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# Determinants of Repo Haircuts and Bankruptcy

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**Abstract**

Variations in repo haircuts play a crucial role in leveraging (or deleveraging) in security markets, as observed in the two major economic events that happened so far in this century, the US housing bubble that burst into the great recession and the European sovereign debts episode. Repo trades are secured but recourse loans. Default triggers insolvency. Collateral may be temporarily exempt from automatic stay but creditors' final reimbursement depends on the bankruptcy outcome. We address existence of bankruptcy equilibria, characterize it and infer how haircuts are related to asset or counterparty risks.

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# 1 Introduction

In a repo trade, a security is pledged as collateral for a cash loan and can then by reused by the cash lender, that is, pledged in a another contract or short-sold. The reuse of the collateral makes repo trades quite different from mortgage loans where the durable good collateral stays put. The resulting leverage was studied in detail by Bottazzi, Luque and Páscoa (2012), under the assumption that agents always fulfilled their financial obligations.

Leverage played a major role in the recent financial crisis of 2008. Leading to the crisis, it was not only households that were highly indebted but also large financial institutions. These large institutions turned to the *shadow banking* system to finance themselves (see e.g. Gorton and Metrick (2010)). The repo market is a crucial part of this system. The haircut applied to the loan given in a repo trade is inversely related to how much agents can build up their positions in a security by using the repo market as a means of financing security positions. Security and repo trades can be combined in way that allows security positions to be increased, as the security gets pledged as collateral in repo, then repledged or short sold by the creditor (then again pledged by the counterparty of the short seller and so on). How do collateral reuse and haircuts determine what leverage is? The former may be an ingredient but does not determine by itself what leverage is (and limitations on reuse do not automatically translate into targeted reductions in leverage). For long agents to lever up to haircut potential, the reuse of the collateral only becomes necessary when the wealth of these agents is high enough and the haircut is low enough that the resulting aggregate leveraged long positions exceed aggregate initial holdings of the security. It is ultimately the haircut that determines what leverage is (and it could be the shorts being leveraged instead). More recently, in the sovereign debt crisis of 2010-12, there was substantial deleverage (also of those short selling) caused by the consecutive hikes in repo margins on bonds issued by several European governments.

Given that leverage and the haircut are inversely related, it is crucial to understand how the latter is determined. The haircut is the difference between the values of the collateral and the respective cash loan, at the time when the repo trade starts. It is usually expressed as a percentage (less than or equal to 1) of the collateral value. Equivalently, the initial margin captures that difference by expressing the collateral value as a percentage (greater than or equal to 1) of the cash loan. A repo trade has a purchase leg and then a repurchase leg at a repurchase price that is locked in at the first leg. The difference between the purchase price (the cash loan) and the repurchase price is the repo interest rate, agreed upon in advance. Hence, in the absence of default, there would be no reason to charge a haircut. The haircut reflects the cash lender's perceived risk of loss in the event of the cash borrower's default.

In this article we model the limited commitment involved in repo trades. In this respect, also, there is a key difference by comparison with what happens in many (but not all) mortgages, as captured in the GE collateral literature. Repo trades are recourse loans, whereas many (but not all) mortgages are non-recourse. If an household that has signed a non-recourse mortgage decides to default, it would just surrender the house and walk away without suffering any other penalties. That is not the case in recourse loans: in the event of default, creditors can be repaid above the collateral liquidation value by forcing the bankruptcy of the faulty borrower and then becoming claimants in the partition of the borrower's estate. It may also happen that creditors end up recovering less than the collateral liquidation value, when that is the outcome from the partition of the estate among all creditors. Repo collateral is exempted from certain provisions of the US Bankruptcy Code that normally apply to pledges, in particular, the automatic stay on enforcement of collateral in the event of insolvency. That is, creditors can keep the collateral that had been pledged to them (and can sell it) but, when the bankruptcy court takes the final decisions, they may get more or less than what their claim was (the promised repayment) and this may be different from the liquidation value of the collateral.

It should be noted that when an agent goes bankrupt, it is not just the repayment of the cash borrowed in repo that is at stake. If a security happened to be pledged to this agent in repo, then this collateral will not be given back to the cash borrowers - a *"fail"* occurs as a result of bankruptcy - and the respective manufactured dividends due to the beneficial owner will not be paid also.

Default is a very serious event and needs to be modeled by taking into consideration the whole bankruptcy process. It is not a decision that can be taken asset by asset, comparing promised payments and collateral values. Debtors can't be assumed to be repaying the minimum of these two, contrary to what happens in non-recourse loans, as shown in a long standing literature emerging from the work by Geanakoplos and Zame in the nineties (see Geanakoplos (1997), Geanakoplos and Zame (1997) and Geanakoplos and Zame (2014)). For the same reason, default can't be avoided by designing contracts so that collateral values never fall below promised payments, as was the case in a contemporaneous literature dating back to Kiyotaki and Moore (1997). Garnishable estates must now be set against total debts (net of credits that the defaulter may be entitled to). This creates a non-convexity in the borrower's budget set that seems to have put off previous research efforts.

There are however interesting results that can be established, in spite of the intrinsic non-convexity of individual decision problems. We consider binomial economies, where just two states of nature, $U$ or $D$, may occur after the initial node (and each of these states may be followed without uncertainty into a third date). Our paper focus on over-the-counter (OTC) repo, that is, trades that are not centrally cleared through an exchange (or central clearing counterparty, CCP), and bilateral (as opposed to tri-party where collateral selection, payment, custody and settlement are outsourced to a third-party agent). Our finite-agent model does not let us explore the convexifying effect of large numbers that has been used in continuum of agents models in several contexts, including in consumer bankruptcy problems with unsecured loans (see Araujo and Pascoa (2002) and Sabarwal (2003)). However, modeling the agents set as a continuum is not appropriate in a context of OTC repo where each trader should anticipate counterparty bankruptcy risk and choose repo haircuts accordingly[2].

For equilibrium to exist, leverage should be bounded. Here there is another important distinction between credit backed by securities and credit backed by houses or productive resources.

[2]A bankruptcy analysis might be doable also for centrally cleared repo, but aggregate default risk should take the place of counterparty risk. The OTC case seems to be more informative and easier to relate to the applied literature on the determinants of repo haircuts.

In the latter, the aggregate supply of collateral is fixed and, therefore, under exogenous collateral margins, borrowing becomes bounded. In the former, the collateral supply is endogenous since it includes short-sales and, therefore, there are no a priori bounds on secured borrowing, even under exogenous margins. However, repo and security positions must be related in another way: the net security title balance held by each agent must be non negative. This is known as the *box constraint* and says that in to order to pledge the agent must be long in the security and in order to short-sell the agent must be long in repo (the security being pledged to him). In the one-security case, the box constraint suffices to bound secured borrowing (and leverage), but in the multi-security case other constraints should be added with the purpose of bounding repo and security trades[3].

In order to gain intuition and allow for a full characterization of equilibria, we start by examining a one-security and two-agent case. In this simple case, equilibria dispense with any a priori bounds on financial positions and we show that bankuptcy can only occur in one of the two states and for only one of the two agents (the leveraged secured borrower). Then, we contemplate the multi-security and multi-agent case and establish existence of equilibria under some bounds on the value of repo positions, more precisely, on what agents can pledge as collateral.

The non-convexity is dealt with by constructing a generalized game where the decision process of each agent of the original economy is now being decomposed into decisions taken by three different fictitious agents: one solves the no-bankruptcy problem, the second assumes bankruptcy has to occur and the third compares the two solutions. Mixing on these two solutions can be avoided if these do not involve being on different vertices of the budget set, as ensured by the bounds on repo trades.

Having characterized and established existence of equilibria, we address the determinants of the repo haircut. On this issue, there are different views in the applied literature. Gorton and Metrick (2012) argue that haircuts depend both on the underlying asset and on who is

[3]see Bottazzi, Luque and Páscoa (2012) on bounds that result from the segregation of haircuts or the distinction between dealers and non-dealers and Bottazzi, Luque and Páscoa (2017) on bounds that follow from equity requirements in the spirit of the Basel regulation of banks.

the counterparty in a repo transaction but that, particularly in times of crisis, the latter gains importance. In contrast, Krishnamurty et al. (2014) report little variation of haircuts across counterparties and place much more weight on the underlying asset. Infante (2015) argues that these observed differences on haircuts arise because two different markets are studied: the bilateral and tri-party repo.

The loss that a lender may suffer from counterparties' default may be related to the collateral falling in value (or being sold in a fire sale) but, since the loan is recourse, the loss cannot be associated to that asset risk in such a simple way. It may happen that there is no *asset risk* but the *counterparty risk* will nevertheless govern what the lender gets back, which does not have to be equal to the collateral liquidation value. What a secured creditor recovers in the bankruptcy process depends on what is the liquidation value of the whole estate of the defaulter and how it will be partitioned among all creditors, even though the exemption from automatic stay allows the creditor to sell the collateral while waiting for the final outcome of the bankruptcy process.

We characterize how haircuts respond to asset and counterparty risks. Suppose there are many traders in the repo market of each security, repo rates are security-specific but haircuts are specific to each pair of traders. In such competitive setting, we should expect counterparty bankruptcy risk to affect pair-specific haircuts but not the repo rate, as opposed to what happens in the two-agent example. Say state $D$ is the state where bankruptcy may occur. Suppose an agent $i$ is solvent in state $D$ and is, in terms of the whole portfolio, a net creditor to a counterparty $j$ (in state $D$) and the expected repayment rate of this counterparty decreases (an increased counterparty risk). Then, agent $i$ would like to raise (lower) the haircut charged to counterparty $j$ when accepting collateral from $j$, for securities whose repo repayment exceeds (falls below) the collateral value. That is, when the asset is risky from the creditor's perspective, haircuts tend to move in the same direction as the counterparty risk. But for the other securities (risky from the debtors' point of view, wary of a repo fail), haircuts move in the opposite direction.

Quite differently, in the two-agent and one-security example, counterparty risk affects the repo rate and his effect is strong enough to make bankruptcy rates decrease as the haircut increases.

In a small numbers context, it is now the other direction that may become more relevant: how are overall solvency rates affected when the haircut charged in one security changes? That is why in such extreme non-competitive case, haircuts and expected repayment rates may move together, contrary to our results for the competitive case. To summarize, the way counterparty risk may impact haircuts depends on how competitive the repo market is and to understand how that impact works in the competitive case we need to couple this risk with asset risk. When faced with a rise in counterparty risk, competitive creditors tend to ask for higher haircuts for securities that exhibit an asset risk from the creditors' perspective, but lower haircuts may arise if the security involves the opposite risk (a fail rather than a default risk).

# 2 The Model

## 2.1 Fundamentals

We consider a binomial economy with three dates. At an initial date (date 0) there is only one node in the event tree, followed by nodes $U$ and $D$ at the second date. Each second date note has a unique successor at the third date: $U^+$ and $D^+$ are the successors of $U$ and $D$, respectively. As we will see, the third date just serves to guarantee that securities retain value at the second date, when borrowing and lending transactions are settled (and we may want to dispende with the third date in some cases, as discussed below).

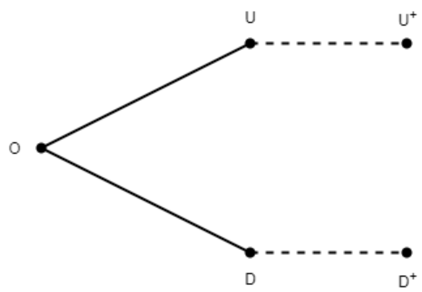


Figure 1: Events tree of the binomial economy.

Binomial models have been used to study the leverage cycle in economies with default on non-recourse loans (see e.g., Fostel and Geanakoplos (2012) and Fostel and Geanakoplos (2014)). Given that repo trades constitute recourse loans, we will model default as a bankruptcy process.

