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INTERNAL RATIONALITY, HETEROGENEITY, AND COMPLEXITY IN THE NEW KEYNESIAN MODEL

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Internal Rationality, Heterogeneity, and Complexity in the New Keynesian Model[*]

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Abstract

This paper studies the dynamic behaviour of a workhorse New Keynesian model in which households and firms can be fully rational or internally rational. First, we derive the model with a fixed proportion n of agents fully rational and a fixed proportion $(1 - n)$ of agents internally rational, in a similar manner to Massaro (2013). In this model, we establish two propositions. First, a decrease in the proportion of fully rational agents does not destabilise the system if the rational expectations determinacy condition for the monetary rule holds. Second, the rational expectations determinacy condition is identical to the stability condition for the model in which all agents are internally rational. We then extend the model to include predictor selection along the lines of Branch and McGough (2010). In this model, we establish two further propositions. First, the rational expectations determinacy condition ensures local determinacy and stability as the cost of being fully rational becomes infinitely negative. Second, if the model starts from a position of indeterminacy, an increase in the fixed cost of being fully rational can lead to the loss of local stability via a Hopf bifurcation. A *rational route to randomness* follows from this, which we explore numerically. Taken as a whole, these results indicate that complex dynamics in the internally rational New Keynesian model are closely associated with monetary policy failures. Finally, we consider the robustness of our results to changes in the monetary policy rule.

Keywords: New Keynesian behavioural model, internal rationality, heterogeneous expectations.

JEL Codes: E03, E12, E32, E70, E71.

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1 Introduction

This paper constructs and explores the monetary policy consequences of the workhorse New Keynesian model with bounded rationality and heterogeneous agents. It departs from existing models of this genre in its approach to bounded rationality and learning. As Graham (2011) has pointed out, two main approaches to bounded rationality have been made by the literature to date. The first of these, known as the Euler equation approach, follows the pioneering work of Evans and Honkapohja (2001), and assumes that agents forecast their own decisions in future periods. Furthermore, agents know the minimum state variable form of the equilibrium, and use direct observations or VAR estimates of these states to update their forecasts each period.

Although the Euler equation approach to bounded rationality responds to what many regard as an extreme assumption of model consistent expectations, the departure is only a modest one as agents still need to know the minimum state variable form of the equilibrium. Introducing simple behavioural heuristics, as is the case in many agent based models, addresses this concern. However, this raises a different concern regarding the bounds on bounded rationality, as agents using simple heuristics may fall considerably short of full rationality. In particular, behavioural heuristics can depart from rationality in an infinite number of ways, leading to the "wilderness" of Sargent (1993).

The second approach to bounded rationality, first introduced by Eusepi and Preston (2011) into an RBC model, assumes that agents are internally rational given their beliefs over aggregate states and prices. Adam and Marcet (2011) applies this approach to asset-pricing, Spelta et al. (2012) applies it to a housing pricing model, Woodford (2013) applies it to a New Keynesian framework, and Adam et al. (2017) applies it to a model of stock market booms. As with the Euler equation approach, agents cannot form model consistent expectations. The two approaches then differ with respect to what agents learn about - their own decisions in the Euler equation approach, and variables exogenous to the agents in the internal rationality approach.

We adopt a general definition of internal rationality used in Adam and Marcet (2011) up to the point where they write: "agents maximize utility under uncertainty, given their constraints and given a consistent set of probability beliefs about payoff-relevant variables that are beyond their control or external". Then rational expectations are both internal and external, the latter meaning model consistent. Within internal rationality beliefs can take the form of a well-defined probability measure over a stochastic process (the fully Bayesian plan), or they can adopt an anticipated utility framework of Kreps (1998). Adam and Marcet (2011) and Adam et al. (2017) adopt the former approach whereas this paper and the other applications mentioned adopt the latter. Cogley and Sargent (2008) compares the two and encouragingly find that anticipated utility can be seen as a good approximation to fully Bayesian optimization (see Branch and McGough (2016) and Deak et al. (2017) for

further discussion).[1]

Recently, Branch and McGough (2010), De Grauwe (2012a), De Grauwe (2012b), Jang and Sacht (2014), and Jang and Sacht (2012), amongst others, have considered models in which agents can be fully rational or boundedly rational using the Euler equation approach. In these models, agents choose their behaviour based on the reinforcement learning approach set out in Young (2004) and pioneered by Brock and Hommes (1997). This approach is widely used in the machine learning literature, and proposes that the different strategies available to agents (e.g. full rationality or bounded rationality) have pay-off levels associated with them. Although adaptation can be slow and there can be a random component of choice, the higher the pay-off from using a strategy in the past, the more likely it will be used in the future.

