort, and $\varepsilon_{t,q} = 1$ for product innovation effort in t. For simplicity, we continue to denote by $\varepsilon_t = 0$ the choice of using the spare time endowment as leisure in t (that is, exerting neither process nor product innovation effort).

[16] The equilibrium value of P_t in (14) is that one that leaves consumers indifferent between buying the higher-quality variety at the price P_t and buying the baseline quality variety (which being the *numeraire* carries a price equal to one). Given our nonhomothetic structure, this indifference price rises with the level of income Y_t.

4.1 (Absence of) Product Innovation Effort in t=1

We show first that an equilibrium where the individuals invest their spare time endowment in product innovation cannot possibly arise in $t = 1$. Such an equilibrium would require that, for any generic individual i, the action $\varepsilon_{i,1,q} = 1$ is a best response to $\varepsilon^*_{1,q} = 1$.

In an equilibrium where $\varepsilon^*_{1,q} = 1$ the lifetime earnings of the cohort alive in $t = 1$ would follow from (15) with $R_{t-1} = 0$. This yields $Y_1 = 1$, which using Lemma 2, implies that firms would *not* be willing to pay anything for a product innovation blueprint. In turn, since investing in product innovation entails giving up the utility of leisure η, no individual would thus find it optimal to do so in this case.

The above result shows that an equilibrium where $\varepsilon^*_{1,q} = 1$ cannot exist. Once we allow for product innovation effort, pinning down the type of equilibrium that arises in $t = 1$ requires also analysing possible individual deviations to $\varepsilon_{i,1,q} = 1$ from situations where either $\varepsilon^*_{1,p} = 1$ or $\varepsilon^*_1 = 0$. Consider, firstly, the case in which the cohort alive in $t = 1$ exert process innovation effort in equilibrium (i.e., $\varepsilon^*_{1,p} = 1$). If some generic individual i decided to deviate to $\varepsilon_{i,1,q} = 1$, then he would generate a product innovation blueprint and sell it to a firm for a payment $\pi_{i,1,q} = (1+\sigma)^{\rho/(1+\rho)} - 1$.[17] Given that when i sets $\varepsilon_{i,1,p} = 1$ he is offered $\pi_{i,1,p} = \sigma$, the deviation to $\varepsilon_{i,1,q} = 1$ is actually not profitable for him.[18] Next, we can also observe that a deviation to $\varepsilon_{i,1,q} = 1$ from $\varepsilon^*_1 = 0$ cannot be optimal for a generic individual i in $t = 1$ either. This result follows from a similar reasoning as the one precluding the existence of an equilibrium with $\varepsilon^*_{1,q} = 1$: when the individuals set $\varepsilon^*_1 = 0$, $Y_1 = 1$ and product innovation blueprints would thus command a price equal $\pi_{i,1,q} = 0$, which cannot lead to a profitable deviation from $\varepsilon^*_1 = 0$.

In sum, this subsection shows that an equilibrium with positive product innovation effort will never arise in $t = 1$. Moreover, it also shows that condition (7) in Section 3 remains still valid when the model incorporates product innovation effort as well. The next section proceeds to study whether an equilibrium with positive product innovation effort may eventually arise at some point along the growth path of the economy.

4.2 Product Innovation Effort in a Generic t

For an equilibrium with product innovation effort to arise two non-deviation conditions must be satisfied when the entire cohort of individuals alive in t set $\varepsilon^*_{t,q} = 1$. The first is that none of

[17] To obtain this using the expression in (13), note that the price that firms would be able to charge for the higher-quality variety when $\varepsilon^*_{1,p} = 1$ is $P_1 = (1+\rho)(1+\sigma)^{\rho/(1+\rho)}$, since in this case we have that $Y_1 = 1 + \sigma$.

[18] Notice that this result entails that, when condition (7) holds, our model (with both process and product innovation effort) will then still deliver an equilibrium with positive process innovation effort in $t = 1$.

them must prefer to deviate to consuming their spare time endowment as leisure. The second is that none of them must prefer to deviate to exerting process innovation effort.

The previous subsection shows that the first condition above fails to be satisfied during $t = 1$. The reason for this is that, in the absence of any previous process innovation effort (that is, when $R_{t-1} = 0$), equation (15) yields $Y_t = 1$, which (given our nonhomothetic preference structure) turns out to be too small to make product innovation effort worthwhile. Notice, however, that according to (15) the level of Y_t when individuals in t undertake product innovation is increasing in R_{t-1}. This suggests that there may exist a value of R_{t-1} large enough to support an equilibrium with product innovation effort. The next lemma lays out this result formally.

Lemma 4 *Consider an economy in period t that has previously undergone R_{t-1} periods in which former cohorts used their spare time endowments as process innovation effort. Suppose the entire cohort of individuals alive in t set $\varepsilon_{t,q} = 1$.*

i) If

$$\rho \ln (1 + \sigma R_{t-1}) > \eta, \tag{16}$$

no single individual in t will prefer to deviate to consuming his spare time endowment as leisure.

ii) If

$$(1 + \sigma R_{t-1})^{1+\rho} > 1 + \sigma R_{t-1} + \sigma, \tag{17}$$

no single individual in t will prefer to deviate to exerting process innovation effort.

Lemma 4 presents the two non-deviation conditions that an equilibrium with product innovation must satisfy in period t. These conditions will only hold when R_{t-1} is sufficiently large. In other words, unless the economy has previously undergone a sufficiently large number of rounds of process innovation, it will *not* be able to sustain an equilibrium with product innovation effort in t. While conditions (16) and (17) ensure the existence of an equilibrium with product innovation in t, they do not rule out the possibility that other types of symmetric equilibria in pure strategies may exist as well. The next proposition provides a more general description of the equilibrium that arises in a generic period t.

Proposition 2 *Consider an economy in period t that has previously undergone R_{t-1} periods in which former cohorts exerted process innovation effort, and for which both (16) and (17) hold true. Then:*

i) There exists an equilibrium in period t in which all individuals in t exert product innovation effort (i.e., $\varepsilon^*_{t,q} = 1$).

ii) The equilibrium with $\varepsilon_{t,q}^* = 1$ is the unique pure-strategy symmetric equilibrium in period t, unless the following two conditions are also verified:

$$\ln\left(1 + \frac{\sigma}{1 + \sigma R_{t-1}}\right) \leq \eta \tag{18}$$

$$\frac{\rho}{1+\rho}\ln(1 + \sigma R_{t-1}) \leq \eta. \tag{19}$$

In particular, when (18) and (19) hold true, alongside with $\varepsilon_{t,q}^* = 1$, there also exists an equilibrium where all individuals in t use their spare time endowment as leisure (i.e., $\varepsilon_t^* = 0$).

Proposition 2 shows that when the two non-deviation conditions stipulated in Lemma 4 are satisfied, the economy will exhibit an equilibrium in period t where individuals undertake product innovation effort. In addition, this equilibrium is also unique (within the class of pure-strategy symmetric equilibria), unless the parametric configuration of the model is such that both (18) and (19) hold (together with the conditions in Lemma 4).

The multiple equilibria case described in Proposition 2 arises because of the possibility of coordination failures under some parametric configurations. In those cases, when the individuals alive in period t expect $\varepsilon_t^* = 0$ to hold, their best response to it turns out to be using their spare time endowment as leisure. Conversely, when they all expect $\varepsilon_{t,q}^* = 1$, their best response to it is to exert product innovation effort as well.

In this paper, we are mostly interested in studying under which conditions will economies be able to support long-run growth, rather than in the possibility of coordination failures preventing (potential) growth from materialising. For this reason, in the next subsection we will disregard the equilibrium with $\varepsilon_t^* = 0$, when it arises as a coordination failure in a context of multiple equilibria. Nevertheless, in Section 4.5 we will return to the issue of multiple equilibria, and provide an intuition for the mechanism leading to coordination failures.[19]

4.3 From Process Innovation to Product Innovation

Although Proposition 2 specifies the conditions that would lead to an equilibrium with product innovation effort, it still leaves one crucial question pending: whether or not an economy will actually be able to (endogenously) generate a level of R_{t-1} large enough to make (16) and (17) hold simultaneously. In fact, if an economy fails to do so (because the incentives to

[19] When the model yields both $\varepsilon_{t,q}^* = 1$ and $\varepsilon_t^* = 0$ as Nash equilibria, a third type of equilibrium would also arise involving mixed-strategies among product innovation effort and leisure. This mixed-strategy equilibrium is, however, locally unstable.

keep undertaking further process innovation wane too quickly), an equilibrium with product innovation effort will never materialise. We proceed to study now the conditions required for a successful transition from a equilibrium with process innovation to one with product innovation.

Some additional notation will prove useful for future reference. Firstly, we will denote by $\underline{R}_1$ the value of R_{t-1} that makes the LHS of (16) be equal to η. Namely:

$$\underline{R}_1 \equiv \frac{e^{\eta/\rho} - 1}{\sigma}. \tag{20}$$

Secondly, we will denote by $\underline{R}_2$ the value of R_{t-1} that equals the LHS of (17) to its RHS. In this case, there is no general explicit solution for $\underline{R}_2$, which is thus defined implicitly by:

$$\frac{(1+\sigma \underline{R}_2)^{1+\rho}}{1+\sigma \underline{R}_2 + \sigma} \equiv 1. \tag{21}$$

Both $\underline{R}_1$ and $\underline{R}_2$ are strictly decreasing in σ.[20] When $R_{t-1} > \underline{R}_1$ and $R_{t-1} > \underline{R}_2$, there is an equilibrium in which individuals alive in t will exert product innovation effort. Since both conditions must hold for this, we can simply combine $\underline{R}_1$ and $\underline{R}_2$ together by defining their envelope:

$$\underline{R}(\sigma) \equiv \max\{\underline{R}_1, \underline{R}_2\}, \tag{22}$$

where in (22) we make it explicit the dependence of $\underline{R}$ on σ. The threshold $\underline{R}(\sigma)$ essentially pins down the *minimum* value that R_{t-1} must reach for the economy to be able to switch to an equilibrium with product innovation effort in period t.