There is only one consumption good. Markets for this commodity open at each event. We denote the price this good at event $e$ by $p_e$. There is a finite set of $I \geq 2$ agents, indexed by $i$. A bundle of commodities consumed by agent $i$ is denoted by $x^i = (x^i_0, x^i_U, x^i_D, x^i_{U^+}, x^i_{D^+})$. There are also $F$ real securities indexed by $f$, each one being characterized by a vector of non-negative real returns $R_f = (R_{fU}, R_{fD}, R_{fU^+}, R_{fD^+})$. Given spot prices $p_e$, the nominal return of security $f$ is $p_e R_{fe}$[4].

Trading of securities occurs at the first and second dates. Each agent chooses a securities portfolio $\phi^i \in \mathbb{R}^{3F}$ consisting of positions in the $F$ securities at the initial nodes and nodes $U$ and $D$. Security prices are denoted by $q \equiv (q^f_e) \in \mathbb{R}^{3F}$. Agents' endowments of commodities are $\omega^i \in \mathbb{R}^5_+$, with $\omega^i_s > 0$ in both states. Agents have initial holdings, at date 0, of each security $f$, $o^i_f > 0$. Preferences are described by utility functions $U^i : \mathbb{R}^5_+ \to \mathbb{R}$. For each security $f$, we normalize its positive net supply to be one: $\sum_i o^i_f = 1$.

For the moment, we assume local non-satiation agents' preferences[5] and that $U^i$ is concave, twice continuous differentiable and such that second date marginal utilities $D_s U^i$ are positive valued on the interior of the positive orthant.

## 2.2 Repo markets

Agents can have negative positions in securities, short-sales are permitted. Short-selling, however, is not the same as issuing (which we take as given this model, having occurred prior to date 0). In order to short-sell a security, an agent must go first in the repo market and *borrow* the desired amount of securities. This is the way short-selling is actually done in reality.

Borrowing of securities actually consists in buying the security and promising to resell it to

[4] We could have considered nominal securities instead.

[5] in the following sense: for any consumption plan $x^i$ and any node $e$, there is some $\bar{x}^i$ arbitrarily close to $x^i$ that satisfies $\bar{x}^i_{e'} = x^i_{e'}$ for all $e' \neq e$, and $U^i(\bar{x}^i) > U^i(x^i)$.

the lender, at a future date and at a predetermined price. There is a difference between the price at which a security is bought, in the first leg of the transaction, and the price at which it is resold to its original owner, in the second leg of the transaction, at a future date. This difference is captured by the *repo rate*. The highest repo rate within its class of securities is referred to as the *general collateral rate* (GC).

The borrower of a security acquires possession rights associated with the security. However, any coupon or dividend paid to the borrower during the term of the transaction is passed through to the original owner; this is called a *manufactured payment* or a *manufactured dividend*.

A repo transaction is actually a collateralized loan. If agent $i$ buys security $f$ from $j$ in the first leg of the transaction, $i$ is at the same time the borrower of the security and a lender of cash, and $j$ is the lender of the security and borrower of cash. A haircut $(1 - h_f^{ij})$ is applied to the market value of the security to compensate the lender of funds for the risk associated to the transaction, so that the cash loan may be lower than the value of the collateral. Haircut is such that $0 < h_f^{ij} \leq 1$.

For simplicity, repo trading takes place only at date 0. We denote by $z_f^{ij}$ agent $i$'s repo position, with counterparty $j$, on security $f$. If $z_f^{ij} > 0$, it means that $i$ is the lender of cash and borrower of the security and we say that he is *long* in repo. At the same time, since markets must clear, we should expect that $z_f^{ji} < 0$, which means that agent $j$ is a borrower of cash, a lender of the security, and we say that he is *short* in repo.

Given that a haircut is applied to every repo transaction, the amount of funds that can be borrowed by pledging one unit of security $f$ in the repo market, in a transaction with counterparties $i$ and $j$, is given by $h_f^{ij} q_{f0}$. As a matter of notation, we use both $h_f^{ij}$ and $h_f^{ji}$ to denote the haircut applied to transactions in the repo market involving security $f$ and counterparties $i$ and $j$, regardless of which one of the agents is long in repo for security $f$ and which one is short. Denote by $\rho_f$ the repo rate of a loan backed by security $f$ and let $r_f = 1 + \rho_f$.

An important feature of a repo transaction is that the borrower of the security has the right to lend it in the repo market or short-sell it in the security market. That is, the collateral can be *rehypothecated* directly or indirectly, and this process may occur many times over for the same settlement period. This is an important feature of repo markets as it it the *reuse* of the collateral that allows agents to leverage their portfolio positions or their cash loans beyond what would be possible if collateral was used only once.

There is a constraint that captures both the need to pledge collateral when borrowing cash in repo and the need to borrow a security (accept it as collateral) when short-selling it. This is the *box constraint*. This restriction states that the agent must hold a nonnegative amount of the security in his possession. That is, the sum of security position and repo trades must be nonnegative. At the initial date (the only node where repo markets are open) the box constraint for security $f$ is:

$$\phi^i_{fe} + \sum_{j \neq i} z^{ij}_f \geq 0 \tag{1}$$

At second date nodes, repo markets are not open and the box constraints reduce to plain no-short-sales constraints:

$$\phi^i_{fU} \geq 0, \;\; \phi^i_{fD} \geq 0 \tag{2}$$

When there is more than one security, without any further assumptions on portfolio or repo positions, leverage can be unbounded (see Bottazzi, Luque and Páscoa (2012) on some institutional arrangements that bound leverage). In this paper we take the simple approach of imposing some bounds on repo positions whenever there is a need to bound leverage[6].

## 2.3 Bankruptcy and feasible market plans

Given market prices and repo rates, $(p, q, r)$, agents decide on a plan $(x^i, \phi^i, z^i)$, consisting of consumption and portfolios in the securities and repo markets. Let us define the budget constraints that these plans must satisfy. At date 0, the repo market opens and agents have

[6]In the one-security example in Section 3, leverage will be bounded and no further constraints will be imposed on financial positions.

initial endowments of goods and securities. Agent $i$'s budget constraint at this date is:

$$p_0(x_0^i - \omega_0^i) + \sum_f q_{f0}(\phi_{f0}^i - o_f^i) + \sum_f \sum_{j\neq i} q_{f0} h_f^{ij} z_f^{ij} \leq 0 \tag{3}$$

Repo markets do not open at the second date but agents can still trade in securities, using real payments from their previous securities positions. We allow for the possibility of agents not fulfilling their obligations in the second date. This only happens if agents become insolvent. For every agent, we need to see how do assets set against liabilities, in each state of the second date. On the assets' side we have a first component which is the new market value plus returns associated with the actual amount of each security that the agent has in his possession when he enters that node. This (non-negative) amount is what the agent had in his date 0 box for the corresponding security. By adding up the current market value and returns, across all securities, we get assets' component pertaining to *the value of the past box positions*:

$$\Xi_s^i = \sum_f \left(\phi_{f0}^i + \sum_{j\neq i} z_f^{ij}\right)(q_{fs} + p_s R_{fs})$$

To evaluate assets or liabilities resulting from repo positions taken at the previous node, we need to remember that repo positions consist of securities that the agent has received (borrowed) or pledged (lent) as collateral backing a cash loan. If agent $i$ borrowed security $f$ from agent $j$ at the initial node ($z_f^{ij} > 0$), he must return the security together with the respective returns to its original owner, that is, pass on the non-negative value $z_f^{ij}(q_{fs}+p_s R_{fs})$ to agent $j$ but at the same time agent $j$ must repay (at the gross rate $r_f$) to $i$ the cash loan he obtained before. Conversely, a repo short agent repays the cash loan and gets back the value of the security he pledged as collateral. The settling of all of agent $i$'s repo transactions with agent $j$ as counterparty is captured by the following term:

$$I_s^{ij} = \sum_f z_f^{ij}\left[q_{f0} h_f^{ij} r_f - (q_{fs} + p_s R_{fs})\right]$$

If all repo transactions are settled as was agreed upon at the initial date, that is, if every agent is solvent in state $s$, agent $i$'s corresponding budget constraint is:

$$p_s(x_s^i - \omega_s^i) + \sum_f q_{fs}\phi_{fs}^i \leq \Xi_s^i + \sum_{j\neq i} I_s^{ij} = \sum_f \left(\phi_{f0}^i (q_{fs} + p_s R_{fs}) + \sum_{j\neq i} z_f^{ij} q_{f0} h_f^{ij} r_f\right)$$

Since repos are recourse loans, insolvency will occur in state $s$ only if the agent's assets are insufficient to cover his liabilities. The assets include the value of the past box positions, plus the positive settlements of his repo transactions, plus the garnishable portion (according to some coefficient $\beta$) of his commodity endowment in that state. Liabilities consist in the negative settlements of his repo trades. That is, insolvency will occur if, and only if, the following condition is satisfied:

$$\beta p_s \omega_s^i + \Xi_s^i + \sum_{j \neq i} \eta_s^j I_s^{ij+} \geq \sum_{j \neq i} I_s^{ij-}$$

Here, $I_s^{ij+}$ and $I_s^{ij-}$ denote the positive and the negative parts[7], respectively, of $I_s^{ij}$, while $\eta_s^j$ denotes the portion of all of agent $j$'s financial obligations that he *effectively* repays given his income in that state. That is, $\eta_s^j = 1$ if $j$ is solvent in state $s$ and $\eta_s^j < 1$ when he declares bankruptcy. In words, agent $i$ declares bankruptcy in state $s$ if, and only if, the garnishable portion of his commodity endowment ($\beta p_s \omega_s^i$), plus the value of the securities in his possession at the beginning of the second date ($\Xi_s^i$), plus the positive repo repayments the agent gets from all his counterparties ($\sum_{j \neq i} \eta_s^j I_s^{ij+}$) is not sufficient to repay all of agent $i$'s repo obligations ($I_s^{ij-}$).

Once we allow bankruptcy, agent $i$'s budget constraint in state $s$ of the second date is:

$$p_s(x_s^i - \omega_s^i) + \sum_f q_{fs} \phi_{fs}^i \leq \max \left\{ -\beta p_s \omega_s^i \ , \ \Xi_s^i + \sum_{j \neq i} I_s^{ij+} \eta_s^j - I_s^{ij-} \right\}, \tag{4}$$

where

$$\eta_s^i = \begin{cases} 1 \ , \ \beta p_s \omega_s^i + \Xi_s^i + \sum_{j \neq i} \eta_s^j I_s^{ij+} \geq \sum_{j \neq i} I_s^{ij-} \\ \frac{\beta p_s \omega_s^i + \Xi_s^i + \sum_{j \neq i} \eta_s^j I_s^{ij+}}{\sum_{j \neq i} I_s^{ij-}} \ , \quad \text{otherwise} \end{cases} \tag{5}$$

At the last date there is no trading of securities. Agents consume from their commodity endowments and security payments according to their security positions constituted at the previous nodes

$$p_{s^+}(x_{s^+}^i - \omega_{s^+}^i) \leq \sum_f \phi_{fs}^i p_{s^+} R_{fs^+} \tag{6}$$

[7] $I_s^{ij+} = \max\{0, I_s^{ij}\}$ and $I_s^{ij-} = -\min\{0, I_s^{ij}\}$.

Given parameters $(p, q, r)$, an agent's plan of consumption, of securities, and of repo positions $(x^i, \phi^i, z^i)$, will be called *feasible* if $x^i \geq 0$ and conditions (1), (2), (3), (4) and (6) hold. We denote by $B^i(p, q, r)$ the set of all feasible plans for agent $i$, and by $B^i_\star(p, q, r)$ the subset of utility maximizing plans in $B^i(p, q, r)$.