The "behavioural New Keynesian model", as this approach has come to be known, incorporates heterogeneity and bounded rationality. In addition, it can generate significantly non-normal equilibrium distributions, and in some cases deterministic limit cycles. However, the reliance on the Euler equation approach to bounded rationality is problematic, as agents forecasting their own decisions does not seem to be empirically plausible. Massaro (2013) and Eusepi and Preston (2016) provide a good discussion of this issue. The former paper constructs a model in which agents can be fully rational or internally rational. However, while internally rational agents choose between competing bounded rational predictors in this model, the proportion of fully rational agents is fixed.

Given the foregoing, the present paper studies the dynamic behaviour of a workhorse New Keynesian model in which households and firms can be fully rational or internally rational. Our major contribution is therefore a reworking of the seminal model of Branch and McGough (2010) using the more plausible learning equilibrium concept of internal rationality rather than Euler learning. In section 2, we derive the model with a fixed proportion n of agents fully rational and a fixed proportion $(1-n)$ of agents internally rational, in a similar manner to Massaro (2013). In this model, we establish two propositions. First, a decrease in the proportion of fully rational agents does not destabilise the system if the rational expectations determinacy condition holds. Second, the rational expectations determinacy condition is identical to the stability condition for the model in which all agents are internally rational. In section 3, we extend the model to include predictor selection along the lines of Branch and McGough (2010). In this model, we establish two further propositions. First, the rational expectations determinacy condition ensures local determinacy and stability as the cost of being fully rational becomes infinitely negative. Second, if the model starts from a position of indeterminacy, an increase in the fixed cost of being fully rational can lead to the loss of local stability via a Hopf bifurcation. A *rational route to randomness* follows from

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[1]Sinitskaya and Tesfatsion (2015) introduce forward-looking optimizing agents into an ACE framework. They use a concept that falls within our general definition of internal rationality which they refer to as "constructive rational decision making". Graham (2011) uses the term "individual rationality" to refer to the same general concept.

this, in which the dynamics quickly become chaotic (Brock and Hommes (1997)). Taken as a whole, these results indicate that complex dynamics in the internally rational New Keynesian model are closely associated with monetary policy failures.

These results rely on a monetary policy rule with no persistence in the interest rate. This allows us to demonstrate our main results analytically - which we feel is an important contribution, given that the existing literature relies on numerical results - but implies a potential loss of generality. Finally, therefore, we consider the robustness of our main results to monetary policy rules with persistence in Section 4. We demonstrate analytically that the rational expectations determinacy condition is not identical to the stability condition for the model in which all agents are internally rational when the interest rate is persistent. In particular we find that under internal rationality, the policy space of parameters describing a saddle-path stable interest rate rule is considerably reduced compared with RE. Given our main results, this suggests that the interaction between changes in the monetary policy rule, learning, and bounded rationality is an important area for future research. Section 5 concludes.

2   The Linearised Model with n Fixed

In this section, we derive a version of the workhorse New Keynesian model with a fixed proportion n of agents fully rational and a fixed proportion $(1 - n)$ of agents internally rational. To provide a useful point of reference, section 2.1 sets out the standard linearised New Keynesian model with rational expectations, and the determinacy condition for this model[2]. Section 2.2 then extends this to incorporate heterogeneity, and section 2.3 establishes two propositions: First, that a decrease in the proportion of fully rational agents does not destabilise the system if the rational expectations determinacy condition holds, and second, that the rational expectations determinacy condition is identical to the stability condition for the model in which all agents are internally rational.