Since the model takes place in discrete time, in order to deal with integer issues, we must also introduce a *period* threshold.

Definition 2 *Let* $\underline{t}(\sigma) \equiv \text{integer}\{\underline{R}(\sigma) + 1\}$*, where* $\underline{R}(\sigma)$ *is defined by (22).*

The value of $\underline{t}(\sigma)$ pins down the *minimum* number of periods that the economy must sustain an equilibrium with process innovation effort before it can switch to an equilibrium with product innovation effort.

Proposition 3 *Consider an economy that satisfies condition (7). Depending on the specific values taken by* $\bar{t}(\sigma)$ *and* $\underline{t}(\sigma)$*, this economy may or may never be able to switch at some point to an equilibrium with product innovation effort. In particular:*

i) If $\bar{t}(\sigma) < \underline{t}(\sigma)$*, the economy will experience process innovation effort until period* $t = \bar{t}$*. From*

[20] The fact that $\underline{R}_1$ decreases with σ can be observed directly from the expression in (20); a formal proof that $\underline{R}_2$ decreases with σ can be found in Appendix C.

$t > \bar{t}$ onwards the economy will stop carrying out any type of innovation effort.

ii) If $\bar{t}(\sigma) \geq \underline{t}(\sigma)$, the economy will experience process innovation effort until period $t = \underline{t}$, and in $t = \underline{t} + 1$ the economy will be able to switch to an equilibrium with product innovation effort.

Proposition 3 shows first that when parametric conditions lead to $\bar{t}(\sigma) < \underline{t}(\sigma)$, the economy will *never* manage to switch to an equilibrium with product innovation. Importantly, in these cases, process innovation and income growth will eventually come to an end. This will happen in $t = \bar{t}$. From then on, the value of R_t will remain constant at $\bar{t}$, implying in turn that incomes will also stay fixed thereafter, at the level $Y_t = 1 + \sigma\bar{t}$ for all $t \geq \bar{t}$.

Conversely, when $\bar{t}(\sigma) \geq \underline{t}(\sigma)$, an equilibrium with product innovation effort will arise in $t = \underline{t} + 1$. The economy will experience an initial phase of growth driven by process innovation effort until $t = \underline{t}$. As incomes rise during this phase, the (implicit) willingness to pay for the higher-quality variety increases. Eventually, at $t = \underline{t} + 1$ the willingness to pay for $q = 1 + \rho$ becomes high enough to turn product innovation effort more profitable than process innovation effort. At this point, the growth-regime switch can take place.

An important question to address is what turns the condition $\bar{t}(\sigma) \leq \underline{t}(\sigma)$ more likely to hold. Recall that $\overline{R}(\sigma)$ is increasing in σ, while $\underline{R}(\sigma)$ is decreasing in σ. In addition, from the expressions in (20) and (21), it follows that the threshold $\underline{R}(\sigma)$ also satisfies $\lim_{\sigma \to 0} \underline{R}(\sigma) = \infty$ and $\lim_{\sigma \to \infty} \underline{R}(\sigma) = 0$, while from (12) we can observe that $\lim_{\sigma \to \infty} \overline{R}(\sigma) = (e^{\eta} - 1)^{-1} > 0$. As a result, given the way $\bar{t}(\sigma)$ and $\underline{t}(\sigma)$ are defined, there must always exist a value of σ high enough to ensure that condition $\bar{t}(\sigma) \leq \underline{t}(\sigma)$ holds true. We state this result more formally in the following corollary.

Corollary 1 *There exists a strictly positive and finite cut-off value, $\widehat{\sigma} > e^{\eta} - 1$, such that when $\sigma > \widehat{\sigma}$, the condition $\bar{t}(\sigma) \leq \underline{t}(\sigma)$ holds true.*

4.4 From Product Innovation (Back) to Process Innovation

In our model, process innovation effort exhibits an inherent tendency to come to a halt. For this reason, it becomes crucial that the switch to an equilibrium with product innovation effort takes place soon enough; otherwise the switch will simply end up not happening at all. Naturally, product innovation effort helps sustaining positive growth while it takes place. However, this is only one part of the positive effect that product innovation exerts on growth: product innovation effort may also boost the incentives to further undertake process innovation effort in future periods.

The underlying reason why the incentives to undertake process innovation effort exhibit an intrinsic decaying tendency rests on the decreasing marginal utility of consumption. As process innovations lead to an expansion of the physical production of the baseline quality variety, the additional utility that individuals obtain from higher consumption levels of that variety declines, hurting in turn the profit derived from another round of process innovation. Product innovation effort works, however, on a rather distinct dimension: it leads to higher utility by each unit of physical consumption. Furthermore, the marginal utility of consumption declines more slowly for higher quality varieties than for lower quality ones. Quality upgrading thus relaxes the depressing effect that decreasing marginal utility of consumption imposes on the incentives to further raise physical production via process innovation.

In the sake of brevity, in this subsection we focus only on the case in which $\bar{t}(\sigma) \geq \underline{t}(\sigma)$ holds. This means that the growth path of the economy is driven by positive process innovation effort until period $t = \underline{t}$, with $R_t = t$ during that phase. Next, at $t = \underline{t} + 1$, the economy is able to switch to an equilibrium with product innovation effort. This will allow the production of the quality variety $q = 1 + \rho$ in $t = \underline{t} + 1$, and also in all $t > \underline{t} + 1$.

Consider now an economy in $t = \underline{t} + 2$, right after an equilibrium with product innovation effort took place. Firms that are active in this period will inherit a technology that allows them to produce $(1 + \sigma\underline{t}) / (1 + \rho)$ units the quality variety $q = 1 + \rho$ with one unit of labour. The question to address now is whether the individuals alive in $t = \underline{t} + 2$ will choose to invest their spare time endowment in process innovation effort, or if they will prefer to consume it as leisure.

When a generic individual i alive in $t = \underline{t} + 2$ exerts process innovation effort, he will be able to sell his blueprint for a price

$$\pi^{p}_{i,\underline{t}+2} = P_{\underline{t}+2} \frac{1 + \sigma(1 + \underline{t})}{1 + \rho} - w_{\underline{t}+2},$$

where,

$$P_{\underline{t}+2} = (1 + \rho) Y^{\rho/(1+\rho)}_{\underline{t}+2} \quad \text{and} \quad w_{\underline{t}+2} = P_{\underline{t}+2} \frac{1 + \sigma\underline{t}}{1 + \rho}.$$

Hence, if i exerts process innovation effort, his lifetime earnings will be given by

$$Y_{i,\underline{t}+2}(\varepsilon_{i,p} = 1) = Y^{\rho/(1+\rho)}_{\underline{t}+2} \left[1 + \sigma(1 + \underline{t})\right]. \tag{23}$$

On the other hand, if i uses his spare time endowment as leisure, he will only obtain the wage $w_{\underline{t}+2}$. Thus,

$$Y_{i,\underline{t}+2}(\varepsilon_{i,p} = 0) = Y^{\rho/(1+\rho)}_{\underline{t}+2} (1 + \sigma\underline{t}). \tag{24}$$

Comparing (23) and (24), leads finally to the following result:

Proposition 4 *Consider an economy for which the condition $\bar{t}(\sigma) \geq \underline{t}(\sigma)$ is satisfied. Then, following an equilibrium with product innovation effort in $t = \underline{t} + 1$, the individuals alive in $t = \underline{t} + 2$ will exert process innovation effort in equilibrium (i.e., $\varepsilon^*_{\underline{t}+2,p} = 1$), if and only if:*

$$(1+\rho)\ln\left(1+\frac{\sigma}{1+\sigma\underline{t}}\right) > \eta. \tag{25}$$

To interpret Proposition 4, it proves insightful to compare condition (25) vis-a-vis (11). Recall that along a growth path with positive process innovation effort we have $R_t = t$. Given this, according to condition (11), an economy that managed to sustain process innovation effort until $t = \underline{t}$ would next need to satisfy

$$\ln\left(1+\frac{\sigma}{1+\sigma\underline{t}}\right) > \eta, \tag{26}$$

in order to be able to keep sustaining further process innovation effort *after* period $\underline{t}$. The situation is slightly different for an economy that exhibited a growth path with process innovation effort until $\underline{t}$, and switched to an equilibrium with product innovation effort in $\underline{t} + 1$. In this case, to be able to support further process innovation effort after period $\underline{t} + 1$, the economy will need to satisfy (25). This condition is actually *weaker* than (26) since $\rho > 0$. Hence, in our model, product innovation effort may also foster growth by reinvigorating the incentives to carry out further process innovation effort in future periods.

4.5 Coordination Failures and Low-Quality Traps

Proposition 2 showed that under certain parametric configurations our model exhibits multiple equilibria. In one equilibrium, individuals' expectations coordinate on $\varepsilon^*_{t,q} = 1$. This is the equilibrium we studied in the previous two subsections, where product innovation keeps the economy on a positive growth path during t (and, possibly, it also fosters future growth by regenerating the incentives for further process innovation after t). The other equilibrium is, instead, the result of a coordination failure: individuals expect no one to use their spare time as product innovation effort, which ends up curbing individuals' incentives to undertake product innovation.

The possibility of coordination failures rests on the combined effect of a pecuniary and a non-pecuniary positive externality associated to exerting product innovation effort. The intuition behind the pecuniary externality is quite straightforward. As more individuals exert product innovation effort (instead of consuming leisure) the value of Y_t increases. This in turn means that the firm selling the higher-quality variety can charge a higher price (P_t) for it, which ultimately translates into a greater value of product innovation blueprints (π_t^q).