### 2.4 Equilibrium

For this economy, equilibrium is defined as follows:

**Definition 1 (Equilibrium).** *An equilibrium is an allocation of bundles, securities and repo positions $(x, \phi, z)$ together with prices $(p, q, r)$ and haircuts implied by $h = (h^{ij}_f)$, such that:*

*(a) for each agent $i$, $(x^i, \phi^i, z^i) \in B^i_\star(p, q, r, h)$,*

*(b) commodity markets clear: $\sum_i \left(x^i_0 - \omega^i_0\right) = 0$, $\sum_i \left(x^i_s - \omega^i_s\right) = \sum_f R_{fs}$ for $s = U, D$, and $\sum_i \left(x^i_{s^+} - \omega^i_{s^+}\right) = \sum_f R_{fs^+}$ for $s^+ = U^+, D^+$,*

*(c) security markets clear: $\sum_i \phi^i_{fe} = 1$ at each event $e = 0, U, D, U^+, D^+$,*

*(d) repo markets clear: $z^{ij}_f + z^{ji}_f = 0$ for all $i, j$ and $f$.*

## 3 A one-security and two-agent model

We can now take our base model and consider the simplest of economies. Suppose there are only two dates, only one security ($F = 1$ and we dispense with the index $f$ for the security) and only two agents indexed $i$ and $j$ with $\omega^i_0 = \omega^j_0 = 0$ and $o^i = o^j = 1$. Agents utilities are simply the expected consumption at the second date: $U^i(x^i_U, x^i_D) = a^i x^i_U + (1 - a^i) x^i_D$ and $U^j(x^j_U, x^j_D) = a^j x^j_U + (1 - a^j) x^j_D$. To simplify we have dispensed with the third date and assumed repo to maturity[8], that is, both repo trades are settled at the maturity date of the security (even though the security does not have a price at the second date it can still serve as collateral, since its second date value consists in its returns given by $R_s$).

[8]This type of repo trades occurs in reality, although some agents (in particular, central banks) are not willing to engage in it.

As there is only one security and one pair of agents, we simplify notation by letting $h = h^{ij} = h^{ji}$, $z^i = z^{ij}$ and $z^j = z^{ji}$. Suppose $R_U > R_D$ and that agents' subjective probabilities are different enough as to guarantee trade in the repo market: $E^iR > E^jR$, where $E^iR \equiv a^iR_U + (1-a^i)R_D$. This is the same as assuming that $a^i > a^j$.

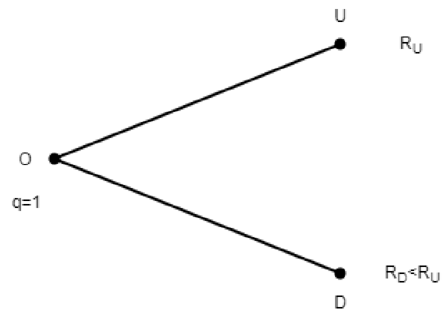


Figure 2: Economy with 2 dates. Security's price and payments.

Normalizing prices so that the security price is one, we can write agent $i$'s constraints as:

$$\phi^i + hz^i = o^i, \qquad \text{(First date)}$$

$$\phi^i + z^i \geq 0 \Leftrightarrow o^i + (1-h)z^i \geq 0, \qquad \text{(Box constraint)}$$

$$\begin{aligned} x^i_s = \omega^i_s + \max\big\{ &-\beta\omega^i_s, (o^i + (1-h)z^i)R_s \\ &+\eta^j_s[(hr - R_s)z^i]^+ - [(hr - R_s)z^i]^-\big\}, \qquad \text{(Second date, state } s) \end{aligned}$$

If we substitute $x^i_U$ and $x^i_D$ into agent $i$'s utility function, we can write his problem as:

Maximize

$$E^i\omega^i + a^i \max\big\{ -\beta\omega^i_U, (o^i + (1-h)z^i)R_U + \eta^j_U[(hr - R_U)z^i]^+ - [(hr - R_U)z^i]^-\big\}$$

$$+ (1-a^i)\max\big\{ -\beta\omega^i_D, (o^i + (1-h)z^i)R_D + \eta^j_D[(hr - R_D)z^i]^+ - [(hr - R_D)z^i]^-\big\}$$

s. t.

$$o^i + (1-h)z^i \geq 0$$

The only decision variable in the problem is $z^i$ and the agent only needs to decide whether to be long ($z^i > 0$) or short ($z^i < 0$) in repo. Given our assumption on security payments and utilities, it is reasonable[9] to search for equilibria in which $hr \in (E^j R, E^i R)$. It is easy to see that if such an equilibrium exists, it is necessary for agent $i$ to be repo short ($z^i < 0$), and for $j$ to be repo long ($z^j > 0$). Being short in repo, agent $i$ can potentially transfer consumption from state $D$ to state $U$, which gives him comparatively more utility. However, taking a short repo position is not guaranteed to increase his consumption in state $U$, or to yield an increase in overall utility, since this depends on the agent's counterparty effective repayment rate in state $U$ ($\eta^j_U$). We can however rule out agent $i$ taking a long repo position as this would transfer consumption from a high utility state to a state with low utility. An analogous argument justifies agent $j$ taking a long repo position.

If agent $i$ decides to be short in repo, his position will be determined by the box constraint:

$$z^i = -\frac{o^i}{1-h} \tag{7}$$

Note that the magnitude of the position in equation (7) can be many times higher than the total initial supply of the security in the economy ($o^i + o^j$). Agent $i$ is able to build such a (large) position because of the re-usability of collateral in repo markets. At the same time, the position is not unbounded because of the haircut[10]: how much an agent can leverage his initial endowment of the security is related inversely to the haircut applied in the repo market. This is one good reason to define the *asset specific leverage* as the inverse of the haircut applied to the security when used as collateral in the repo market: $\frac{1}{1-h}$.

The next thing to note is that agents $i$ and $j$ are solvent in states $U$ and $D$, respectively. In fact, in state $U$ agent $i$ has a non-negative financial income: $(o^i + (1-h)z^i)R_U + \eta^j_U[(hr - R_U)z^i]^+ - [(hr - R_U)z^i]^- = (o^i + (1-h)z^i)R_U + \eta^j_U(R_U - hr)|z^i| \geq 0 > -\beta\omega^i_U$. Therefore, agent $i$ does not become insolvent, actually makes $x^i_U \geq \omega^i_U$ (and analogously for agent $j$ in state $D$).

However, agent $i$ is decreasing consumption in state $D$ and we cannot be sure of his solvency in that state. The same applies for agent $j$ in state $U$. If we let $\alpha^i_s = 1$ iff agent $i$ is solvent in

[9] In fact, it is easy to see that if $hr > E^i R$ or $hr < E^j R$ both agents will want to take either long or short repo positions, and there cannot be market clearing.

[10] See Bottazzi, Luque and Páscoa (2012) for a complete example of how these positions are built in repo markets with one security.

state $s$ and $\alpha^i_s = 0$ when he declares bankruptcy, we have four possible cases to consider:

| Case | $\alpha^i_U$ | $\alpha^i_D$ | $\alpha^j_U$ | $\alpha^j_D$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 |
| 2 | 1 | 0 | 1 | 1 |
| 3 | 1 | 1 | 0 | 1 |
| 4 | 1 | 1 | 1 | 1 |

Before we proceed studying each case, it will be useful to write the necessary[11] first order conditions for agents $i$ and $j$ problems. If we let $\mu^i$ be the multiplier for agent $i$'s box constraint, the conditions are as follow[12]:

$(z^i < 0)$ :

$$a^i[(1-h)R_U + \eta^j_U(hr - R_U)] + (1-a^i)\alpha^i_D[(1-h)R_D + (hr - R_D)] + \mu^i(1-h) = 0 \tag{8}$$

$(z^j > 0)$ :

$$a^j\alpha^j_U[(1-h)R_U + (hr - R_U)] + (1-a^j)[(1-h)R_D + \eta^i_D(hr - R_D)] = 0 \tag{9}$$

From these conditions, it follows that there is no equilibrium in which agent $j$ declares bankruptcy in state $U$. If that were the case, equation (9) would imply:

$$(1-h)R_D + \eta^i_D(hr - R_D) = 0$$

This is not possible since we have assumed $h < 1$ and $hr > R_D$. This immediately rules out cases 1 and 3. This is interesting enough to be written as a proposition:

**Proposition 1.** *In this economy, if bankruptcy occurs, it occurs only in one state. Moreover, it can only be the agent that is long leveraged in the security (and short in repo) who may go bankrupt.*

[11] Necessity follows from Slater's condition (as an interior point for the constraint set can be constructed by making $z = 0$). However, first order conditions are not sufficient, due to the non-convexity.

[12] Notice some agent must have the box constraint not binding. It must be the long in repo and short-selling the security. In fact, if his box were binding, this joint operation would generate cash at the initial date (due to the benefit of charging haircut) but there is no way to spend it, since there is no consumption at this date.

Let us then study the equilibrium where bankruptcy occurs. We have two equations from the first order conditions:

$$a^i[(1-h)R_U + \eta^j_U(hr - R_U)] + \mu^i(1-h) = 0 \tag{10}$$

$$a^j[(1-h)R_U + (hr - R_U)] + (1-a^j)[(1-h)R_D + \eta^i_D(hr - R_D)] = 0 \tag{11}$$

We also have the equation that determines the effective bankruptcy rate of agent $i$ in state $D$ (equation (14)). Since we know that this agent's position is $z^i = -\frac{o^i}{1-h} = -\frac{1}{1-h}$, this equation can be written as:

$$\eta^i_D = \frac{\beta\omega^i_D(1-h)}{(hr - R_D)} \tag{12}$$

It seems like we have a nonlinear system with three equations ((10), (11), (12)) and four unknowns ($\mu^i$, $h$, $r$, $\eta^i_D$). In fact, equation (10) provides little information: given values for $h$ and $r$, it simply says that $\mu^i = a^i h/(1-h)(R_U - r)$. Since $\mu^i$ must be non-negative we can at most say that a necessary condition for optimality is having $R_U > r$, a condition that is not guaranteed by our initial assumption that $E^i R > hr$.

Given that equation (10) does not pin down the exact value of any variable, we can focus on the nonlinear system of two equations ((11), (12)) and three unknowns ($h$, $r$, $\eta^i_D$). There is still indeterminacy. Equilibrium repo rates and repayment rates are related to equilibrium haircuts by the following equations:

$$r = \frac{1}{h}[R_U - (1-h)\frac{E^j R + \beta\omega^i_D}{a^j}] \tag{13}$$

$$\eta^i_D = \frac{\beta\omega^i_D}{\frac{R_U - R_D}{1-h} - \frac{E^j R + \beta\omega^i_D}{a^j}} \tag{14}$$

We can for example, study an economy with initial parameters:

| | | | | |
|---|---|---|---|---|
| $\beta = 0.35$ | $R_U = 1.4$ | $\omega^i_U = 4$ | $\omega^j_U = 6$ | $a^i = 0.9$ |
| | $R_D = 0.1$ | $\omega^i_D = 2$ | $a^j_D = 4$ | $a^j = 0.2$ |

The following are examples of equilibrium values of $h$, $r$ and $\eta^i_D$:

| $h$ | $r$ | $\eta^i_D$ | $\phi^+$ |
|---|---|---|---|
| 0.89 | 1.0045 | 0.0969 | 9.09 |
| 0.90 | 1.0444 | 0.0833 | 10 |
| 0.91 | 1.0835 | 0.0711 | 11.11 |

The third column reports leverage $1/(1-h)$, which is the long position in the security. Short sales are given by $2-\phi^+$, which in the three outcomes is equal to 7.09, 8 and 9.11, respectively.

To check the consistency of the equilibrium, we need to check that the value of $\beta\omega^j_U + 2R_U - (R_U - hr)/(1-h)$ is positive, which means that agent $j$ is solvent in state $U$, as we assumed before computing the equilibrium. This expression is indeed positive in the three cases reported in the table (equal to 0.3, 0.2999 and again 0.3, respectively).

It is also necessary to check that for all these equilibria, $hr \in (E^jR, E^iR)$ and that $r < R_U$. All these conditions are satisfied by the equilibria shown on the table.