2.1   The standard linearised New Keynesian model

2.1.1   Households

Households maximise discounted lifetime utility. Let $C_t(j)$ be consumption and $H_t(j)$ be hours worked of the jth household. The within-period utility function is,

$$U_t(j) = U(C_t(j), H_t(j)) = \log(C_t(j)) - \kappa \frac{H_t(j)^{1+\phi}}{1+\phi}, \tag{1}$$

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[2]The full non-linear model is provided in appendix A.

and the value function of the representative household at time t is,

$$V_t(j) = V_t(B_{t-1}(j)) = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s U(C_{t+s}(j), H_{t+s}(j)) \right]. \tag{2}$$

The individual household's problem at time t is to choose paths for consumption $C_t(j)$, labour supply $H_t(j)$, and holdings of financial assets $B_t(j)$ to maximize $V_t(j)$ given its flow budget constraint in period t,

$$B_t(j) = R_t B_{t-1}(j) + W_t H_t(j) + \Gamma_t(j) - C_t(j), \tag{3}$$

where W_t is the wage rate, Γ_t is distributed profits, and R_t is the ex post real interest rate paid on assets held at the beginning of period t.

The first order conditions are,

$$U_{C,t}(j) = \beta \mathbb{E}_t \left[R_{t+1} U_{C,t+1}(j) \right], \tag{4}$$

$$\frac{U_{L,t}(j)}{U_{C,t}(j)} = W_t. \tag{5}$$

Expressing variables in log deviation from the steady state in lower case, the consumption Euler equation (4) and the household's utility function imply that the linearised consumption decision satisfies a standard linearised Euler equation,

$$c_t(j) = \mathbb{E}_t \left[c_{t+1}(j) - r_{t+1} \right], \tag{6}$$

where c_t is consumption and r_t is the ex post real interest rate. An alternative form of the decision rule, which is useful in deriving the internally rational solution, solves the household budget constraint forward in time and imposes the Euler and transversality conditions. In symmetric equilibrium with zero net financial assets, this yields a consumption function for the representative household of the form,

$$\mathrm{PV}_t(C_t) = \frac{1}{\kappa^{\frac{1}{\phi}}} \mathrm{PV}_t \left(\frac{W_t^{1+\frac{1}{\phi}}}{C_t^{\frac{1}{\phi}}} \right) + \mathrm{PV}_t(\Gamma_t). \tag{7}$$

(7) simply states that the present value of consumption is equal to the present value of total income, where present value (PV_t) is defined in the usual manner. Using exogenous point

expectations, appendices A and B demonstrate that the linearised consumption function corresponding to (7) is given by,

$$\alpha_1 c_t(j) = \alpha_2 w_t + \alpha_3(\omega_{2,t} + r_t) + \alpha_4 \omega_{1,t}, \tag{8}$$

where,

$$\omega_{1,t} = \alpha_5 \mathbb{E}_t w_{t+1} - \alpha_6 \mathbb{E}_t r_{t+1} + \beta \mathbb{E}_t \omega_{1,t+1},$$

$$\omega_{2,t} = (1 - \beta)\gamma_t - r_t + \beta \mathbb{E}_t \omega_{2,t+1},$$

$$\gamma_t = \frac{1}{1 - \alpha} c_t - \frac{\alpha}{1 - \alpha}(w_t + h_t).$$

Consumption is therefore a function of the current wage and profit income, and expected wage and profit income, where w_t is the real wage, h_t is labour supply in hours, and γ_t is profit per household, all in log deviation from the steady state. The parameters and composite parameters for the model are defined in table 1.

Finally, under standard assumptions the linearised optimal labour supply of household j is an intra-temporal decision, given by,

$$h_t^s(j) = \frac{1}{\phi}(w_t - c_t(j)). \tag{9}$$

2.1.2 Firms

Firms in the retail sector uses a homogeneous wholesale good to produce a basket of differentiated goods for aggregate consumption,

$$C_t = \left(\int_0^1 C_t(m)^{(\zeta-1)/\zeta} dm \right)^{\zeta/(\zeta-1)}, \tag{10}$$

where ζ is the elasticity of substitution. For each m, the consumer chooses $C_t(m)$ at a price $P_t(m)$ to maximize (10) given total expenditure $\int_0^1 P_t(m)C_t(m)dm$. This results in a set of consumption demand equations for each differentiated good m with price $P_t(m)$ of the form,