The intuition for the non-pecuniary externality in more subtle. It relates to a higher probability of expansion in the set of *available* quality varieties (Q_t) as more individuals exert product innovation effort. Recall that only one blueprint (s_t^q) in the set of feasible blueprints (B_t^q) will turn out to be successful in generating quality upgrading. As consequence, the larger the mass of individuals who exert product innovation effort, the greater the probability that one of those individuals will generate the blueprint s_t^q. Individuals derive more utility from the higher-quality variety than from the baseline quality variety. Hence, an increased probability that the higher-quality variety will be marketed in period t will *indirectly* raise as well the incentives for individuals to use their spare time endowment as product innovation effort.[21]

5 Unbounded Number of Quality Varieties and Long-Run Growth

So far we have studied an environment with only two levels of quality; i.e., $Q = \{1, 1+\rho\}$. While this simplified framework is able to convey our main insights in terms of feedbacks between process and product innovation effort, it cannot generate dynamics with rising incomes in the long run. In particular, when Q is bounded above, innovation and growth will eventually stop, no matter the parametric configuration of the model.[22] This section extends the previous model by allowing an infinite number of quality varieties. Interestingly, we show that in this case those economies that manage to switch to an equilibrium with quality upgrading in $t = \underline{t} + 1$ (as described in Proposition 3) will turn out to be able to sustain positive growth forever.

Before moving on to show this result formally, it should be first straightforward to note that allowing an infinite number of quality varieties will not alter any of our previous results for economies that *fail* to reach at some point an equilibrium with product innovation. In other

[21] While the pecuniary externality linked to product innovation effort has a similar flavour to those present in the Big Push literature [e.g., Murphy, Shleifer and Vishny (1989)], the non-pecuniary externality is conceptually very different from those studied before in the context of coordination failures and poverty traps. This externality is caused by an expansion in the consumption space when the economy experiences a product innovation, which in turn alters agents' marginal rate of substitution between leisure and commodity consumption in favour of the latter (thereby indirectly enhancing agents' incentives to undertake product innovation effort).

[22] See Appendix B for a description of how growth eventually stop in our previous version of the model with Q=$\{1, 1+\rho\}$, even when an economy manages to switch to an equilibrium with product innovation effort at some point along the growth path.

words, if the parametric configuration of the model is such that $\bar{t}(\sigma) < \underline{t}(\sigma)$, the results in Proposition 3 will remain intact even when Q is not bounded above. That is, the economy will experience process innovation effort until period $t = \bar{t}$, and it will stop carrying out any type of innovation effort after period $\bar{t}$.

For the rest of this section we will then focus on the case when $\bar{t}(\sigma) \geq \underline{t}(\sigma)$. As we have already seen, this economy will feature an equilibrium with process innovation until period $t = \underline{t}$, and it will switch to an equilibrium with product innovation in $t = \underline{t} + 1$.[23] The question to address now is two-fold: whether the economy will be able to sustain an equilibrium with *some* type of innovation effort in $t = \underline{t}+2$, and in case it is able to do so *which* type of innovation effort it will be. Naturally, if the economy experiences some type of innovation effort in $\underline{t} + 2$, the very same question will arise again in $\underline{t} + 3$, and so on and so forth.

The following lemma shows an important preliminary result regarding the income path in the long run when the set Q comprises an infinite number of quality varieties.

Lemma 5 *When* $Q = \{1, 1+\rho, 1+2\rho, ...\}$*, economies that satisfy the condition* $\bar{t}(\sigma) \geq \underline{t}(\sigma)$ *will always be able to sustain an equilibrium with some type of innovation effort.*

Lemma 5 essentially states that economies which are able to switch to an equilibrium with product innovation in period $\underline{t} + 1$, will also be able sustain an equilibrium with positive innovation effort and income growth in all periods after $\underline{t} + 1$. The reason behind this result is the following: if the non-deviation condition (16) holds in $\underline{t} + 1$ when $R_{t-1} = \underline{t}$, then the analogous non-deviation conditions that would apply to a quality variety $q > 1 + \rho$ in future periods will always hold true for any $R_{t-1} \geq \underline{t}$. This in turn means that, in any period $t \geq \underline{t} + 2$, the action $\varepsilon_{i,q} = 1$ will *strictly dominate* the action $\varepsilon_i = 0$ when all the other individuals $j \neq i$ are choosing $\varepsilon_{j,q} = 1$, ruling out the possibility that leisure consumption arises as a unique equilibrium in a period $t \geq \underline{t} + 2$.

Lemma 5 addresses the question of whether an economy may sustain an equilibrium with *some* type of innovation effort in the periods that follow $t = \underline{t} + 1$. We proceed to study now *which* type of innovation effort takes place in those periods. We first show that a growth path where only product innovation effort takes place in all $t > \underline{t} + 1$ can never arise in equilibrium. Next, we show that a growth path along $t > \underline{t} + 1$ that relies only on process innovation effort cannot take place in equilibrium either. Given the result in Lemma 5, we finally show that the growth path followed during the horizon $t \geq \underline{t} + 1$ will display finite spells with product innovation, alternating with finite spells with process innovation.

[23] In the sake of brevity, in this section we disregard again the possibility that coordination failures may prevent an equilibrium with product innovation effort from taking place when this equilibrium actually exists.

Consider first the situation in which the only type of innovation effort undertaken during $t > \underline{t} + 1$ is in product innovation. In this case, we would have $R_{t-1} = \underline{t}$ for all $t > \underline{t} + 1$. In addition, the highest quality variety available in t (when $t > \underline{t} + 1$) would be $q_t = 1 + (t - \underline{t})\rho$. These results in turn imply that, when $\varepsilon_{t,q} = 1$ for all $t > \underline{t} + 1$, the income in a generic period $t > \underline{t} + 1$ will be given by

$$Y_t(\widetilde{\varepsilon}_{q,t>\underline{t}+1}) = (1 + \sigma \underline{t})^{1+(t-\underline{t})\rho}\,, \tag{27}$$

where we use $\widetilde{\varepsilon}_{q,t>\underline{t}+1}$ to denote the *hypothetical* path in which $\varepsilon_{t,q} = 1$ for all $t > \underline{t} + 1$. For this to be an equilibrium path no single individual alive in any period $t > \underline{t} + 1$ would prefer to deviate from it to exerting process innovation effort. Notice that if a generic individual i alive in $t > \underline{t} + 1$ chose instead to set $\varepsilon_{i,t,p} = 1$, his earnings would be

$$Y_t(\varepsilon_{i,t,p} = 1 \,|\widetilde{\varepsilon}_{q,t>\underline{t}+1}) = Y_t(\widetilde{\varepsilon}_{q,t>\underline{t}+1})^{(t-1-\underline{t})\rho/[1+(t-1-\underline{t})\rho]}\,(1 + \sigma \underline{t} + \sigma)\,. \tag{28}$$

Comparing (27) and (28), we can observe that $Y_t(\widetilde{\varepsilon}_{q,t>\underline{t}+1}) > Y_t(\varepsilon_{i,t,p} = 1 \,|\widetilde{\varepsilon}_{q,t>\underline{t}+1})$ requires that $(1 + \sigma \underline{t})^{1+(t-\underline{t})\rho} > (1 + \sigma \underline{t} + \sigma)^{1+[(t-1)-\underline{t}]\rho}$, which will fail to hold when t becomes sufficiently large (i.e., when t departs sufficiently from $\underline{t}$).

Suppose now that during $t > \underline{t}+1$ all individuals use their spare time endowments as process innovation effort. Such a growth path would be characterised by $R_t = t - 1$ and $q_{t-1} = 1 + \rho$, for all $t > \underline{t} + 1$. The income in a generic period $t > \underline{t} + 1$ will be thus given by

$$Y_t(\widetilde{\varepsilon}_{p,t>\underline{t}+1}) = [1 + \sigma\,(t - 1)]^{1+\rho}\,, \tag{29}$$

where $\widetilde{\varepsilon}_{p,t>\underline{t}+1}$ denotes the *hypothetical* path in which $\varepsilon_{t,p} = 1$ for all $t > \underline{t} + 1$. If individual i alive in $t > \underline{t} + 1$ deviates to exerting product innovation effort (i.e., $\varepsilon_{i,t,q} = 1$), he would obtain

$$Y_t(\varepsilon_{i,t,q} = 1 \,|\widetilde{\varepsilon}_{p,t>\underline{t}+1}) = Y_t(\widetilde{\varepsilon}_{p,t>\underline{t}+1})^{2\rho/(1+2\rho)}\,[1 + \sigma\,(t - 2)]\,. \tag{30}$$

Now, comparing (29) and (30), it follows that $Y_t(\widetilde{\varepsilon}_{p,t>\underline{t}+1}) > Y_t(\varepsilon_{i,t,q} = 1 \,|\widetilde{\varepsilon}_{p,t>\underline{t}+1})$ requires $[1 + \sigma\,(t - 1)]^{1+\rho} > [1 + \sigma\,(t - 2)]^{1+2\rho}$, which will also fail to hold when t is large enough.

We can now present the main result of this section, describing the growth path followed by economies that manage to sustain positive growth in the long run.