Observe that haicuts move together with repayment rates (this must always happen in this 2-agent and 1-security economy, as seen in equation (14)). The way repo rates $(r-1)$ change allows for this outcome, which is not surprising in a two-agent and one-security case. A higher haircut makes the repo short become more solvent. His solvency rate is $\eta^i_D = \frac{\beta\omega^i_D}{|z^i|(rh-R_D)}$. An increase in the haircut $1-h$ reduces leverage $\frac{1}{1-h} = |z|$. To reinforce this, $hr$ is decreasing in the haircut (even though $r$ might not always be decreasing), as seen in equation (23).

As we will see in section 5, in a competitive setting, where many agents trade many securities, the impact of the haircut in one security on the insolvency of an agent becomes less noticeable. It is the other direction that becomes more relevant: haircuts rise in response to lower repayment rates of the counterparty, for securities that involve a risk from the creditor's point of view (have a collateral liquidation value below the promised repo loan settlement). That is, in a competitive setting, creditors tend to focus on how to protect their individual credits rather than trying to influence the solvency of the counterparty.

# 4 Existence of equilibrium in the general case

In this section we strengthen our assumptions on agents' characteristics,

**Assumption 1.** ***Agents characteristics.*** *For every agent $i$, $(\omega_0^i, (\omega_s^i)) >> 0$ and marginal utilities ratios $DU_s^i/DU_0^i$ have positive minimum on $\{x; U^i(x) \geq U^i(\omega^i) \wedge x \leq \sum_i \omega^i\}$.*

Notice that the bankruptcy rule ensures strict positivity of consumption in each state of the second date. This assumption has to be extended to the third date when repo is not done up to maturity and that date is needed: $(\omega_{s+}^i)) >> 0$ and marginal utilities ratios $DU_{s+}^i/DU_s^i$ have positive minimum on the same set.

**Lemma 1.** *Under assumption 1, there are positive-real-valued functions $\hat{M}_f^i$ defined on repo rates, date 0 security prices and date 1 commodity prices, such that there is an equilibrium under the constraints $q_{f0} z_f^{ij} \geq -\theta \hat{M}_f^i(p_U, q_{f0}, r_f)$, $\forall i, j, f$, and $\forall \theta \in (0, 1]$.*

*Proof.* See the Appendix. □

**Theorem 1.** *Under assumption 1, there are constants $\bar{M}_f^i > 0$ ($i = 1, \cdots, I$ and $f = 1, \cdots, F$) such that, for all $0 < \bar{N} \leq \min_{i,f}\{\bar{M}_f^i\}$, there are positive bounds $N_f^i < N$, such that there is equilibrium under the constraints:*

$$q_{f0} z_f^{ij} \geq -N_f^i, \quad \forall i, \forall j, \forall f$$

*Proof.* From Lemma 1, under Assumption 1, there is an equilibrium $(x, \phi, z, p, q, r, h)$ in which for each $i$, $j$ and $f$, the market value of the repo position $z_f^{ij}$ satisfies the constraint:

$$q_{f0} z_f^{ij} \geq -\hat{M}_f^i(p_U, q_{f0}, r_f)$$

Let $\bar{M}_f^i \equiv \hat{M}_f^i(p_U, q_{f0}, r_f)$, then $(x, \phi, z, p, q, r, h)$ is also an equilibrium under the constraints: $q_{f0} z_f^{ij} \geq -\bar{M}_f^i$, $\forall i, \forall j, \forall f$. Now, setting fixed bounds $q_{f0} z_f^{ij} \geq -\bar{N}_f^i$, $\forall i$, $\forall j$, $\forall f$, in the proof of Lemma 1, we find an equilibrium $(x', \phi', z', p', q', r', h')$ that satisfies:

$$q'_{f0} z_f^{ij} \geq -\min\{\hat{M}_f^i(p'_U, q'_{f0}, r'_f), \bar{N}_f^i\}$$

If we now let $N_f^i \equiv \min\{\hat{M}_f^i(p'_U, q'_{f0}, r'_f), \bar{N}_f^i\}$ and we have that $(x', \phi', z', p', q', r', h')$ is an equilibrium under the constraints $q'_{f0} z_f^{ij} \geq -N_f^i$, $\forall i, \forall j, \forall f$. □

# 5 Haircuts

Haircuts in pairwise repo trades are endogenously determined in the equilibrium that we defined and whose existence was just established (and characterized for the 2-agent and 1-security case). We discuss now what may govern haircuts, that is, how should we expect haircuts to be set in equilibrium, depending on what are the parameters and other equilibrium variables for the relevant pair of repo traders.

Suppose agent $i$ has a possession value for security $f$ at the initial node, that is, a binding box constraint for security $f$ at the initial node - more precisely, the shadow value $\mu_{f0}^i$ of this constraint is positive. Denoting by $\lambda_e^i$ agent $i$'s multiplier for the budget constraint at each node $e$ and $\nu_f^{ij}$ the multiplier for the lower bound on repo positions value, from the first order conditions for agent $i$'s problem, we get the following expression for $h_f^{ij}$

$$h_f^{ij} = \frac{\tau_f^{ij}(p,q) + \frac{\nu_f^{ij}}{\lambda_0^i}}{1 - r_f \sum_s \frac{\lambda_s^i}{\lambda_0^i} \alpha_s^i \kappa_s^{ij}} \tag{15}$$

where

$$\tau_f^{ij}(p,q) = \begin{cases} \sum_s \frac{\lambda_s^i}{\lambda_0^i} \alpha_s^i (1 - \kappa_s^{ij}) \frac{(q_{fs} + p_s R_{fs})}{q_{f0}}, & \text{if } \mu_{f0}^i = 0 \\ 1 - \sum_s \frac{\lambda_s^i}{\lambda_0^i} \alpha_s^i \kappa_s^{ij} \frac{(q_{fs} + p_s R_{fs})}{q_{f0}}, & \text{if } \mu_{f0}^i > 0 \end{cases}$$

and $\kappa_s^{ij} = \gamma_s^{ij} \eta_s^j + (1 - \gamma_s^{ij})$, $\gamma_s^{ij} = 1$ if $I_s^{ij} > 0$, $\gamma_s^{ij} = 0$ if $I_s^{ij} < 0$, $\alpha_s^i = 1$ if agent $i$ is solvent in state $s$, $\alpha_s^i = 0$ if $i$ goes bankrupt in state $s$, and $\eta_s^i$ satisfies (14).

It is worth recalling that $\eta_s^j$ is the effective percentage of his debt that agent $j$ pays to all of his counterparties, so it can be used as a measure of counterparty risk: the lower $\eta_s^j$ is, the riskier (or less solvent) agent $j$ is in state $s$, and this must be taking into account by agents deciding having $j$ and counterparty and, in particular, in setting the terms of repo contracts $(h_f^{ij})$.

Equation (15) is true in any equilibrium and can be used to study *the incentives* that counterparties $i$ and $j$ have to either increase or decrease the haircut $(1 - h_f^{ij})$ associated to their repo transactions in response to an increase in the risk of one of the counterparties. Let's look at the derivative of $h_f^{ij}$ with respect to $\eta_D^j$, under the assumption that agents' marginal rates of income substitution remain unchanged. To be more precise,

*Assumption ($\Lambda$)*: agent $i$'s marginal rates of substitution of income across the first two dates, $\lambda_s^i/\lambda_0^i$, are not affected by a change in the counterparty $j$'s effective repayment rate $\eta_D^j$.

Although we might not want to take this assumption literally, it is useful to get a sense of how haircuts move with counterparty risk in a context where agent $i$ is trading in many securities and has many counterparties, so that a small variation in the default rate of one of them in some state won't affect the optimal inter-nodes deflators of agent $i$.

This assumption holds for linear utilities (recall that the bankruptcy structure ensures the positivity of consumption in each state, which implies that $DU_s^i(x^i) = \lambda_s^i p_s$) in the case of repo to maturity (dispensing with the third date and allowing for $p_s = 1$) and provided that agent $i$ is consuming at the initial date (so that $DU_0^i(x^i) = \lambda_0^i p_0$) and that the equilibrium commodity price $p_0$ is not affected by a small change in the effective repayment rate $\eta_D^j$ of counterparty $j$ in state $D$.

$$(16) \qquad \frac{\partial h_f^{ij}}{\partial \eta_D^j} = \frac{-\alpha_D^i \gamma_D^{ij} \cdot \frac{1}{q_{f0}} \cdot \frac{\lambda_D^i}{\lambda_0^i} \cdot [(q_{fD} + p_D R_{fD}) - h_f^{ij} q_{f0} r_f] + h_f^{ij} \frac{\partial r_f}{\partial \eta_D^j} \sum_s \frac{\lambda_s^i}{\lambda_0^i} \alpha_s^i \kappa_s^{ij}}{1 - r_f \sum_s \frac{\lambda_s^i}{\lambda_0^i} \alpha_s^i \kappa_s^{ij}}$$

When most of the response to a variation in counterparty risk is channeled into a change in haircuts, rather than a change in the repo rate, we can be more specific about the direction of change. We say that *repo rates are competitive* if actions by a pair of agents $i$ and $j$, in particular actions that change their solvency rates ($\eta_s^i$ and $\eta_s^j$) do not affect equilibrium repo rates. This is a reasonable assumption if there are many agents (and therefore, many pairs of counterparties) in the economy, but not to be expected in an economy with only two (or very few) agents, as in the example of section 3. We have,

**Proposition 2.** *Suppose repo rates are competitive. Let us evaluate the impact of $\eta_D^j$ on $h_f^{ij}$), under a scenario where agents' marginal rates of substitution are not affected. Say $I_D^{ij} > 0$ (i is a net creditor in the repo market with respect to j) and i is solvent in state D ($\alpha_D^i = \eta_2^i = 1$). If* *agent $j$'s expected repayment rate $\eta_D^j$ decreases, agent $i$ will want to:*

- *Increase the haircut $(1-h_f^{ij})$ he charges (pays) in his repo long (short) positions with agent $j$, of securities $f$ such that $h_f^{ij} q_{f0} r_f > q_{fD} + p_D R_{fD}$.*
- *Decrease haircuts paid to (charged to) agent $j$ for his short (long) repo positions in securities $g$ such that $h_g^{ij} q_{g0} r_g < q_{gD} + p_D R_{gD}$.*

*If $I_D^{ij} < 0$ ($i$ is a net debtor in the repo market with respect to $j$), or if $i$ is insolvent in state $D$, he has no incentives to increase or decrease the haircut $(1-h_f^{ij})$ in response to expected changes in $\eta_D^j$.*

**Remark 1.** The last part of the proposition reflects the fact that if $i$ were a net debtor to agent $j$ instead, he would not be entitled to any share in the liquidation of agent $j$'s estate in the event of agent $j$'s bankruptcy. Note that as long as agents $i$ and $j$ trade in the repo market, one of them must be a net creditor and proposition 2 applies to either $i$ or $j$, as long as the agent is solvent in state $D$.

*Proof.* See the appendix. □

Suppose that $i$ is a net creditor with counterparty $j$, and that $z_f^{ij} > 0$, and that $h_f^{ij} q_{f0} r_f > q_{fD} + p_D R_{fD}$. If agent $i$ anticipates a decrease in $j$'s expected repayment rate, $\eta_D^j$, then $i$ would like to charge $j$ a higher haircut (by lowering $h_f^{ij}$). To understand why this is so, note that the magnitude of $j$'s net debt to $i$, is given by $z_f^{ij}[q_{f0} r_f h_f^{ij} - (q_{fD} + p_D R_{fD})]$ and lowering $h_f^{ij}$ would reduce this debt and, therefore, the loss resulting from agent $j$'s bankruptcy.