$$C_t(m) = \left(\frac{P_t(m)}{P_t} \right)^{-\zeta} C_t, \tag{11}$$

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Parameter	Definition
α	Elasticity of output with respect to labour input ($\alpha > 0$)
β	Representative household discount rate ($0 < \beta < 1$)
Υ	Fixed cost of rational expectations predictor ($-\infty < \Upsilon < \infty$)
ζ	Elasticity of substitution between consumption goods ($\zeta \geq 0$)
θ_π	Monetary policy rule elasticity of inflation ($\theta_\pi \geq 0$)
θ_y	Monetary policy rule elasticity of output ($\theta_y \geq 0$)
μ	Intensity of choice parameter ($-\infty < \mu < \infty$)
ξ	Calvo probability that firms change price ($0 \leq \xi \leq 1$)
ϕ	Inverse Frisch elasticity of labour supply ($\phi > 0$)
α_1	$\alpha_1 = 1 + \alpha/\phi$
α_2	$\alpha_2 = \alpha(1 - \beta)\left(1 + 1/\phi\right)$
α_3	$\alpha_3 = 1 - \alpha$
α_4	$\alpha_4 = \alpha\beta$
α_5	$\alpha_5 = (1 - \beta)\left(1 + 1/\phi\right)$
α_6	$\alpha_6 = 1 + 1/\phi$
δ	$\delta = (1 - \xi)(1 - \beta\xi)^{-1}$
κ	$\kappa = (1 - \xi)(1 - \beta\xi)(1 + \phi)(\alpha\xi)^{-1}$
ψ	$\psi = (1 - \beta\xi)^{-1}$
A	$A = (\theta_\pi\kappa - \theta_\pi\kappa\psi)(\beta\theta_y)^{-1}$
B	$B = (\theta_y + \theta_\pi\kappa\psi)(\beta\theta_y)^{-1}$
C	$C = (\kappa - \delta\beta\theta_y - \kappa\psi)(\beta\theta_y)^{-1}$
D	$D = (\delta\beta\theta_y + \kappa\psi)(\beta\theta_y)^{-1}$

Table 1: Parameters, parameter definitions, and composite parameter definitions.

implying,

$$Y_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} Y_t, \tag{12}$$

where $P_t = \left[\int_0^1 P_t(m)^{1-\zeta} dm\right]^{\frac{1}{1-\zeta}}$ is the aggregate price index, and C_t and P_t are Dixit-Stigliz aggregates - see Dixit and Stiglitz (1977).

For each variety m the retail good is produced costlessly from wholesale production

$$Y_t(m) = Y_t^W = A_t H_t(m)^\alpha. \tag{13}$$

Following Calvo (1983), we now assume that there is a probability of $1-\xi$ at each period that the price of each retail good m is set optimally to $P_t^0(m)$. If the price is not re-optimized, then it is held fixed.[3] For each retail producer m, given its real marginal cost MC_t, the objective is at time t to choose $\{P_t^0(m)\}$ to maximize discounted profits,

$$\mathbb{E}_t \sum_{k=0}^\infty \xi^k \Lambda_{t,t+k} Y_{t+k}(m) \left[P_t^0(m) - P_{t+k} MC_{t+k}\right], \tag{14}$$

subject to (12), where $\Lambda_{t,t+k} \equiv \beta^k \frac{U_{C,t+k}/P_{t+k}}{U_{C,t}/P_t}$ is the *nominal* stochastic discount factor over the interval $[t, t+k]$. The solution to this is,

$$\mathbb{E}_t \sum_{k=0}^\infty \xi^k \Lambda_{t,t+k} Y_{t+k}(m) \left[P_t^0(m) - \frac{1}{(1-1/\zeta)} P_{t+k} MC_{t+k}\right] = 0, \tag{15}$$

and by the law of large numbers the evolution of the price index is given by,

$$P_{t+1}^{1-\zeta} = \xi P_t^{1-\zeta} + (1-\xi)(P_{t+1}^0)^{1-\zeta}. \tag{16}$$

In a zero-inflation steady state, it is shown in appendices A and B that the linear choice for the optimizing retail firm m given the above can be written as,

$$p_t^o(m) - p_t = \omega_{4,t} - \omega_{3,t}, \tag{17}$$

where $p_t^o(m)$ is the optimal price for firm m, p_t is the aggregate price level, and,

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[3]Thus we can interpret $\frac{1}{1-\xi}$ as the average duration for which prices are left unchanged.