Proposition 5 *When* $Q = \{1, 1 + \rho, 1 + 2\rho, ...\}$, *only those economies that satisfy the condition* $\overline{t}(\sigma) \geq \underline{t}(\sigma)$ *will be able to sustain an equilibrium growth path with positive growth in the long run. Along such a growth path, the economy will experience the following growth sequence:*

1. *There is an initial growth phase driven by process innovation effort starting in* $t = 1$ *until period* $t = \underline{t} \geq 1$

2. *There is a second growth phase driven product innovation effort starting in* $t = \underline{t} + 1$*, and lasting for a finite number of periods until period* $\widehat{t} \geq \underline{t} + 1$*.*

3. *After* $\widehat{t}$*, the growth path exhibits finite spells of growth driven by process innovation effort, alternating with finite spells of growth driven by product innovation effort.*

Proposition 5 shows that economies whose σ turns out to be large enough to make the condition $\bar{t}(\sigma) \geq \underline{t}(\sigma)$ hold true (i.e., $\sigma > \widehat{\sigma}$ as defined in Corollary 1) will be able to exhibit positive income growth in the long run. The growth path of these economies is characterised by an initial phase driven by process innovation effort, followed by a sequence of finite spells of growth driven by product innovation effort and process innovation effort that alternate each other indefinitely. This result showcases the interplay between process and product innovations present in our model. On one side, the quantity expansion brought about by process innovations bolsters the incentives to start investing in quality-upgrading innovations. On the other side, the ensuing quality expansion stemming from product innovations relaxes the inherent tendency of profit prospects from further process innovations to decay. The alternation of equilibria with process and product innovation efforts exploits this feedback loop, and is thus instrumental to keeping income growth alive in the long run.

As a final remark, it is worth noting that $\bar{t}(\sigma) \geq \underline{t}(\sigma)$ is a *necessary and sufficient* condition for an economy to exhibit sustained income growth. As a consequence, Proposition 5 entails the possibility of income divergence in the long run across different economies. In particular, those economies whose σ lie below the cutoff value $\widehat{\sigma}$ in Corollary 1 will eventually fall trapped in an equilibrium without income growth. On the other hand, when $\sigma > \widehat{\sigma}$, income per head will be able to keep rising unboundedly at a positive rate in the long run.

6 Concluding Remarks

We presented a model where the combined effect of process and product innovations steer the economy along a growth path featuring both quantity and quality expansion. At early stages of development, when willingness to pay for quality upgrading is low, growth must be driven by the cost-cutting effect of process innovations. However, an economy cannot rely exclusively on process innovations in order to achieve long-lasting growth, as their profits tend to decrease as physical production keeps expanding. Sustained growth necessitates that the economy becomes also able to generate product innovations as it moves along the development path. In addition, quality-upgrading innovations boost the incentives to keep expanding physical production. Therefore, while process innovations are necessary to turn product innovations sufficiently profitable, product innovations are able to regenerate profit prospects from further process innovations. This implicit feedback loop may keep growth alive in the long run.

Our model has restricted the consumption space to a very specific case: one single final good available in different quality varieties, which are all perfect substitutes among each other. One important type of innovation effort that our model has then ruled out is that one that leads to a *horizontal* expansion in the set of final goods, as in Judd (1985), Romer (1990), and Grossman and Helpman (1991c, Ch.3). In principle, these types of innovations may also be able to keep growth alive in the long run. In particular, as profit prospects from cost-cutting innovations dwindle owing to decreasing marginal utility in a *given* good category, individuals may at some point find it worthwhile to introduce a completely *new* good category (whose marginal utility remains still relatively high). This new final good would offer initially large profit prospects from process innovations, which would tend to diminish with subsequent rounds process innovations. We see this mechanism leading to a horizontal expansion of the set of consumption goods as complementary to the interplay between quantity and quality expansion studied by our model. Certainly, a model in which growth features a simultaneous expansion in quantity, quality and variety of consumption, with positive feedbacks between all three dimensions, could yield a more encompassing description of growth in mature economies, and we see this as an appealing avenue of future research.

Finally, our model studies the case of a closed economy in autarky. An interesting question that we cannot then address here is whether our framework, adapted to include open economies and trade, could possibly lead to some sort of international specialisation of innovation effort by type. In particular, in the presence of trade costs, it may be the case that process innovation effort tends move to middle-income economies, while richer economies specialise mainly in generating product innovations. Such result would be somehow reminiscent of the Linder's hypothesis of quality specialisation in trade, and could therefore provide an explanation of that theory originated from a fully-fledged endogenous growth model.

Appendix A: Proofs

Proof of Lemma 1. An individual in t will optimally set $\varepsilon_t^* = 1$ iff the utility obtained from consuming $1 + \sigma R_{t-1} + \sigma$ units of the baseline quality good is strictly greater than the utility derived from consuming $1 + \sigma R_{t-1}$ units of it plus the utility of leisure, η. Using (9) and (10), together with the utility function (4) when $Q_t = \{1\}$, condition (11) obtains. ■

Proof of Lemma 2. We first notice that in any feasible equilibrium of our model, we will have that $[q\,x(q)]^q \geq q\,x(q)$, and hence we can focus only on that part of the utility function (4).[24] Next, we carry out the proof in three separate steps.

Step 1) $P_t > (1+\rho)\,Y_t^{\rho/(1+\rho)}$ cannot hold in equilibrium.

Using (4) we can observe that the utility obtained by an individual alive in t if he chooses to consume the variety of the good with quality $q = 1 + \rho$ is given by:

$$U_t(q = 1+\rho) = \ln\left[(1+\rho)\,\frac{Y_t}{P_t}\right]^{1+\rho}. \tag{31}$$

Instead, if he chooses to consume the baseline quality variety, he would obtain:

$$U_t(q = 1) = \ln\left(Y_t\right). \tag{32}$$

Hence, comparing (31) and (32), we can observe that $P_t > (1+\rho)\,Y_t^{\rho/(1+\rho)}$ implies $U_t(q = 1+\rho) < U_t(q = 1)$, and therefore no one would consume the higher-quality variety.

Step 2) $P_t < (1+\rho)\,Y_t^{\rho/(1+\rho)}$ cannot hold in equilibrium.

Suppose in equilibrium $P_t = \widetilde{P}_t < (1+\rho)\,Y_t^{\rho/(1+\rho)}$. Since an equilibrium must also necessarily satisfy the zero profit condition, it must then be the case that product innovators are being paid

$$\widetilde{\pi}_t = \widetilde{P}_t \frac{1+\sigma R_{t-1}}{1+\rho} - (1+\sigma R_{t-1})$$

for their blueprints. Suppose now some firm decides to offer product innovators $\widehat{\pi}_t$ for their blueprints, where

$$\widehat{\pi}_t \equiv \left(\widetilde{P}_t + \widehat{\varepsilon}\right)\frac{1+\sigma R_{t-1}}{1+\rho} - (1+\sigma R_{t-1})\,, \quad \text{and} \quad \widehat{\varepsilon} > 0.$$

This firm would then attract all product innovation blueprints created in t. Furthermore, this firm could charge a price $P_t' \equiv \widetilde{P}_t + \varepsilon' < (1+\rho)\,Y_t^{\rho/(1+\rho)}$, where $\varepsilon' > \widehat{\varepsilon} > 0$, for the higher-quality variety of the final good, obtaining positive (expected) profits. As a consequence, a situation

[24] For a formal proof of this result, see Appendix C.

where a firm charges a price $P_t < (1+\rho)\, Y_t^{\rho/(1+\rho)}$ for the higher-quality variety while it also satisfies the zero profit condition cannot arise in equilibrium.

Step 3) Using again (31) and (32), we can first observe that when $P_t = (1+\rho)\, Y_t^{\rho/(1+\rho)}$ individuals alive in t are indifferent between the baseline quality variety and the higher-quality variety. Moreover, when (13) holds, there exist no profitable deviation to any firm. In particular, in order to outcompete a firm whose strategy is characterised by (14) and (13), another firm should either offer the higher-quality variety for a lower price or, alternatively, offer product innovators a higher payment for their blueprints while keeping the price of the higher quality variety fixed (since $P_t > (1+\rho)\, Y_t^{\rho/(1+\rho)}$ cannot hold in equilibrium, as shown before in *Step 1*). Both deviations, however, lead to a loss. ■

Proof of Proposition 1. The proof follows immediately from the derivations in the main text, together with the fact that (1) implies that after a sequence of t consecutive periods featuring an equilibrium with positive process innovation effort we have that $R_t = t$. ■

Proof of Lemma 3. Using (13), we obtain that when individuals alive in t exert product innovation effort:

$$Y_t = P_t \frac{1+\sigma R_{t-1}}{1+\rho}. \tag{33}$$

Replacing (33) into (14) yields

$$P_t = (1+\rho)\left(P_t \frac{1+\sigma R_{t-1}}{1+\rho}\right)^{\frac{\rho}{1+\rho}},$$

from where we may solve for P_t and obtain:

$$P_t = (1+\rho)\,(1+\sigma R_{t-1})^{\rho}. \tag{34}$$

Lastly, plugging (34) into (33) yields (15). ■

Proof of Lemma 4. Part *i*). First of all, notice that equation (15) implies that when the entire cohort alive in t undertake product innovation, $Y_t = (1+\sigma R_{t-1})^{1+\rho}$. In this situation, the level of utility achieved by any generic individual i alive in t is given by

$$U_{i,t}(\varepsilon_{i,t,q} = 1 |\, \varepsilon_{t,q} = 1) = \ln\left[(1+\rho)\,\frac{(1+\sigma R_{t-1})^{1+\rho}}{P_t}\right]^{1+\rho}. \tag{35}$$

Using the expression in (14), together with $Y_t = (1+\sigma R_{t-1})^{1+\rho}$, (35) yields:

$$U_{i,t}(\varepsilon_{i,t,q} = 1 |\, \varepsilon_{t,q} = 1) = (1+\rho)\ln(1+\sigma R_{t-1}). \tag{36}$$

Suppose now this generic individual i alive in t would deviate from $\varepsilon_{i,t,q} = 1$ to setting $\varepsilon_{i,t} = 0$. In this case, $Y_{i,t}(\varepsilon_{i,t} = 0) = w_t = 1 + \sigma R_{t-1}$. Notice now that the price (14) leaves indifferent an individual with $Y_t = (1 + \sigma R_{t-1})^{1+\rho}$ between the two varieties of the consumption good. Since, $1 + \sigma R_{t-1} < (1 + \sigma R_{t-1})^{1+\rho}$, it must then be the case that, if setting $\varepsilon_{i,t} = 0$, individual i will then strictly prefer to consume the baseline quality rather than (the more expensive) higher-quality variety. Moreover, when $w_t = 1 + \sigma R_{t-1}$, there will always be a firm willing to offer the baseline quality variety (i.e., $q = 1$), as it would break even by doing so. Hence, when a generic individual i alive in t sets $\varepsilon_{i,t} = 0$, within a context where the rest are setting $\varepsilon_{t,q} = 1$, i achieves

$$U_{i,t}(\varepsilon_{i,t} = 0 | \, \varepsilon_{t,q} = 1) = \ln(1 + \sigma R_{t-1}) + \eta \tag{37}$$

Finally, comparing (36) and (37), condition (16) ensures that $U_{i,t}(\varepsilon_{i,t,q} = 1 | \, \varepsilon_{t,q} = 1) > U_{i,t}(\varepsilon_{i,t} = 0 | \, \varepsilon_{t,q} = 1)$, completing the proof.