Now suppose $h_f^{ij} q_{f0} r_f < q_{fD} + p_D R_{fD}$. If everything else is as in the previous paragraph, a decrease in $\eta_D^j$ will be an incentive for $i$ to collect a lower haircut from $j$ (by raising $h_f^{ij}$). Even though agent $i$ is a net creditor to agent $j$ when adding up all of his repo transactions with $j$, he has now a debt to $j$ associate to his position on security $f$ with absolute value $z_f^{ij}[(q_{fD} + p_D R_{fD}) - q_{f0} r_f h_f^{ij}]$. That is, the collateral kept by $i$ when lending cash to $j$ has now a higher market value than what $j$ owes to $i$. If $h_f^{ij}$ increases, he gets to keep more of the collateral in the event of $j$'s bankruptcy.

In both cases, $i$ has incentives to respond in a way that counteracts the loss in income when $j$ becomes more insolvent in state $D$. The appropriate response depends on the relationship of the value of $j$'s debt $(q_{f0}r_f h_f^{ij})$ with the market value of the collateral $(q_{fD} + p_D R_{fD})$. This relationship is what is usually understood as *asset risk*, the risk is precisely that the value of the collateral might decrease in some future date and become insufficient to cover the value of the debt that it is backing. In practice, haircuts are set so that it is expected that[13] $h_f^{ij} q_{f0} r_f < (q_{fD} + p_D R_{fD})$. This should be considered the most relevant case in the context the proposition.

Proposition 2 is useful to understand agents' incentives when a perceived increase in counterparty risk occurs. It should not be interpreted as a comparative statics analysis. One should expect to observe different haircuts but also different repo positions in equilibria with different repayment rates. One could expect that even the roles of net creditor and net debtor might get reversed when repayment rates change.

# 6 Conclusions

Repo markets have attracted a lot of attention in the applied macro and finance literatures, particularly since the 2008 financial crisis. These literatures have focused on how leverage in these markets has impacted on the whole economy, how this leverage depends on repo haircuts and what determines these margins. However, at the theoretical level, these important issues had not been addressed yet, possibly due to the complexity and intrinsic non-convexities involved in a bankruptcy model where counterparty risk could be understood and margins could be explained as endogenous variables.

We contribute in that direction, providing an example, adressing existence of equilibrium and charactering how counterparty and asset risks interact to determine repo haircuts. Full recognition of the recourse nature of repo loans is crucial. This implies also that default must be modelled in terms of insolvency and according to bankruptcy rules. The model can be made

[13]Or if the value of the collateral decreases and doesn't cover the value of the debt, a margin call is issued by the lender of funds.

richer, incorporating more detailed institutional aspects, or allowing also for centrally cleared repo, but the main drivers of margins seem to identifiable already in a simple model. Competitiveness versus small numbers of traders or asset risk in a creditor's perspective (suggesting default tensions) versus a debtor's risk perspective (concerned with repo fails instead) are some of the key issues in assessing how counterparty risk impacts on repo haircuts and the resulting leverage.

# A Appendix

**Lemma 2.** *Let $0 < \hat{\beta} < \beta$. A sufficient condition that guarantees that no agent goes bankrupt in $s = U$ is that the value of agents' repo positions is bounded as follows:*

$$q_{f0} z_f^{ij} \geq -\hat{M}_f^i(p_U, q_{f0}, r_f) \equiv -\frac{q_{f0}\hat{\beta} p_U \omega_U^i}{(I-1)[F + \sum_f (R_{fU} + r_f)]} \tag{17}$$

To prove this lemma, recall that the net income from agent $i$'s securities and repo positions at $s = U$ is given by $\sum_f (\phi_{f0}^i (q_{fU} + p_U R_{fU}) + \sum_{j \neq i} z_f^{ij} q_{f0} r_f h_f^{ij})$. From the bilateral market clearing conditions, if repo positions are bounded from below by $-M$, they are bounded from above by $M$. From the box constraint, short security positions are bounded from below by $-(I-1)M$. When this is the case, the magnitude of the debt is then bounded from above by:

$$\begin{aligned}\sum_f \left( |\phi_{f0}^i| (q_{fU} + p_U R_{fU}) + \sum_{j \neq i} |z_f^{ij}| q_{f0} r_f h_f^{ij} \right) \leq & \sum_f (I-1) M [(q_{fU} + p_U R_{fU}) + q_{f0} r_f h_f^{ij}] \\ \leq & (I-1) M [F + \sum_f (R_{fU} + r_f)]\end{aligned}$$

where the last inequality follows from the fact that $(p_U, q_U)$ is in the simplex, $q_{f0} \leq 1$ and $h_f^{ij} \leq 1$. We want any possible debt in $s = U$ to be covered by the garnishable income of the agent in that state, that is, we want $\beta p_U \omega_U^i > (I-1) M [F + \sum_f (R_{fU} + r_f)]$. This will certainly be achieved if the lower bound for the value of repo positions satisfies condition (17).

***Proof of Lemma 1.***

**1.** *The $n$-th Auxiliary economy.*

We will construct a sequence of auxiliary economies $\mathcal{E}(n)$. The economy will be truncated in such a way that agents' consumption plans go beyond the corresponding attainability bounds by a small amount $\epsilon > 0$. These attainability bounds exist as long as repo positions are bounded from below (see part 3 of this proof).

Now we need to make some change of variables (and extend what repo trades might be) together with modifying some constraints in order to prove the lower semi-continuity of the constraint correspondence of each agent. First, suppose that at least a small portion of the repo transactions agent $i$ has with agent $j$ as counterparty can be done under the same terms each of the other agents get when dealing with $j$. More precisely, for all $k \neq j$, agent $i$ is allowed to hold the repo position $z_f^{ikj}$ in exchange for a cash loan with haircut $(1 - h^{kj}) = (1 - h^{jk})$ instead of $(1 - h^{ij})$. Remember that, as a matter of notation, $h_f^{lk}$ and $h_f^{kl}$ both denote the same haircut.

Next, we note that repo rates are decided at the initial date, when repos are negotiated. Define $P_f \equiv 1/r_f$ and make the following change of variables: $\tilde{z}_f^{ikj} = q_{f0} r_f z_f^{ikj}$ for all $i$, $j$, $k$, and $f$.

After these changes of variables and new bounds on repo positions, we modify agent $i$'s budget constraints. At the first date we have:

$$p_0(x_0^i - \omega_0^i) + \sum_f q_{f0}(\phi_{f0}^i - o_f^i) + \sum_f P_f \sum_{j \neq i} \sum_{k \neq j} h_f^{kj} \tilde{z}_f^{ikj} \leq 0$$

The box constraint is rewritten in term of the new variables and relaxed a bit:

$$q_{f0}\phi_{f0}^i + \sum_{j \neq i} \sum_{k \neq j} P_f \tilde{z}_f^{ikj} \geq -1/n$$

Agent $i$'s net income from his repo positions with counterparty $j$, at state $s$ is given by:

$$I_s^{ij}(\tilde{z}^i) = \sum_f \sum_{k \neq j} \tilde{z}_f^{ikj} \left[ h_f^{kj} - Q_{fs} P_f \right]$$

Where the price $Q_{fs}$ will be set as close as possible (and for some $n$ onward will be exactly equal) to $(q_{fs} + p_s R_{fs})/q_{f0}$. This will become apparent after we define who sets this price and the criterion that is used to set it.

Payments from the agent's possession of securities at the beginning of the second date, in state $s$, are given by:

$$\Xi_s^i = \sum_f \phi_{f0}^i (q_{fs} + p_s R_{fs}) + \sum_f \sum_{j \neq i} \sum_{k \neq j} \tilde{z}_f^{ikj} Q_{fs} P_f$$

Agents behave as if bankruptcy in each state is given. The decision is taken by a fictitious agent that sets the variable $\alpha_s^i = 1$ when the agent is solvent and $\alpha_s^i = 0$ when the agent declares bankruptcy, according to the following criterion:

$$(\eta_s^i, \alpha_s^i) = \begin{cases} (1,1)\ ,\ \Xi_s^i + \sum_{j \neq i}[\eta_s^j I_s^{ij+} - I_s^{ij-}] \geq -\beta p_s \omega_s^i \\ \left( \frac{\beta p_s \omega_s^i + \Xi_s^i + \sum_{j \neq i} \eta_s^j I_s^{ij+}}{\sum_{j \neq i} I_s^{ij-}}, 0 \right)\ ,\quad \text{otherwise} \end{cases} \tag{18}$$

Note that we have also written the criterion that defines the repayment rate of the agents in each state. The repayment of agent $i$ in state $s$ is defined by $\eta^i_s \in [0,1]$, taken as given by each agent. We can write the budget constraint for the second date, state $s$, as:

$$p_s(x^i_s - \omega^i_s) + \sum_f q_{fs}\phi^i_{fs} \le \alpha^i_s\Big(\Xi^i_s + \sum_{j\neq i}\eta^j_s I^{ij+}_s - I^{ij-}_s\Big) + (1-\alpha^i_s)\big(-\beta p_s\omega^i_s\big) \tag{19}$$

We make an additional modification to equation (19). Suppose that there is a fictitious agent that sets $\gamma^{ij}_s = 1$ if $I^{ij}_s > 0$ and $\gamma^{ij}_s = 0$ if $I^{ij}_s \le 0$. Then $I^{ij+}_s = \gamma^{ij}_s I^{ij}_s$, $I^{ij-}_s = -(1-\gamma^{ij}_s)I^{ij}_s$. Agent $i$ takes $\gamma^{ij}_s$ as given for each $j \neq i$ and we rewrite state $s$'s budget constraint as:

$$p_s(x^i_s - \omega^i_s) + \sum_f q_{fs}\phi^i_{fs} \le \alpha^i_s\Big(\Xi^i_s + \sum_{j\neq i}\kappa^{ij}_s I^{ij}_s\Big) - (1-\alpha^i_s)\big(\beta p_s\omega^i_s\big)$$

where $\kappa^{ij}_s = \gamma^{ij}_s\eta^j_s + (1-\gamma^{ij}_s) \in [0,1]$. Agent $i$'s box constraint for the second date is:

$$\phi^i_{fs} \ge -1/n$$

Budget constraints at the third date remain unaltered, given by (6). The value of agent $i$'s repo trade with counterparty $j$ is constrained in the following manner:

$$P_f\tilde{z}^{iij}_f \ge -M^i_f(p_U, q_{f0}, P_f) + P_f/n,\ \forall j \neq i \tag{20}$$

where

$$M^i_f(p_U, q_{f0}, P_f) = \frac{q_{f0}\hat{\beta}p_U\omega^i_U}{(I-1)\Big(F + \sum_f(R_{fU} + 1/(P_f + 1/n)\Big)}$$

and $\hat{\beta} \in (0,\beta)$ is fixed and given. For the moment, we impose also exogenous bounds:

$$\begin{aligned} &\tilde{z}^{iij}_f \ge -n, \\ &\text{and } \tilde{z}^{ikj}_f \in \left[-\frac{1}{n}, \frac{1}{n}\right],\ \forall j \neq i, \forall k \in \{1,\cdots,I\}\backslash\{i,j\} \end{aligned} \tag{21}$$

**Remark 2.** Note that, as $n \to \infty$, $M^i_f(p_U, q_{f0}, P_f) \to \hat{M}^i_f(p_U, q_{f0}, r_f)$. In the limit $P_f\tilde{z}^{iij}_f \ge -M^i_f(p_U, q_{f0}, P_f) \Leftrightarrow q_{f0}z^{iij}_f \ge -\hat{M}^i_f(p_U, q_{f0}, r_f)$.