$$\omega_{3,t} = \xi\beta\mathbb{E}_{t+1}\left[(\zeta-1)\pi_{t+1} + \omega_{3,t+1}\right] + (1-\beta\xi)(y_t + u_{C,t}),$$

$$\omega_{4,t} = \xi\beta\mathbb{E}_{t+1}\left[\zeta\pi_{t+1} + \omega_{4,t+1}\right] + (1-\beta\xi)(y_t + u_{C,t} + mc_t + ms_t),$$

where π_t is the aggregate inflation rate, y_t is aggregate output, $u_{C,t}$ is household marginal utility, mc_t is marginal cost, and ms_t is an exogenous supply shock. Substituting $\omega_{3,t}$ and $\omega_{4,t}$ into (17), we have,

$$p_t^o(m) - p_t = \beta\xi\mathbb{E}_t[\pi_{t+1} + p_{t+1}^o(m) - p_{t+1}] + (1-\beta\xi)(mc_t + ms_t). \tag{18}$$

Finally, for the wholesale sector we have,

$$y_t = \alpha h_t^d, \tag{19}$$

$$mc_t = w_t - y_t + h_t^d, \tag{20}$$

where h_t^d is labour demand. Note that labour productivity is assumed to be constant.

2.1.3 Aggregation

Assuming a unit measure of households and retail firms, aggregation entails,

$$c_t(j) = c_t, \tag{21}$$

$$h_t^s(j) = h_t^s, \tag{22}$$

$$p_t^o(m) = p_t^o, \tag{23}$$

$$\xi\pi_t = (1-\xi)(p_t^o - p_t). \tag{24}$$

2.1.4 Equilibrium, the Fisher equation, and the policy rule

Equilibrium in the output and labour markets requires,

$$y_t = c_t, \tag{25}$$

$$h_t^s = h_t^d = h_t. \tag{26}$$

The model is completed with a Fisher equation,

$$r_t = r_{n,t-1} - \pi_t, \tag{27}$$

where $r_{n,t}$ is the nominal interest rate, and a policy rule of the form,

$$r_{n,t} = \theta_\pi \pi_t + \theta_y y_t. \tag{28}$$

In this simple model, we confine our attention to implementable policy rules with no persistence in the interest rate, as this simplifies the analysis considerably. As noted in the introduction, we consider monetary policy rules with persistence in section 4. In addition, we only specify a single shock process (the supply shock ms_t), as we mainly focus on determinacy and stability conditions in the sequel. All variables are expressed in log-deviations from the steady-state, with the linearisation details presented in appendix B.

2.1.5 Reduced form

Imposing the aggregation conditions, we arrive at the familiar linearised New Keynesian Phillips curve,

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi}(mc_t + ms_t), \tag{29}$$

which, on substituting $mc_t = (1+\phi)y_t/\alpha$, gives (31) with κ defined in table 1. We therefore arrive at the workhorse New Keynesian three equation model,

$$y_t = \mathbb{E}_t y_{t+1} - (r_{n,t} - \mathbb{E}_t \pi_{t+1}), \tag{30}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa y_t + \kappa ms_t, \tag{31}$$

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$$r_{n,t} = \theta_\pi \pi_t + \theta_y y_t. \tag{32}$$

Before presenting the determinacy condition, two points about this formulation need to be made. First, there is no lagged output in the demand curve (30), nor lagged inflation in the Phillips curve (31). These can enter through the introduction of external habit in households' utility functions and price indexing, respectively. But we choose to focus on bounded rationality as a persistence mechanism, so both of these features are omitted. Second, even without these persistence terms, the linearisation is only correct about a zero inflation steady state.

2.1.6 Determinacy and stability condition

To find the determinacy and stability condition for the rational expectations model in (30) - (32), we write the model in state space form, setting $ms_t = 0$ and substituting out $r_{n,t}$ from (30) using (32). We then have,

$$\begin{bmatrix} \mathbb{E}_t y_{t+1} \\ \mathbb{E}_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \theta_y + \kappa/\beta & \theta_\pi - 1/\beta \\ -\kappa/\beta & 1/\beta \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}. \tag{33}$$

Denote the trace of the system in (33) by τ, and the determinant by Δ. We then have,

$$\tau = 1 + \theta_y + \kappa/\beta + 1/\beta,$$

$$\Delta = \frac{1 + \theta_y + \kappa\theta_\pi}{\beta}.$$

For stability, we simply require a stable shock process ms_t. For determinacy, we require that both of the eigenvalues of the system in (33) lie outside the unit circle, as both y_t and π_t are jump variables (Blanchard and Kahn (1980)). Necessary and sufficient conditions are (Woodford (2003)),

1. $\Delta > 1$,

2. $1 - \tau + \Delta > 0$,

3. $1 + \tau + \Delta > 0$.