Part *ii*) Since both $\varepsilon_{i,t,q} = 1$ and $\varepsilon_{i,t,p} = 1$ entail the leisure loss of η, a necessary and sufficient condition for $U_{i,t}(\varepsilon_{i,t,p} = 1 | \, \varepsilon_{t,q} = 1) < U_{i,t}(\varepsilon_{i,t,q} = 1 | \, \varepsilon_{t,q} = 1)$ will be that $Y_{i,t}(\varepsilon_{i,t,p} = 1) < Y_{i,t}(\varepsilon_{i,t,q} = 1)$. Using then (9) and (15), condition (17) immediately obtains. ■

Proof of Proposition 2. Part *i*) The proof that when (16) and (17) hold there exists an equilibrium in t with $\varepsilon^*_{t,q}$ follows directly from Lemma 4.

Part *ii*) We prove this part of the proposition in three separate steps.

Step 1) When condition (17) holds, an equilibrium with $\varepsilon^*_{t,p} = 1$ does not exist.

Proof. Using (14) and (13), we can observe that a necessary condition for an equilibrium with $\varepsilon^*_{t,p} = 1$ to exist is that: $Y_t^{(1+\rho)/\rho}(1 + \sigma R_{t-1}) - (1 + \sigma R_{t-1}) < \sigma$. This leads to the following expression:

$$Y_t < \left(\frac{1 + \sigma + \sigma R_{t-1}}{1 + \sigma R_{t-1}} \right)^{\frac{1+\rho}{\rho}}. \tag{38}$$

Consider then the case when all the individuals alive in t set $\varepsilon_{t,p} = 1$. In this situation Y_t is given by (9). Plugging this value into the LHS of (38) leads after some simple algebra to $(1 + \sigma + \sigma R_{t-1}) > (1 + \sigma R_{t-1})^{1+\rho}$, contradicting (17).

Step 2) When condition (17) holds and condition (18) does not hold, an equilibrium with $\varepsilon^*_t = 0$ does not exist.

Proof. Notice first that (18) not holding means that (11) holds true. Therefore, when (18) fails to hold no individual i alive in t would thus set $\varepsilon_{i,t} = 0$ in equilibrium. Moreover, the fact that (17) holds in turn implies that the unique equilibrium in this case must feature $\varepsilon^*_{t,q} = 1$.

Step 3) When, alongside (16) and (17), also both (18) and (19) hold true, an equilibrium with $\varepsilon_t^* = 0$ also exists.

When all the individuals alive in t set $\varepsilon_t = 0$, the expression in (10) applies, and thus $Y_t = 1+\sigma R_{t-1}$. In this situation, if some generic individual alive in t deviates to exerting product innovation effort, the price of the higher-quality variety would be $P_t = (1+\rho)\left(1+\sigma R_{t-1}\right)^{\rho/(1+\rho)}$, and a product innovation blueprint would sell for $\pi_{i,t}^q = \left(1+\sigma R_{t-1}\right)^{(1+2\rho)/(1+\rho)} - (1+\sigma R_{t-1})$. As a result, by deviating to exerting product innovation, individual i would have as lifetime income: $Y_{i,t} = \left(1+\sigma R_{t-1}\right)^{(1+2\rho)/(1+\rho)}$. Notice now that since individual i has measure zero, after his unilateral deviation to $\varepsilon_{i,t,q} = 1$, the probability that he ends up generating the successful product innovation blueprint (i.e., s_t^q) is actually zero. Therefore, when individual i is contemplating the possibility to deviate unilaterally to $\varepsilon_{i,t,q} = 1$ (from a situation where all individuals alive in t set $\varepsilon_t = 0$), he is also aware that (almost surely) he will not end up generating s_t^q, and thus the only variety that will be offered by firms is the baseline quality variety, $q = 1$. As a result, the utility that individual i expects to obtain should he deviate unilaterally to $\varepsilon_{i,t,q} = 1$ is given by:

$$U_{i,t}(\varepsilon_{i,t,q} = 1 |\, \varepsilon_t = 0) = \ln\left(1+\sigma R_{t-1}\right)^{(1+2\rho)/(1+\rho)}. \tag{39}$$

On the other hand, by sticking to $\varepsilon_{i,t} = 0$, individual i would obtain:

$$U_{i,t}(\varepsilon_{i,t} = 0 |\, \varepsilon_t = 0) = \ln(1+\sigma R_{t-1}) + \eta. \tag{40}$$

An equilibrium where $\varepsilon_t^* = 0$ will exist if $U_{i,t}(\varepsilon_{i,t,q} = 1 |\, \varepsilon_t = 0) \leq U_{i,t}(\varepsilon_{i,t} = 0 |\, \varepsilon_t = 0)$. Therefore, using (39) and (40), we can obtain (19). This completes the proof that when both (18) and (19) are verified by an economy that also satisfies (16) and (17), then two equilibria exist (among the class of symmetric Nash equilibria in pure strategies): $\varepsilon_{t,q}^* = 1$ and $\varepsilon_t^* = 0$. ■

Proof of Proposition 3. Part *i)* From Proposition 1, it follows that the economy will keep growing through process innovation effort until $t = \bar{t}(\sigma)$, and reach a level of $R_{\bar{t}} = \bar{t}(\sigma)$ in that period. Consider now what happens in $t = \bar{t}(\sigma) + 1$. We know from Proposition 1 that no individual alive in $t = \bar{t}(\sigma) + 1$ will invest in process innovation. Also, the fact that $\underline{t}(\sigma) > \bar{t}(\sigma)$ means that $\bar{t}(\sigma) \leq \underline{t}(\sigma) - 1$. Furthermore, from Definition 2, it follows that $\underline{R}(\sigma) > \underline{t}(\sigma) - 1$. Therefore, $\bar{t}(\sigma) < \underline{R}(\sigma)$, in turn implying that no individual alive in $t = \bar{t}(\sigma) + 1$ will invest in product innovation either. Since individuals will then consume their spare time endowment as leisure in $t = \bar{t}(\sigma) + 1$, the same situation will repeat itself in $t = \bar{t}(\sigma) + 2$, and thereafter.

Part *ii)* Given the results in Proposition 1, we can observe that the economy will keep growing through process innovation effort until $t = \underline{t}(\sigma)$, and reach a level of $R_{\underline{t}} = \underline{t}(\sigma)$ in that period. Consider now what happens in period $t = \underline{t}(\sigma) + 1$. Using again Definition 2, $\underline{t}(\sigma) > \underline{R}(\sigma)$. Therefore, the conditions in Lemma 4 must hold true in $t = \underline{t}(\sigma) + 1$, and the economy will therefore exhibit an equilibrium with product innovation effort in that period. Finally, owing to Proposition 2, in this situation the economy cannot possibly exhibit an equilibrium in $t = \underline{t}(\sigma) + 1$ with process innovation effort, which completes the proof. ■

Proof of Proposition 4. From Assumption 2, we can observe that firms active in $t = \underline{t} + 2$ will inherit a technology that will allow them to produce $(1 + \sigma\underline{t})/(1 + \rho)$ units of the quality variety $q = 1 + \rho$ with one unit of labour. Letting $P_{\underline{t}+2}$ denote the price of the quality variety $q = 1 + \rho$ in period $t = \underline{t} + 2$, it then follows that the equilibrium wage in that period will be $w_{\underline{t}+2} = P_{\underline{t}+2}(1 + \sigma\underline{t})/(1 + \rho)$, which using the result in (14) leads to:

$$w_{\underline{t}+2} = Y_{\underline{t}+2}^{\rho/(1+\rho)}(1 + \sigma\underline{t}). \tag{41}$$

Notice now that since $Y_{\underline{t}+2}^{\rho/(1+\rho)} > 1$, no firm active in $t = \underline{t} + 2$ will, in equilibrium, offer the baseline quality variety (if one of these firms did so, it would make a loss). As a result, in $t = \underline{t} + 2$ the only quality variety that will be actively offered in the market is $q = 1 + \rho$.