**2.** *Definition of quasi-equilibrium for the n-th auxiliary economy.*

A quasi-equilibrium is an allocation of bundles, securities and modified repo positions $(x, \phi, \tilde{z})$ together with prices $(p, q, P)$ and haircuts implied by $h$, such that:

(a) for each agent $i$, the plan $(x^i, \phi^i, \tilde{z}^i)$ is optimal, *given the values of the bankruptcy viariables* $\alpha = (\alpha_s^i)$ $i = 1, \cdots, I$ *and* $s = U, D$, that he observes,

(b) commodity markets clear: $\sum_i \left(x_0^i - \omega_0^i\right) = 0$ and $\sum_i \left(x_s^i - \omega_s^i\right) = \sum_f R_{fs}$, $\sum_i \left(x_{s+}^i - \omega_{s+}^i\right) = \sum_f R_{fs+}$ for all $s$,

(c) stock markets clear: $\sum_i \phi_{fe}^i = 1$ at each event $e$,

(d) repo markets clear: $\sum_{k \neq j} z^{kij} + \sum_{l \neq i} z^{lji} = 0$ for each $i < I$ and $j > i$.

It is important to note that a consumption plan that is optimal in an auxiliary economy, where agents take bankruptcy as given and out of their hands, might not be optimal in the original economy, where agents decide their own bankruptcy in each state. This problem will disappear in the limit.

**3.** *If repo positions are bounded as in* (21)*, then individually-feasible and market clearing allocations are bounded.*

To see this, let $M$ be such that $\tilde{z}_f^{iij} \geq -M$, $\forall i$. From the market clearing condition in repo markets we have that:

$$\tilde{z}_f^{jji} = -\sum_{l \notin \{i,j\}} \tilde{z}_f^{lji} - \sum_{k \notin \{i,j\}} \tilde{z}_f^{kij} - \tilde{z}_f^{iij} \leq 1/n(I-2) + 1/n(I-2) + M = M + 1/n[2(I-2)]$$

From the box constraint we have that:

$$q_{f0}\phi_{f0}^i \geq -1/n - \sum_{j \neq i} \tilde{z}_f^{iij} - \sum_{j \neq i} \sum_{k \notin \{i,j\}} \tilde{z}_f^{ikj} \geq -\Big[(I-1)M + 1/n\big[3(I-1)(I-2)+1\big]\Big] \equiv -\underline{M}_n$$

Then $q_{f0}\phi_{f0}^{i-} \leq \underline{M}_n$. For any attainable node 0−market-clearing portfolio allocation we have $q_{f0}\phi_{f0}^i + q_{f0}\sum_{j \neq i} \phi_{f0}^j \equiv q_{f0}\phi_{f0}^i + \sum_{j \neq i} \left(q_{f0}\phi_{f0}^{j+} - q_{f0}\phi_{f0}^{j-}\right) = q_{f0}$. So we have $q_{f0}\phi_{f0}^i \leq q_{f0} + \sum_{j \neq i} q_{f0}\phi_{f0}^{j-} \leq q_{f0} + (I-1)\underline{M}_n$. At node $s$ we have $0 \leq \phi_{fs}^i \leq 1$. Finally, attainable consumption is bounded by: $0 \leq x_0^i \leq \sum_i \omega_0^i$, by $0 \leq x_s^i \leq \sum_i \omega_s^i + \sum_f R_{fs}$ and by $0 \leq x_{s+}^i \leq \sum_i \omega_{s+}^i + \sum_f R_{fs+}$.

**4.** *Generalized game for the n-th auxiliary economy.*

We define a generalized game played by consumers, auctioneers and some additional agents whose choice sets and objectives we describe now:

An auctioneer at the first date chooses $(p_0, q_0, P)$ in the simplex $\Delta_+^{1+2F}$ to maximize:

$$p_0 \sum_i (x_0^i - \omega_0^i) + \sum_f q_{f0} \sum_i (\phi_{f0}^i - o_f^i) + \sum_f P_f \sum_{i<I} \sum_{j>i} h_f^{ij} \left( \sum_{k \neq j} \tilde{z}_f^{kij} + \sum_{l \neq i} \tilde{z}_f^{lji} \right)$$

Since repo markets are bilateral and must clear for each pair of agents, while repo rates are assumed to be just security-specific, we still at the first date, and for each security $f$, another auctioneer choosing $h_f = (h_f^{ij})_{i<I, j>i}$ on the simplex $\Delta_+^{I(I-1)/2}$ to maximize:

$$\sum_{i<I} \sum_{j>i} h_f^{ij} \left( \sum_{k \neq j} \tilde{z}_f^{kij} + \sum_{l \neq i} \tilde{z}_f^{lji} \right)$$

At state $s$ of the second date, there are several agents to consider. First, an auctioneer chooses $(p_s, q_s)$ in the simplex $\Delta_+^{1+F}$ to maximize:

$$p_s \left[ \sum_i (x_s^i - \omega_s^i) - R_{fs} \right] + \sum_f q_{fs} \sum_i (\phi_{fs}^i - \phi_{f0}^i)$$

For each $f$, there is an agent that chooses $Q_{fs}$ in the set $\left[0, \frac{(q_{fs} + p_s R_{fs})}{q_{f0} + 1/2^n} + 2^n\right] \cap \left[0, 2^{n+1}\right]$ to minimize:

$$\left(q_{f0} Q_{fs} - (q_{fs} + p_s R_{fs})\right)^2$$

To deal with the budget non-convexity caused by bankruptcy, for each agent $i$, there are three added fictitious agents in state $s$. Each of these agents take the choices of every other agent in the whole economy as given; this includes the bankruptcy variable for consumer $i$ in state $s' \neq s$.

(i) The first of these agents, which we denote $i_s^0$, solves consumer $i$'s problem under the presumption that $i$ declares bankruptcy in state $s$. That is, $i_s^0$ has the same utility function and endowments as consumer $i$, but the budget constraints differ, only in state $s$, where $i_s^0$ assumes that the state $s$ bankruptcy variable $\alpha_s^{i_s^0}$ takes value 0. Let's denote the utility obtained by this agent when he solves his problem by $U_{\star s}^{i_s^0}$

(ii) The second of these agents, $i_s^1$, solves consumer $i$'s problem while being solvent in state $s$, taking the value of the own bankruptcy variable $\alpha_s^{i_s^1}$ to be 1. Denote this agent's maximal utility by $U_{\star s}^{i_s^1}$

(iii) A third agent chooses $\alpha_s^i$ in $[0,1]$ in order to maximize:

$$\alpha_s^i \left[ U_{\star s}^{i_s^1} - U_{\star s}^{i_s^0} \right] \tag{22}$$

Finally we need to construct the effective payments, which depend on reimbursement rates. For each pair $(i,j)$, there is a fictitious agent that chooses $\gamma_s^{ij}$ in $[0,1]$ in order to maximize:

$$\gamma_s^{ij} I_s^{ij}$$

For each $i$, and each $s$, there is a fictitious agent that chooses $\eta_s^i$ in $[0,1]$ in order to minimize

$$\left( \left[ \sum_{j \neq i} (1-\gamma_s^{ij}) I_s^{ij} \right] \eta_s^i - \left[ \beta p_s \omega_s + \Xi_s^i + \sum_{j \neq i} \gamma_s^{ij} \eta_s^j I_s^{ij} \right] \right)^2$$

At date 2, state $s^+$, we can set $p_{s^+} = 1$.

Finally, consumers choose plans $(x^i, \phi^i, \tilde{z}^i)$ satisfying the budget and box constraints, in the above modified forms, in order to maximize utility. Note that the consumers only take into account the given values of the bankruptcy variables $\alpha_s^i$ ($i = 1, \cdots, I$ and $s = U, D$), that is, they only consider the values obtained in (22).

**5.** $\alpha_s^i$ *is such that:*

$$\alpha_s^i \begin{Bmatrix} = 1 \\ \in (0,1) \\ = 0 \end{Bmatrix} \Rightarrow \left[ \Xi_s^i + \sum_{j \neq i} [\eta_s^j I_s^{ij+} - I_s^{ij-}] \right] \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} - \beta p_s \omega_s^i$$

This will ultimately imply that $\alpha_s^i$ is such that the condition in equation (18) is satisfied.

Denote by $\mathrm{RHS}^i$ the expression $\Xi_s^i + \sum_{j \neq i} [\eta_s^j I_s^{ij+} - I_s^{ij-}]$ evaluated at the at plan agent $i$ picks once $\alpha_s^i$ has been chosen for him (as in (22)). Denote by $\mathrm{RHS}^{i_s^0}$ and $\mathrm{RHS}^{i_s^1}$ the same expression but now evaluated at the plans that agents $i_s^0$ and $i_s^1$ pick, respectively (taking for $\alpha_s^{i_s^0}$ and $\alpha_s^{i_s^1}$ the values 0 and 1, respectively).

Consider first the case when $\alpha_s^i = 1$. By the way $\alpha_s^i$ is chosen, we know that it must be the case that $U_{\star s}^{i_s^1} \geq U_{\star s}^{i_s^0}$. If $\mathrm{RHS}^i < -\beta p_s \omega_s^i$ then agent $i_s^0$ could choose the same plan as the one chosen by agent $i$ when solving his own problem (where $\alpha_s^{i_s^0} = 0$). Indeed, this plan would satisfy all budget constraints in agent $i_s^0$'s problem but, crucially, would yield strictly more consumption in state $s$. Denote by $\hat{U}_s^{i_s^0}$ the utility attained by $i_s^0$ when choosing this plan. Since utilities are monotone and, in particular, utilities increase with consumption in state $s$, we would have $\hat{U}_s^{i_s^0} > U_{\star s}^{i_s^1} \geq U_{\star s}^{i_s^0}$. This contradicts the optimality of $U_{\star s}^{i_s^0}$.

The case when $\alpha_s^i = 0$ is analogous and we conclude that, if $\alpha_s^i = 0$, we cannot have $\mathrm{RHS}^i > -\beta p_s \omega_s^i$.

If $\alpha_s^i \in (0,1)$ it is necessary that $U_{\star s}^{i_s^1} = U_{\star s}^{i_s^0}$. In this case both arguments apply and $\mathrm{RHS}^i$ can neither be strictly lower, nor strictly greater than $-\beta p_s \omega_s^i$.

In short, the monotonicity of agents' preferences have the consequence that even though, for each $i$ and $s$, $\alpha_s^i$ is chosen using a *utility criterion*, it is consistent with a choice made using a *solvency criterion* (as in (18)).

**6.** *Consumers' constraint correspondence is continuous and the generalized game has a an equilibrium.*

Upper semi-continuity of constraint correspondences is immediate. To prove the lower semi-continuity it suffices to show that, any parameters, the interior of the consumers' constraint set is non-empty, since consumers have now a modified constraint set (taking the values of $\alpha_s^i$ as given) which is convex. We proceed to find a plan that belongs to this interior:

If $p_0 \neq 0$ take $x^i = 0$, $\phi^i = 0$ and $\tilde{z}^i = 0$.

If $p_0 = 0$ and $q_0 \neq 0$, there is at least one security $f$ such that $q_{f0} > 0$. Let

$$\phi_{\min} \equiv -\frac{1}{2} \min_{f,s} \left\{ \frac{\min\{1, p_{s+} \omega_{s+}^i\}}{\max\{1, p_{s+} R_{fs+}\} \cdot n \cdot F} \right\}$$

Choose the consumption plan $x^i = 0$ and repo positions $\tilde{z}^i = 0$. At date 0 choose the portfolio $\phi_{g0}^i = 0$ for all $g \neq f$, $\phi_{f0}^i = o_f^i/2$. Finally, at date 1 set $\phi_{gs}^i = \phi_{\min}$ $\forall g, s$.

If $p_0 = 0$ and $q_0 = 0$, then there is some $f$ such that $P_f > 0$. There is some $k < I$ and some

$j > k$ such that[14] $h_f^{kj} > 0$. Now, $\kappa_s^{ij} \in [0,1]$ implies $P_f Q_{fs}(1-\kappa_s^{ij}) + \kappa_s^{ij} h_s^{kj} \leq [P_f Q_{fs} + h_s^{kj}]$. Let $x^i = 0$; $\tilde{z}_g^{ilv} = 0$, $\forall (l,v,g) \neq (k,j,f)$; $\phi_{g0}^i = 0$, $\forall g$; $\phi_{gs}^i = \phi_{\min}$, $\forall g \forall s$, and make

$$\tilde{z}_f^{ikj} = -\frac{1}{2} \min_s \left\{ \frac{\min\{1, (1-\beta)p_s\omega_s^i - \phi_{\min}\sum_f q_{fs}\}}{\max\{1, P_f Q_{fs} + h_f^{kj}\} \cdot n} \right\}$$

Notice that $(q_s, p_s) \in \Delta_+^{1+F}$ ensures that the numerator of the fraction is positive.