As $\beta < 1$ and $\theta_y + \kappa\theta_\pi > 0$, condition 1 is always satisfied, and the binding condition is condition 2. Substituting in the trace and determinant, we arrive at the familiar condition,

$$\theta_\pi + \left(\frac{1-\beta}{\kappa}\right)\theta_y > 1. \tag{34}$$

2.2 The New Keynesian model with internal rationality

We now extend the basic model set out in section 2.1 to include both internally rational and fully rational households and firms. This allows us to demonstrate our first and second propositions in section 2.3, and forms the basis of the model with reinforcement learning in section 3.

2.2.1 Households

We now distinguish between the consumption of fully rational households, c_t^{RE}, and internally rational households, c_t^{IR}. The consumption of fully rational households is pinned down by the rational expectations Euler equation as before,

$$c_t^{RE} = \mathbb{E}_t\left[c_{t+1}^{RE} - r_{t+1}\right], \tag{35}$$

where we now omit the household index variable for simplicity (and hopefully without causing confusion). With Euler learning, the consumption of bounded rational households would be pinned down by the Euler equation,

$$c_t^{EL} = \mathbb{E}_t^*\left[c_{t+1}^{EL} - r_{t+1}\right],$$

where $\mathbb{E}_t^*$ denotes a bounded rational (backwards looking) expectations operator. Hence households base their consumption decisions on forecasts of the same decision in future periods, which for the reasons given in the introduction we regard as unsatisfactory.

Instead of Euler learning, the consumption function for internally rational households is as follows,

$$\alpha_1 c_t^{IR} = \alpha_2 w_t + \alpha_3(\omega_{2,t} + r_{n,t-1} - \pi_t) + \alpha_4 \omega_{1,t}, \tag{36}$$

with,

$$\omega_{1,t} = \frac{1}{1-\beta}\left[\alpha_5\mathbb{E}_t^*w_{t+1} + \alpha_6\mathbb{E}_{h,t}^*\pi_{t+1}\right] - \alpha_6\left(r_{n,t} + \frac{\beta}{1-\beta}\mathbb{E}_t^*r_{n,t+1}\right),$$

$$\omega_{2,t} = (1-\beta)\gamma_t^{IR} + \beta\mathbb{E}_t^*\gamma_{t+1}^{IR} - \left(r_{n,t-1} + \beta r_{n,t} + \frac{\beta^2}{1-\beta}\mathbb{E}_t^*r_{n,t+1}\right) + \pi_t + \left(\frac{\beta}{1-\beta}\right)\mathbb{E}_{h,t}^*\pi_{t+1},$$

where the budget constraint is iterated forward in time and the Euler and transversality conditions imposed, as in (8) above, but with bounded rational expectations. Hence internally rational households base their consumption decisions on forecasts of the variables exogenous to them - wages, profits, interest rates, and inflation rates.

The internally rational consumption function in (36) is very similar to the rational expectations consumption function in (8), but is somewhat more complicated as we assume that internally rational households do not know that they are identical (as explained in appendices A and B). More importantly, as there are now two types of household, we have to differentiate between the profit flows accruing to internally and fully rational households. In the general case, with a fully specified market for the ownership of firms, an individual household's profit earnings would depend on their entire history of strategy choice over fully rational and internally rational behaviour, leading to a complicated distribution over households. To avoid this - and ensure tractability - Massaro (2013) assumes that profit is distributed equally across households. We take a different approach, and assume that profits accrue to households in proportion to their economic activity, i.e.,

$$\gamma_t^{RE} = \frac{1}{1-\alpha}c_t^{RE} - \frac{\alpha}{1-\alpha}(w_t + h_t^{RE}), \tag{37}$$

$$\gamma_t^{IR} = \frac{1}{1-\alpha}c_t^{IR} - \frac{\alpha}{1-\alpha}(w_t + h_t^{IR}). \tag{38}$$

This assumption is obviously artificial, but arguably less so than assuming equal distribution, and also ensures $\gamma_t = n\gamma_t^{RE} + (1-n)\gamma_t^{IR}$ in each period, where n is the proportion of fully rational agents. Equally importantly, it allows us to derive particularly straightforward analytical expressions of the model's reduced form below.