Consider now the effect of a process innovation in period $t = \underline{t}+2$. This would allow the firm that implements the process innovation to produce $(1 + \sigma\underline{t} + \sigma)/(1 + \rho)$ units of the quality variety $q = 1 + \rho$ with one unit of labour. As a consequence, process innovation blueprints will command a price:

$$\pi_{\underline{t}+2}^{p} = Y_{\underline{t}+2}^{\rho/(1+\rho)}\sigma. \tag{42}$$

Using (41) and (42), we can then obtain for a generic individual i alive in period $t = \underline{t} + 2$ the following expressions:

$$Y_{i,\underline{t}+2}(\varepsilon_{i,\underline{t}+2}^{p} = 1) = Y_{\underline{t}+2}^{\rho/(1+\rho)}(1 + \sigma\underline{t} + \sigma), \tag{43}$$

$$Y_{i,\underline{t}+2}(\varepsilon_{i,\underline{t}+2}^{p} = 0) = Y_{\underline{t}+2}^{\rho/(1+\rho)}(1 + \sigma\underline{t}). \tag{44}$$

Finally, using (43), (44) and the utility function (4), bearing in mind $P_{\underline{t}+2} = (1 + \rho)Y_{\underline{t}+2}^{\rho/(1+\rho)}$, we can observe that individuals in $t = \underline{t} + 2$ will set $\varepsilon_{\underline{t}+2}^{p} = 1$ if and only if (25) holds true. ■

Proof of Lemma 5. We carry out this proof by showing that when $\overline{t}(\sigma) \geq \underline{t}(\sigma)$ holds, in a context where the set Q comprises an infinite number of quality varieties, then for any generic individual i alive in $t > \underline{t}$ the action $\varepsilon_{i,t,q} = 1$ strictly dominates the action $\varepsilon_{i,t} = 0$, when all other individuals $j \neq i$ alive in t are choosing $\varepsilon_{j,t,q} = 1$.

Step 1. Period $t = \underline{t}+1$*:* The fact that when $\overline{t}(\sigma) \geq \underline{t}(\sigma)$ holds, $\varepsilon_{i,t,q} = 1$ strictly dominates the action $\varepsilon_{i,t}=0$ when all other individuals $j \neq i$ alive in t are choosing $\varepsilon_{j,t,q}=1$ follows directly from Proposition 3.

Step 2. Generalisation of Lemma 2 when Q is unbounded above: When we let $Q=\{1, 1+\rho, 1+2\rho, ...\}$, it follows that if a firm wishes to purchase the product innovation designed by a generic individual i alive in period t, it will have to pay him:

$$\pi_{i,t}^{q} = P_t(q_t) \frac{1+\sigma R_{t-1}}{q_t} - w_t, \tag{45}$$

where $P_t(q_t)$ is the price of the (newly designed) quality variety $q_t \in Q$. To compute the equilibrium value of $P_t(q_t)$, notice that, based on (4),this will follow from the condition

$$\ln\left(q_t \frac{Y_t}{P_t(q_t)}\right)^{q_t} \geq \ln(Y_t), \tag{46}$$

from where we can obtain

$$P_t(q_t) = q_t \, Y_t^{(q_t-1)/q_t} \tag{47}$$

when (46) holds with equality. Lastly, the fact that all firms active in t inherit a technology that allows producing $(1+\sigma R_{t-1})/q_{t-1}$ units of the quality variety $q_{t-1} \in Q$ with one unit of labour in turn implies that:

$$w_t = P_t(q_{t-1}) \frac{1+\sigma R_{t-1}}{q_{t-1}} = Y_t^{(q_{t-1}-1)/q_{t-1}} \left(1+\sigma R_{t-1}\right), \tag{48}$$

where $q_{t-1} = q_t - \rho$ when all individuals alive in t exert product innovation effort.

Step 3. Generalisation of Lemma 3 when Q is unbounded above: If all individuals alive in t set $\varepsilon_{t,q}=1$, using (45), (47) and (48), we obtain:

$$Y_t(\varepsilon_{t,q}=1) = (1+\sigma R_{t-1})^{q_t} \,. \tag{49}$$

Step 4. Periods $t \geq \underline{t}+2$: Due to Proposition 3, when $\overline{t}(\sigma) \geq \underline{t}(\sigma)$ holds, we must have that $R_{t-1} \geq \underline{t}$ and $q_{t-1} \geq 1+\rho$. Suppose also that all individuals set $\varepsilon_{t,q}=1$ in each period $t \geq \underline{t}+2$. Then, using (49), it follows that

$$Y_t(\varepsilon_{t,q}=1) = \left[1+\sigma\left(\underline{t}+\delta\right)\right]^{q_t} , \tag{50}$$

where $\delta \geq 0$. Using next the utility function (4), together with (50) and (47), we may obtain:

$$U_{i,t}(\varepsilon_{i,t,q}=1|\, \varepsilon_{t,q}=1) = q_t \ln\left[1+\sigma\left(\underline{t}+\delta\right)\right], \tag{51}$$

which denotes the level of utility achieved by a generic individual i alive in $t \geq \underline{t}+2$ when he sticks to the choice $\varepsilon_{i,t,q}=1$, given that all other individuals are choosing $\varepsilon_{t,q}=1$. On the other hand, if in such same circumstance individual i deviates to $\varepsilon_{i,t}=0$, he will achieve:

$$U_{i,t}(\varepsilon_{i,t}=0|\,\varepsilon_{t,q}=1)=q_{t-1}\ln\left[1+\sigma\left(\underline{t}+\delta\right)\right]+\eta=\left(q_t-\rho\right)\ln\left[1+\sigma\left(\underline{t}+\delta\right)\right]+\eta. \tag{52}$$

Comparing (51) and (52), we obtain:

$$U_{i,t}(\varepsilon_{i,t,q}=1|\,\varepsilon_{t,q}=1)>U_{i,t}(\varepsilon_{i,t}=0|\,\varepsilon_{t,q}=1)\quad\Longleftrightarrow\quad\rho\ln\left[1+\sigma\left(\underline{t}+\delta\right)\right]>\eta. \tag{53}$$

Finally, since $\delta \geq 0$, it immediately follows that when condition (16) holds in period $t=\underline{t}+1$ (which is the period when $R_{t-1}=\underline{t}$), then condition (53) will hold true in any period $t \geq \underline{t}+2$. Bearing in mind that $\rho\ln\left(1+\sigma\underline{t}\right)>\eta$ is a necessary condition for $\bar{t}(\sigma)\geq\underline{t}(\sigma)$ to hold, this last step completes the proof that the action $\varepsilon_{i,t,q}=1$ strictly dominates $\varepsilon_{i,t}=0$ when all other individuals alive in t set $\varepsilon_{t,q}=1$ in any generic period $t \geq \underline{t}+2$. ■

Proof of Proposition 5.

Part 1. The fact that there is an initial growth phase, between $t=1$ and $t=\underline{t}\geq 1$ driven by process innovation is already proven in Proposition 3.

Part 2. The fact that there is a second growth phase starting in $t=\underline{t}+1$ that is driven by product innovation effort also follows directly from Proposition 3. Next, the fact that this growth phase lasts for a finite number of periods (i.e., it lasts until $t=\widehat{t}\geq\underline{t}+1$) follows from equations (27) and (28), together with the ensuing discussion in the main text.

Part 3. Lastly, to prove that after $t=\widehat{t}$ the economy will be able to sustain positive growth forever by alternating finite spells where the equilibrium features process innovation effort with finite spells where the equilibrium features product innovation effort, we proceed by contradiction, while bearing in mind the result in Lemma 5.

Consider first the case of a hypothetical economy that for all periods $t \geq t'$ features process innovation effort in equilibrium, where we let $t'>\widehat{t}$. Such a growth path would be characterised by $R_t=t-\left(q_{t'-1}-1\right)/\rho$ and $q_t=q_{t'-1}$, for all $t\geq t'$. The income in a generic period $t \geq t'$ will be thus given by

$$Y_t(\widetilde{\varepsilon}_{p,t\geq t'})=\left\{1+\sigma\left[t-\left(q_{t'-1}-1\right)/\rho\right]\right\}^{q_{t'-1}}, \tag{54}$$

where $\widetilde{\varepsilon}_{p,t\geq t'}$ denotes the *hypothetical* growth path in which $\varepsilon_{t,p}=1$ for all $t\geq t'$. If a generic individual i alive in $t\geq t'$ deviates to exerting product innovation effort (i.e., $\varepsilon_{i,t,q}=1$), he will obtain

$$Y_t(\varepsilon_{i,t,q}=1\,|\widetilde{\varepsilon}_{p,t\geq t'})=Y_t(\widetilde{\varepsilon}_{p,t\geq t'})^{(q_{t'-1}+\rho-1)/(q_{t'-1}+\rho)}\left\{1+\sigma\left[t-1-\left(q_{t'-1}-1\right)/\rho\right]\right\}. \tag{55}$$

Now, comparing (54) and (55), it follows that $Y_t(\widetilde{\varepsilon}_{p,t\geq t'}) > Y_t(\varepsilon_{i,t,q} = 1\,|\widetilde{\varepsilon}_{p,t\geq t'})$ requires

$$\{1+\sigma\left[t-(q_{t'-1}-1)/\rho\right]\}^{q_{t'-1}} > \{1+\sigma\left[t-1-(q_{t'-1}-1)/\rho\right]\}^{q_{t'-1}+\rho}$$

which will fail to hold when t becomes sufficiently large. As a consequence, the economy cannot possibly sustain an equilibrium growth path where individuals exert process innovation effort during an infinitely long sequence of consecutive periods.