Existence of equilibrium for the generalized game follows from the continuity of the constraint correspondences, by a standard argument.

**7.** *First order conditions.*

At an equilibrium for the generalized game, the following first order conditions of consumers' optimization problems must hold (since the reverse-convex constraint qualification holds, as the modified constraints are linear, once the values for $\alpha_s^i$ are taken as given): there exist ${}^n\lambda_e^i$ (multiplier for agent $i$'s budget constraint in node $e$), ${}^n\mu_{fe}^i$ (multiplier for the box constraint for security $f$ in node $e$), and ${}^n\overline{\nu}_{\tilde{z}_f^{ikj}}$, ${}^n\underline{\nu}_{\tilde{z}_f^{ikj}}$ (multipliers for the upper and lower bound for $\tilde{z}_f^{ikj}$ such that:

$$\begin{aligned}
{}^n x_e^i : &\quad \frac{\partial U^i({}^n x^i)}{\partial {}^n x_e^i} \leq {}^n\lambda_e^i\, {}^n p_e, \text{ with "=" if } {}^n x_e^i > 0 \\
{}^n\phi_{f0}^i : &\quad {}^n q_{f0} = \sum_s {}^n\alpha_s^i \cdot \frac{{}^n\lambda_s^i}{{}^n\lambda_0^i - {}^n\mu_{f0}^i} \cdot \left({}^n q_{fs} + {}^n p_s R_{fs}\right) \\
{}^n\phi_{fs}^i : &\quad {}^n q_{fs} = \frac{{}^n\lambda_{s+}^i}{{}^n\lambda_s^i}\, {}^n p_{s+} R_{fs+} + \frac{{}^n\mu_{fs}^i}{{}^n\lambda_s^i} \\
{}^n\tilde{z}_f^{ikj} : &\quad {}^n P_f = \frac{\sum_s {}^n\alpha_s^i\, {}^n\lambda_s^i\, \kappa_s^{ij}\, {}^n h_f^{kj} + {}^n\underline{\nu}_{\tilde{z}_f^{ikj}} - {}^n\overline{\nu}_{\tilde{z}_f^{ikj}}}{{}^n\lambda_0^i\, {}^n h_f^{kj} - \sum_s {}^n\alpha_s^i\, {}^n\lambda_s^i \left(1 - {}^n\kappa_s^{ij}\right) {}^n Q_{fs} - {}^n\mu_{f0}^i} \\
&\quad \text{,with } {}^n\overline{\nu}_{\tilde{z}_f^{ikj}} = 0 \text{ if } k = i
\end{aligned}$$

The last equation can be written also as:

$${}^n\lambda_0^i\, {}^n h_f^{kj}\, {}^n P_f - \sum_s {}^n\alpha_s^i\, {}^n\lambda_s^i \left(1 - {}^n\kappa_s^{ij}\right) {}^n P_f\, {}^n Q_{fs} - {}^n P_f\, {}^n\mu_{f0}^i = \sum_s {}^n\alpha_s^i\, {}^n\lambda_s^i {}^n\kappa_s^{ij}\, {}^n h_f^{kj} + {}^n\underline{\nu}_{\tilde{z}_f^{ikj}} - {}^n\overline{\nu}_{\tilde{z}_f^{ikj}}$$

[14]Note that if we hadn't extended repo trades to include $z_f^{ikj}$ (which we will eventually get rid off), we could not guarantee the positivity of some $h^{kj}$ for some agent $k$. The extension allowed us to assume that $h$ is in a simplex.

The first order conditions for agents $i_s^0$ and $i_s^1$ are analogous, with the values for the respective $\alpha_s^{i^0}$ and $\alpha_s^{i^1}$ set as explained in part 4.

From the problem of the agent that chooses $Q_{fs}$ we have that:

$$^nQ_{fs} = \frac{(^nq_{fs} + {}^np_s R_{fs})}{^nq_{f0}} + \frac{1}{2 \cdot (^nq_{f0})^2}(^n\underline{\nu} - {}^n\overline{\nu})$$

**8.** *The equilibrium for the generalized game is such that markets clear in the $n$-th auxiliary economy.*

Prices in the $n$-th auxiliary economy clear markets. For now, lets ignore the superscripts $n$.

The usual arguments apply to commodities and securities markets. We focus our attention on repo markets. For each $f$, we must have $\sum_{i<I}\sum_{j>i} h_f^{ij}\left(\sum_{k\neq j}\tilde{z}_f^{kij} + \sum_{l\neq i}\tilde{z}_f^{lji}\right) = 0$. If the expression on the left hand side were strictly positive, the auctioneer would choose $P_f = 1$ and Walras' law would not hold. On the other hand, if it were strictly negative, the auctioneer would choose $P_f = 0$ leading agents positions $\tilde{z}_f^{ikj}$ to its upper bound (see part 7 of this proof), which would imply $\sum_{i<I}\sum_{j>i} h_f^{ij}\left(\sum_{k\neq j}\tilde{z}_f^{kij} + \sum_{l\neq i}\tilde{z}_f^{lji}\right) > 0$, a contradiction.

Now, haircuts are such that, for each pair $(i,j)$ we have $\left(\sum_{k\neq j}\tilde{z}_f^{kij} + \sum_{l\neq i}\tilde{z}_f^{lji}\right) = 0$. If we had that for some pair $\left(\sum_{k\neq j}\tilde{z}_f^{kij} + \sum_{l\neq i}\tilde{z}_f^{lji}\right) > 0$, then the auctioneer that sets haircuts would choose $h_f^{ij} = 1$ and we would have a contradiction with what we showed in the previous paragraph. If we had $\left(\sum_{k\neq j}\tilde{z}_f^{kij} + \sum_{l\neq i}\tilde{z}_f^{lji}\right) < 0$, the auctioneer would set $h_f^{ij} = 0$, which would imply that all agents set $\tilde{z}_f^{kij}$ equal to the allowed upper limit (again, see part 7 of this proof), which in turn would imply $\left(\sum_{k\neq j}\tilde{z}_f^{kij} + \sum_{l\neq i}\tilde{z}_f^{lji}\right) > 0$, a contradiction.

**9.** *Consumers' consumption plans are optimal (taking bankruptcy in each state as given) if we remove the bounds that truncate the $n$-th auxiliary economy.*

Here the usual argument applies[15]: consumption plan $(x^i, \phi^i, z^i)$ is optimal for the problem where consumption and securities positions are not bounded from above. If this was not the case, there would be $(\bar{x}^i, \bar{\phi}^i, \bar{z}^i)$ that would be budget feasible at prices $(p,q,P,Q)$ and $U^i(\bar{x}^i) > U^i(x^i)$. A convex combination of the plans $(t\cdot x^i + (1-t)\cdot\bar{x}^i, t\cdot\phi^i + (1-t)\cdot\bar{\phi}^i, t\cdot z^i + (1-t)\cdot\bar{z}^i)$ with

[15] Recall that consumers' problems have convex constraint sets since the second date budget constraints have been modified by taking as given the values of $\alpha_s^i$.

$t \in (0,1)$ and close enough to 1 would still be strictly preferred by agent $i$ to $(x^i, \phi^i, z^i)$, would be budget feasible and would satisfy the upper bounds imposed in the auxiliary economy. This would contradict the optimality of $(x^i, \phi^i, z^i)$ in the auxiliary economy.

Given that markets clear and agent's choices are optimal (*when taking bankruptcy in each state as given*), we have found a quasi-equilibrium for the $n$-th auxiliary economy.

**10.** *The sequence of quasi-equilibria for the auxiliary economies has a cluster point.*

Let $n \to \infty$. We will prove that there is a cluster point for the sequence of equilibria $({}^n x, {}^n \phi, {}^n \tilde{z}, {}^n p, {}^n q, {}^n P, {}^n Q, {}^n h)$. First, by compactness, $({}^n x, {}^n \phi, {}^n p, {}^n q, {}^n h)$ has a cluster point.

Now we argue that ${}^n p_0 \to p_0 > 0$. Otherwise, denoting by $E_0$ the canonical vector in the direction of node 0, the consumption plan $(1 - {}^n p_0/({}^n p_0 + \sum_f {}^n q_{f0})) \cdot ({}^n x, {}^n \phi, {}^n \tilde{z}) + bE_0$ would be budget feasible for $b = \min\{\omega_0^i, o_{10}^i, \cdots, o_{F0}^i\}$. To see this, note that at date 0, this new consumption plan will be feasible if

$$\begin{aligned}
{}^n p_0 \cdot b &\leq \frac{{}^n p_0}{({}^n p_0 + \sum_f {}^n q_{f0})} \cdot \left[ {}^n p_0 \cdot {}^n x_0^i + \sum_f {}^n q_{f0} {}^n \phi_{f0}^i + \sum_f {}^n P_f \sum_{j \neq i} \sum_{k \neq j} h_f^{kj} \; {}^n \tilde{z}_f^{ikj} \right] \\
&\leq \frac{{}^n p_0}{({}^n p_0 + \sum_f {}^n q_{f0})} \cdot \left[ \sum_l {}^n p_0 \cdot \omega_0^i + \sum_f {}^n q_{f0} o_f^i \right]
\end{aligned}$$

Which is satisfied when $b = \min\{\omega_0^i, o_{10}^i, \cdots, o_{F0}^i\}$. Clearly, the new portfolio satisfies repo bounds and budget constraints at first and second dates. The box constraint is also satisfied since $\left|(1 - {}^n p_0/({}^n p_0 + \sum_f {}^n q_{f0})) \cdot \left({}^n q_{f0} {}^n \phi_{f0}^i + \sum_{j \neq i} \sum_{k \neq j} {}^n P_f {}^n \tilde{z}_f^{ikj}\right)\right| \leq \left|{}^n q_{f0} {}^n \phi_{f0}^i + \sum_{j \neq i} \sum_{k \neq j} {}^n P_f {}^n \tilde{z}_f^{ikj}\right|$. So if ${}^n p_0 \to 0$ we have that for $n$ big enough, the utility that agent $i$ gets from consuming $(1 - {}^n p_0/({}^n p_0 + \sum_f {}^n q_{f0})) \cdot ({}^n x, {}^n \phi, {}^n \tilde{z}) + bE_0$ would be strictly greater than the utility from the quasiequilibrium plan $({}^n x, {}^n \phi, {}^n \tilde{z})$, which contradicts the optimality of the latter.

An analogous argument shows that, for $s = U, D$, ${}^n p_s \to p_s > 0$. In the case of the sequence ${}^n p_s$, we would be able to show that for some $n$ large enough, the consumption plan $(1 - {}^n p_s/({}^n p_s)) \cdot ({}^n x, {}^n \phi, {}^n \tilde{z}) + \omega_{ks}^i \cdot E_s$ would be preferable to the equilibrium consumption plan.