Consider next the case of a hypothetical economy that for all periods $t \geq t'$ features product innovation effort in equilibrium, where again we let $t' > \widehat{t}$. Such a growth path would be characterised by $R_{t-1} = (t'-1) - (q_{t'-1}-1)/\rho$ and $q_t = q_{t'-1} + (t-t'+1)\rho$, for all $t \geq t'$. These results in turn imply that the income in a generic period $t \geq t'$ will be given by

$$Y_t(\widetilde{\varepsilon}_{q,t\geq t'}) = \{1+\sigma\left[(t'-1)-(q_{t'-1}-1)/\rho\right]\}^{q_{t'-1}+(t-t'+1)\rho}\,, \tag{56}$$

where we use $\widetilde{\varepsilon}_{q,t\geq t'}$ to denote the *hypothetical* path in which $\varepsilon_{t,q} = 1$ for all $t \geq t'$. If a generic individual i alive in $t \geq t'$ deviates to process innovation effort (i.e., $\varepsilon_{i,t,p} = 1$), he will get

$$Y_t(\varepsilon_{i,t,p} = 1\,|\widetilde{\varepsilon}_{q,t\geq t'}) = Y_t(\widetilde{\varepsilon}_{q,t\geq t'})^{[q_{t'-1}+(t-t')\rho-1]/[q_{t'-1}+(t-t')\rho]}\,\{1+\sigma\left[t'-(q_{t'-1}-1)/\rho\right]\}\,. \tag{57}$$

Now, comparing (56) and (57), it follows that $Y_t(\widetilde{\varepsilon}_{q,t\geq t'}) > Y_t(\varepsilon_{i,t,p} = 1\,|\widetilde{\varepsilon}_{q,t\geq t'})$ requires

$$\{1+\sigma\left[(t'-1)-(q_{t'-1}-1)/\rho\right]\}^{q_{t'-1}+(t-t'+1)\rho} > \{1+\sigma\left[t'-(q_{t'-1}-1)/\rho\right]\}^{q_{t'-1}+(t-t')\rho}$$

which will fail to hold when t becomes sufficiently large. As a result, the economy cannot possibly sustain an equilibrium growth path where individuals exert product innovation effort during an infinitely long sequence of consecutive periods.

The previous two contradictions imply thus that there cannot exist an equilibrium growth path featuring either infinitely long spells of process innovation effort or infinitely long spells of product innovation effort. Lemma 5 stipulates that an economy satisfying the condition $\overline{t}(\sigma) \geq \underline{t}(\sigma)$ is always able to sustain an equilibrium with some type of innovation effort. Hence, it must be the case that the growth path followed by an economy satisfying $\overline{t}(\sigma) \geq \underline{t}(\sigma)$ will exhibit finite spells where individuals exert process innovation effort in equilibrium, alternating with finite spells where they exert product innovation effort in equilibrium. ■

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Online Appendices

Appendix B: A diagrammatic depiction of the dynamic path in the two-quality-variety model

This appendix provides a diagrammatic description of the dynamic path of an economy where $Q = \{1, 1+\rho\}$, depending on the value of σ that applies to it. The dynamics for innovation effort will follow from the two panels in Figure 1. The upper panel plots the threshold functions $\overline{R}(\sigma)$, $\underline{R}_1(\sigma)$ and $\underline{R}_2(\sigma)$, as specified in (12), (20) and (21), respectively. The lower panel plots again the threshold function $\overline{R}(\sigma)$, and it includes the period-level threshold functions $\bar{t}(\sigma)$ and $\underline{t}(\sigma)$; the former as (thicker) dashed lines and the latter as (thinner) solid lines. (Notice that $\bar{t}(\sigma)$ is weakly increasing in σ, while $\underline{t}(\sigma)$ is weakly decreasing in σ.) The lower panel in Figure 1 includes also another function of σ, denoted by $T(\sigma)$. The function $T(\sigma)$ pins down the values of $\underline{t}$ for which the LHS of (25) equals η.[1]

Consider first the case when product innovation is barred from the model (as in Section 3). Economies whose $\sigma \leq e^{\eta} - 1$ will never undertake process innovation, and their incomes will remain fixed at $Y_t = 1$. Economies with $e^{\eta} - 1 < \sigma \leq \sigma_a$ will undertake process innovation only in $t = 1$, and their incomes will be equal to $Y_t = 1 + \sigma$ for all $t \geq 1$. Next, economies with $\sigma \in (\sigma_a, \sigma_b]$ will undertake process innovation in $t = 1$ and $t = 2$; the incomes for this set of economies will be $Y_1 = 1 + \sigma$ and $Y_t = 1 + 2\sigma$ for any $t \geq 2$. Similarly, economies with $\sigma \in (\sigma_b, \sigma_c]$ will undertake process innovation in every period until $t = 3$, exhibiting an income path with $Y_1 = 1 + \sigma$, $Y_2 = 1 + 2\sigma$, and $Y_t = 1 + 3\sigma$ for any $t \geq 3$.[2]

Let us move on now to the case when product innovation is allowed into the model. Notice first that for the sets of economies discussed in the above paragraph, the dynamics remain unaltered. This is because, for any $\sigma \leq \sigma_c$ we have that $\underline{t}(\sigma) > \bar{t}(\sigma)$. Instead, when $\sigma > \sigma_c$ dynamics will be affected by the introduction of product innovation, as in these cases $\underline{t}(\sigma) \leq \bar{t}(\sigma)$. In particular, economies for which $\sigma \in (\sigma_c, \sigma_d]$ will undertake process innovation until $t = 4$, and in $t = 5$ they will switch to an equilibrium with positive product innovation effort.

[1] The formal expression of this function is thus:

$$T(\sigma) = \frac{1 + \sigma - e^{\eta/(1+\rho)}}{\sigma\left(e^{\eta/(1+\rho)} - 1\right)}.$$

Comparing this expression to (12), we can observe that $T(\sigma) > \underline{R}(\sigma)$ for any σ.

[2] A similar reasoning straightforwardly applies to economies whose $\sigma \in (\sigma_c, \sigma_e]$ and $\sigma \in (\sigma_e, \sigma_g]$. The former undertake process innovation until $t = 4$ and reach an income level $Y_t = 1+4\sigma$ in any $t \geq 4$; the latter undertake process innovation until $t = 5$ and reach an income level $Y_t = 1 + 5\sigma$ in any $t \geq 5$.

Similarly, economies whose $\sigma \in (\sigma_d, \sigma_f]$ will exhibit an equilibrium with $\varepsilon_{t,p} = 1$ until $t = 3$, and will switch to an equilibrium with $\varepsilon_{t,q} = 1$ in $t = 4$.[3]

The lower panel in Figure 1 also informs us of what happens in the period right after the economy switches to an equilibrium with product innovation. Consider, for example, an economy with $\sigma = \overline{\sigma} \in (\sigma_c, \sigma_d)$. This economy switches to an equilibrium with product innovation in $t = 5$. What happens next in $t = 6$? Since $\underline{t}(\overline{\sigma}) = 4$ and $T(\overline{\sigma}) > 4$, condition (25) holds for this economy in $t = 6$. Therefore, an economy with $\sigma = \overline{\sigma}$ will undertake process innovation effort again in $t = 6$. Interestingly, notice that $\overline{t}(\overline{\sigma}) = 4$, hence condition (26) would fail to hold for this economy. This means that, in the absence of product innovation effort, an economy with $\sigma = \overline{\sigma}$ would *not* be able to sustain *five* subsequent rounds of process innovation effort as equilibrium outcomes. Only when product innovation effort is allowed, will this economy be able to display a fifth round of process innovation effort (which will take place in $t = 6$, after the equilibrium with product innovation effort in $t = 5$). In other words, product innovation effort in $t = 5$ allows an economy with $\sigma = \overline{\sigma}$ to sustain positive process innovation and growth for longer.

Figure 2 depicts how the ability/inability to switch to an equilibrium with product innovation effort ends up magnifying income disparities among economies with slightly different levels of σ. This figure plots the *income path* for two economies differing in σ: the solid lines show the income path when $\sigma = \overline{\sigma}$, whereas the dashed lines do so for $\sigma = \underline{\sigma}$. Notice from Figure 1 that $\underline{\sigma} \in (\sigma_b, \sigma_c)$. Up until $t = 3$ the income paths diverge from each other due to gaps in labour productivity increases when $\overline{\sigma} > \underline{\sigma}$. After $t = 3$, growth stops forever in the $\underline{\sigma}-$economy, while it continues in the $\overline{\sigma}-$economy until $t = 6$. In $t = 4$ due to process innovation, next in $t = 5$ owing to product innovation effort and, lastly, in $t = 6$ again as a result of process innovation effort. Notice that Figure 2 displays also the income path that the $\overline{\sigma}-$economy would have followed in the *absence* of product innovation effort in $t = 5$: this is shown by the dotted line at the level $Y_t = 1 + 4\overline{\sigma}$. As we can see, in the absence of product innovation effort in $t = 5$ income disparities between the two economies would have ended up being narrower, and stop widening beyond $t = 4$.

[3] Analogously, an economy whose σ is just above σ_f will switch to an equilibrium with $\varepsilon_{t,q} = 1$ in $t = 3$.

Appendix C: Additional Proofs

1. Proof that $[q\,x(q)]^q \geq q\,x(q)$ always holds true in equilibrium for $q = 1+\rho$.

Suppose that for $q = 1+\rho$ we have that $[q\,x(q)]^q < q\,x(q)$. This would mean that the utility function (4) could be simplified in the case to:

$$U_t = \ln\left[x_1 + (1+\rho)x_{(1+\rho)}\right] + \eta(1-\varepsilon),$$

where we are using x_q to denote the quantity of the variety $q = 1, (1+\rho)$ consumed by the individual. Given the above expressions, it follows that the maximum price that could be charged for each unit of the variety $q = 1+\rho$ will be

$$P_t = 1+\rho. \tag{C.1}$$

Since firms intend to maximise profits, (C.1) would be also the equilibrium price. Recall now that the inherited technology from the pre-historic period $t = 0$ allows the production of one unit of the baseline quality variety with one unit of labour, and all individuals are endowed with one labour time. As a consequence of this, the minimum wage level that could possibly hold in equilibrium is one. Since the income of an individual cannot be smaller that his wage, also the lowest income level that could possibly hold in equilibrium is equal to one. In other words, in equilibrium, $Y_t \geq 1$ for any t.