We also have that ${}^n q_{f0} \to q_{f0} > 0$. To see this, take an agent that consumes at note $s^+$ and notice that from the first order conditions (see part 7) we have that:

$$^{n}q_{fs} = \frac{^{n}\lambda^{i}_{s+}}{^{n}\lambda^{i}_{s}} {}^{n}p_{s+} R_{fs+} + \frac{^{n}\mu^{i}_{fs}}{^{n}\lambda^{i}_{s}} \Rightarrow {}^{n}q_{fs} \geq \frac{^{n}\lambda^{i}_{s+}}{^{n}\lambda^{i}_{s}} {}^{n}p_{s+} R_{fs+}$$

Since $\frac{^{n}\lambda^{i}_{s+}}{^{n}\lambda^{i}_{s}} = \frac{D_{s+}U^{i}(^{n}x^{i})}{D_{s}U^{i}(^{n}x^{i})} \frac{^{n}p_{s}}{^{n}p_{s+}}$. Given Assumption 1, $\frac{D_{s+}U^{i}(^{n}x^{i})}{D_{s}U^{i}(^{n}x^{i})}$ has a positive minimum on $\{x; U^{i}(x) \geq U^{i}(\omega^{i}) \wedge x \leq \sum_{i} \omega^{i}\}$. On the other hand we have shown that the cluster point for $^{n}p_{s}$ is strictly greater than zero. We conclude that the cluster point for $^{n}q_{fs}$ is also strictly positive. An analogous argument applies to the sequence $^{n}q_{f0}$ but in this case consider the first order conditions of the agent $i^{1}_{U}$ and note that:

$$^{n}q_{f0} = \sum_{s} {}^{n}\alpha^{i^{1}_{U}}_{s} \cdot \frac{^{n}\lambda^{i^{1}_{U}}_{s}}{^{n}\lambda^{i^{1}_{U}}_{0} - {}^{n}\mu^{i^{1}_{U}}_{f0}} \cdot \left(^{n}q_{fs} + {}^{n}p_{s}R_{fs}\right) \geq \frac{^{n}\lambda^{i^{1}_{U}}_{U}}{^{n}\lambda^{i^{1}_{U}}_{0}} \cdot \left(^{n}q_{fU} + {}^{n}p_{U}R_{fU}\right)$$

Then the argument continues as before. We can use almost the exact same argument to show that $^{n}P \to P > 0$ (by taking $i$ such that $x^{i}_{0} > 0$). From agent $i^{1}_{U}$'s first order condition on $^{n}z^{i^{1}_{U}i^{1}_{U}j}_{f}$ (see part 7 of this proof) we have that:

$$^{n}P_{f} = \sum_{s} \frac{^{n}\lambda^{i^{1}_{U}}_{s}}{^{n}\lambda^{i^{1}_{U}}_{0}} {}^{n}\alpha^{i^{1}_{U}}_{s} {}^{n}\kappa^{i^{1}_{U}j}_{s} + \frac{^{n}P_{f}}{^{n}h^{i^{1}_{U}j}_{f}} \cdot \left[\sum_{s} \frac{^{n}\lambda^{i^{1}_{U}}_{s}}{^{n}\lambda^{i^{1}_{U}}_{0}} {}^{n}\alpha^{i^{1}_{U}}_{s} {}^{n}Q_{fs}\,(1 - {}^{n}\kappa^{i^{1}_{U}j}_{s}) + \frac{^{n}\mu^{i^{1}_{U}}_{f0}}{^{n}\lambda^{i^{1}_{U}}_{0}}\right] + \frac{^{n}\underline{\nu}_{z^{i^{1}_{U}i^{1}_{U}j}_{f}}}{^{n}h^{i^{1}_{U}j}_{f}\,{}^{n}\lambda^{i^{1}_{U}}_{0}}$$

Which implies that along the sequence we must have:

$$^{n}P_{f} \geq \sum_{s} \frac{^{n}\lambda^{i^{1}_{U}}_{s}}{^{n}\lambda^{i^{1}_{U}}_{0}} {}^{n}\alpha^{i^{1}_{U}}_{s} {}^{n}\kappa^{i^{1}_{U}j}_{s} \geq \frac{^{n}\lambda^{i^{1}_{U}}_{U}}{^{n}\lambda^{i^{1}_{U}}_{0}}$$

We have already shown that the sequence $\frac{^{n}\lambda^{i^{1}_{U}}_{U}}{^{n}\lambda^{i^{1}_{U}}_{0}}$ is bounded away from zero so that the cluster point for $^{n}P_{f}$ is strictly positive. Since the sequence $^{n}P_{f}$ has a non-zero limit, we have by (20) that the sequence $^{n}z^{ikj}_{f}$ is bounded and it has a cluster point $z^{ikj}_{f}$.

Finally note that since the sequences $^{n}q_{f0}$, $^{n}q_{fs}$ and $^{n}p_{s}$ all have positive limits, the sequence $\frac{(^{n}q_{fs} + {}^{n}p_{s}R_{fs})}{^{n}q_{f0}}$ converges and for $n$ large enough we have $^{n}Q_{fs} = \frac{(^{n}q_{fs} + {}^{n}p_{s}R_{fs})}{^{n}q_{f0}}$ and $^{n}Q_{fs} \to \frac{(q_{fs} + p_{s}R_{fs})}{q_{f0}}$.

**11.** *The cluster point is a quasi-equilibrium for the limit of the auxiliary economies.*

A quasi-equilibrium for the limiting economy, as $n \to \infty$, is defined by the same conditions that were enumerated in part 2. The fact that a cluster point of quasi-equilibria for the n-auxiliary economies is quasi-equilibrium for the limiting economy follows from the following considerations:

1. Market clearing along the sequence implies market clearing in the limiting economy.

2. Given assumption 1 on utilities, we have that for each $i$, the (sufficient) first order conditions of the problems of the agents $i^1_U, i^0_U, i^1_D, i^0_D$ are satisfied in the limit and the choices of these agents are optimal.

3. For each $i$ and $s$, the sequence ${}^n\alpha^i_s$ also has a cluster point $\alpha^i_s$ that maximizes (22): for example, if $\alpha^i_s = 1$, this means that for $n$ large enough, ${}^n\alpha^i_s > 0$, which in turn means that ${}^nU^{i^1_s}_{\star s} \geq {}^nU^{i^0_s}_{\star s}$, so that $U^{i^1_s}_{\star s} \geq U^{i^0_s}_{\star s}$. This implies in particular that what was shown in part 5 of this proof also applies to $\alpha^i_s$.

4. For each $i$, again by assumption 1, first order conditions are satisfied in the limit and consumers choices are optimal when taking as given the value of each $\alpha^i_s$, $s = U, D$.

**12.** *The quasi-equilibrium in the cluster point is an equilibrium for the original economy.*

Revert to the original variables: $r_f = 1/P_f$ and $z^{ikj}_f = \tilde{z}^{ikj}_f/(q_{f0}r_f)$ for all $i$, $j$, $k$, and $f$. Let $z^{ij} = z^{iij}_f$. The plans of the agents in the cluster point are feasible in the original economy. In particular, the condition $q_{f0}z^{ij}_f \geq \hat{M}^i_f(p_U, q_{f0}, r_f)$ is satisfied. From Lemma 2 in the Appendix, this implies that no agent can declare bankruptcy in state $U$.

Market clearing according to the definition of quasi-equilibrium implies market clearing according to the equilibrium definition in the original economy. This follows because $z^{ikj} = 0$ for every $k \neq i$. Securities and good markets also clear.

The final thing to note is that consumers' plans are optimal, according to the definition of equilibrium. This follows from the fact that bankruptcy cannot occur in state $U$ and the optimal decision of agent $i$ in the original economy, where the agent decides whether or not to declare bankruptcy in state $s = D$, coincides with the decision taken by the same agent in the quasi-equilibrium, when he takes the value of $\alpha^i_D$ as given. This is true because of how $\alpha^i_D$ was chosen: for example if $\alpha^i_D = 1$ it means that the consumption plan and porfolio of agent $i$ are such that there is no bankruptcy (see part 5 of this proof) and that the utility attained dominates the utility he could get by declaring bankruptcy. In fact, these two utilities where compared when

$\alpha_D^i$ was chosen. The agent takes the bankruptcy decision as given but it coincides with the decision he himself would have taken.

A final note: The last argument is valid because bankruptcy can only occur in one of the states. If bankruptcy were possible in both states, $\alpha_U^i$ and $\alpha_D^i$ would be optimal for the respective agents that choose their values but not necessarily for agent $i$. In that case the optimal decision for $i$ involves deciding bankruptcy in both states *at the same time*, while the agents that choose $\alpha_U^i$ and $\alpha_D^i$ make the decision taking the decision of the other as given, and coordination problems can arise.

□

***Proof of Proposition 2.*** From (16), if $r_f$ is competitive, the derivative of $h_f^{ij}$ with respect to $\eta_s^j$ reduces to:

$$\frac{\partial h_f^{ij}}{\partial \eta_D^j} = -\alpha_D^i \gamma_D^{ij} \cdot \frac{1}{q_{f0}} \cdot \frac{\lambda_D^i}{\lambda_0^i} \cdot \frac{(q_{fD} + p_D R_{fD}) - h_f^{ij} q_{f0} r_f}{1 - r_f \left[\frac{\lambda_U^i}{\lambda_0^i} \alpha_U^i \kappa_U^{ij} + \frac{\lambda_D^i}{\lambda_0^i} \alpha_D^i \kappa_D^{ij}\right]}$$

Moreover, we have that $\frac{dh_f^{ij}}{d\eta_s^j}$ has the same sign as $[h_f^{ij} q_{f0} r_f - (q_{fs} + p_s R_{fs})]$ and $\frac{d(1-h_f^{ij})}{d\eta_s^j}$ has the same sign as $[(q_{fs} + p_s R_{fs}) - h_f^{ij} q_{f0} r_f]$. To see that this is the case, look at the first order condition of agent $i$'s problem with respect to $z_f^{ij}$:

$$r_f = \frac{\lambda_0^i}{\sum_s \lambda_s^i \alpha_s^i \kappa_s^{ij}} - \frac{1}{q_{f0} h_f^{ij}} \cdot \frac{\sum_s \lambda_s^i \alpha_s^i [(1 - \kappa_s^{ij})(q_{fs} + p_s R_{fs})]}{\sum_s \lambda_s^i \alpha_s^i \kappa_s^{ij}} - \frac{1}{h_f^{ij}} \cdot \frac{(\mu_{f0}^i / q_{f0}) + \nu_f^{ij}}{\sum_s \lambda_s^i \alpha_s^i \kappa_s^{ij}} \tag{23}$$

Lets focus on the term in the middle of the right hand side of (23):

$$\frac{\lambda_U^i \alpha_U^i [(1 - \kappa_U^{ij})(q_{fU} + p_U R_{fU})] + \lambda_D^i \alpha_D^i [(1 - \kappa_D^{ij})(q_{fD} + p_D R_{fD})]}{\lambda_U^i \alpha_U^i \kappa_U^{ij} + \lambda_D^i \alpha_D^i \kappa_D^{ij}} \tag{24}$$

In an equilibrium as the one which existence we have proven, we have $\alpha_U^i = 1$. The theorem assumes that $i$ is a net creditor so we have $\gamma_D^{ij} = 1$. We have that agent $j$ is solvent in state $U$. We must have $\kappa_U^{ij} = 1$. We have assumed that $i$ is solvent in state $D$, so that $\alpha_D^i = 1$, we can suppose that $\eta_D^j \in (0, 1)$ (so that $j$ was risky to begin with and so that it makes sense to take the derivative with respect to $\eta_D^j$ which belongs in the set $[0, 1]$). From all these considerations, we have that (24) is actually positive:

$$\frac{\lambda_D^i[(1-\eta_D^j)(q_{fD}+p_D R_{fD})]}{\lambda_U^i+\lambda_D^i\eta_D^j} > 0 \tag{25}$$

We can conclude from equation (23) that $r_f < \lambda_0^i / \sum_s \lambda_s^i \alpha_s^i \kappa_s^{ij}$ as long as at least one of the following conditions holds:

(a) Agent $i$ values the possession of security $f$ ($\mu_{f0}^i > 0$).

(b) Agent $i$'s position $z_f^{ij}$ has a market value at the lower bound.

(c) Counterparty $j$ is *somewhat* risky in state $D$ ($\eta_D^j \in (0,1)$).

When at least one of the conditions is satisfied we have $r_f \sum_s \frac{\lambda_s^i}{\lambda_0^i}\alpha_s^i\kappa_s^{ij} < 1$. □

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