Let us use now the fact that $Y_t \geq 1$, together with the equilibrium price when $[q\,x(q)]^q < q\,x(q)$ holds for $q = 1+\rho$ given by (C.1). Bearing in mind that $x_{(1+\rho)} = Y_t/P_t$, it follows that $x_{(1+\rho)} \geq 1/(1+\rho)$. But, when $x_{(1+\rho)} \geq 1/(1+\rho)$, we have that $(1+\rho)x_{(1+\rho)} \geq 1$, and therefore it must be that $\left[(1+\rho)\,x_{(1+\rho)}\right]^{1+\rho} \geq (1+\rho)\,x_{(1+\rho)}$, contradicting our initial assumption. ■

2. Proof that the threshold $\underline{R}_2$ is a decreasing function of σ.

Notice that an alternate way to implicitly define $\underline{R}_2$ is by applying logarithms on (21). This leads to:

$$\underbrace{(1+\rho)\ln(1+\sigma\underline{R}_2) - \ln(1+\sigma\underline{R}_2+\sigma)}_{\Gamma(\rho,\sigma,\underline{R}_2)} = 0. \tag{C.2}$$

From (C.2) we can observe several important properties of the function $\Gamma(\rho,\sigma,\underline{R}_2)$: *i)* $\Gamma'_{\underline{R}_2}(\cdot) > 0$, *ii)* $\Gamma'_{\rho}(\cdot) > 0$, *iii)* $\Gamma(\rho,\sigma,\underline{R}_2 = 0) = -\ln(1+\sigma) < 0$, *iv)* $\lim_{\underline{R}_2\to\infty}\Gamma(\rho = 0,\sigma,\underline{R}_2) = 0$. A first result to notice is that since $\Gamma(\rho = 0,\sigma,\underline{R}_2 = \infty) = 0$ for any $\sigma > 0$, and we have that $\Gamma'_{\rho}(\cdot) > 0$, it must then be that for any $\rho > 0$ there is a unique, finite and strictly positive value of $\underline{R}_2$ such that it satisfies $\Gamma(\rho,\sigma,\underline{R}_2) = 0$. Furthermore, combining this with $\Gamma'_{\underline{R}_2}(\cdot) > 0$, it follows that $\partial\underline{R}_2/\partial\rho < 0$. Using now the full expression $\Gamma'_{\rho}(\cdot) = \ln(1+\sigma\underline{R}_2)$, we can also see that $\partial\Gamma'_{\rho}(\cdot)/\partial\sigma > 0$. Since $\Gamma(\rho = 0, \sigma, \underline{R}_2 = \infty) = 0$ for any $\sigma > 0$, and $\Gamma'_{\rho}(\cdot) > 0$, it must then be the case that, considering two generic $\underline{\sigma} < \overline{\sigma}$ and letting $\Gamma(\rho, \underline{\sigma}, \underline{R}_2(\underline{\sigma})) = 0$ and $\Gamma(\rho, \overline{\sigma}, \underline{R}_2(\overline{\sigma})) = 0$, we must have $\underline{R}_2(\overline{\sigma}) < \underline{R}_2(\underline{\sigma})$. ■

3. Proof of existence of a single and simultaneous crossing point for all three threshold functions $\overline{R}(\sigma)$, $\underline{R}_1(\sigma)$ and $\underline{R}_2(\sigma)$ when $\sigma = e^{\eta/q}(e^{\eta} - 1)$.

From (12) and (20) it follows that $\overline{R}(\sigma) = \underline{R}_1(\sigma)$ if and only if $\sigma = \widetilde{\sigma} \equiv e^{\eta/\rho}(e^{\eta} - 1)$. At that crossing point, we have that $\underline{R}_1(\widetilde{\sigma}) = (e^{\eta/\rho} - 1)/e^{\eta/\rho}(e^{\eta} - 1)$. Next, plugging the RHS of (20) into the LHS of (21), we can observe that $\underline{R}_1(\sigma) = \underline{R}_2(\sigma)$ if and only if the following equality holds: $\left(e^{\eta/\rho}\right)^{1+\rho} = e^{\eta/\rho} + \sigma$. Since this equality holds if and only if $\sigma = e^{\eta/\rho}(e^{\eta} - 1)$, it then follows that all the three threshold functions [$\overline{R}(\sigma)$, $\underline{R}_1(\sigma)$ and $\underline{R}_2(\sigma)$] must necessarily cross each other once, and only once, and all at the same value of σ, given by $\widetilde{\sigma}$. ■

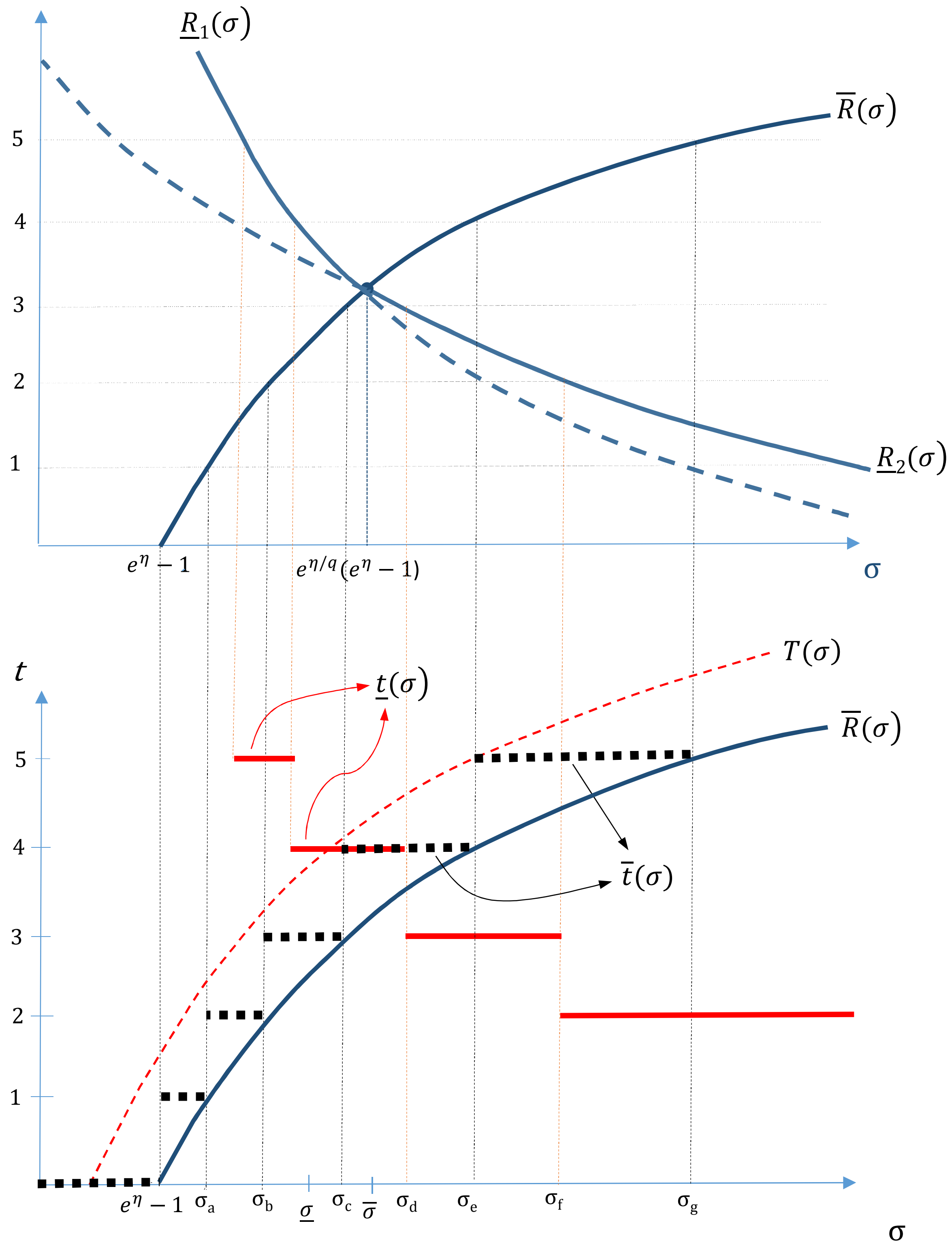

Figure 1: Equilibrium Innovation Effort

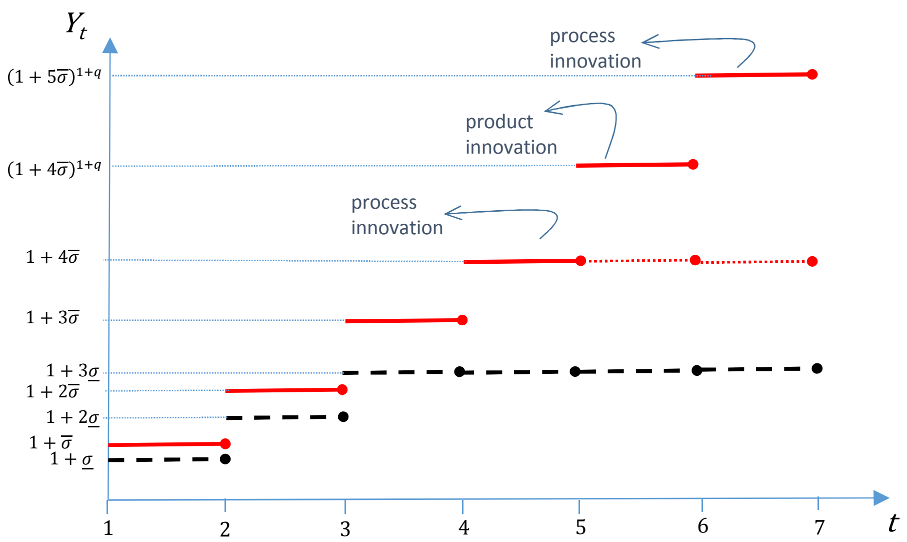

Figure 2: Income Paths