As in section 2.1.1 above, optimal labour supply is an intra-temporal decision, so we have,

$$h_t^{s,RE} = \frac{1}{\phi}\left(w_t - c_t^{RE}\right), \tag{39}$$

$$h_t^{s,IR} = \frac{1}{\phi}\left(w_t - c_t^{IR}\right), \tag{40}$$

where $h_t^{s,RE}$ is the labour supply of fully rational households, and $h_t^{s,IR}$ is the labour supply of internally rational households.

2.2.2 Firms

Following (18) in section 2.1.2, optimal price setting for the internally rational retail firms is given by,

$$(p_t^o - p_t)^{IR} = \beta\xi\mathbb{E}_t^*[\pi_{t+1} + (p_{t+1}^o - p_{t+1})^{IR}] + (1 - \beta\xi)(mc_t + ms_t). \tag{41}$$

If firms know they are identical, they can use the aggregate relationship, $p_{t+1}^o - p_{t+1} = p_{t+1}^o - p_{t+1} = \frac{\xi}{1-\xi}\pi_{t+1}$, and arrive at,

$$(p_t^o - p_t)^{IR} = \frac{\beta\xi}{1 - \xi}\mathbb{E}_t^*\pi_{t+1} + (1 - \beta\xi)(mc_t + ms_t),$$

as for RE. But we avoid this assumption and use (41). Solving forwards, this yields,

$$(p_t^o - p_t)^{IR} = \mathbb{E}_t^* \sum_{i=0}^{\infty} (\beta\xi)^i[\beta\xi\pi_{t+i+1} + (1 - \beta\xi)(mc_{t+i} + ms_{t+i})]. \tag{42}$$

Note internal rationality for retail firms is somewhat more straightforward than for households, as the rational expectations solution is already in recursive form and there is no retail firm budget constraint.

2.2.3 Aggregation

Without loss in generality (for reasons given in section 2.2.6), suppose that the proportion n of fully rational households in the economy is equal to the proportion of fully rational firms. Assuming a unit measure of households, aggregation entails,

$$nc_t^{RE} + (1 - n)c_t^{IR} = c_t, \tag{43}$$

$$nh_t^{s,RE} + (1 - n)h_t^{s,IR} = h_t^s, \tag{44}$$

$$n(p_t^o - p_t)^{RE} + (1 - n)(p_t^o - p_t)^{IR} = p_t^o - p_t, \tag{45}$$

$$\xi\pi_t = (1 - \xi)(p_t^o - p_t). \tag{46}$$

2.2.4 Equilibrium, the Fisher equation, and the policy rule

The equilibrium conditions, Fisher equation, and the monetary policy rule are exactly the same as in the basic rational expectations model, i.e. (25) - (28).

2.2.5 Expectation formation of internally rational agents

To close the model, we need to specify the manner in which internally rational households and firms form their expectations. To do so, we assume that variables which are local to the agents, in a geographical sense, are observable within the period, whereas variables that are strictly macroeconomic are only observable with a lag. This categorization regarding information about the current state of the economy follows Nimark (2014). He distinguishes between the local information that agents acquire directly through their interactions in markets and statistics that are collected and summarised, usually by governments, and made available to the wider public[4]. The only exception to this is the nominal interest rate, which we assume is observable within the period given the timing structure of New Keynesian models. Given this, we assume a strict form of naive expectations. Thus internally rational household expectations are given by,

$$\mathbb{E}_t^* w_{t+1} = w_t, \tag{47}$$

$$\mathbb{E}_t^* \gamma_{t+1} = \gamma_t, \tag{48}$$

$$\mathbb{E}_t^* r_{n,t+1} = r_{n,t}, \tag{49}$$

$$\mathbb{E}_t^* \pi_{t+1} = \pi_{t-1}. \tag{50}$$

Internally rational households can observe their wage and profit income within the period, and observe aggregate inflation with a lag. Similarly, internally rational firm expectations are given by,

$$\mathbb{E}_{f,t}^* \pi_{t+1} = \pi_{t-1}, \tag{51}$$

$$\mathbb{E}_t^* mc_{t+1} = mc_t. \tag{52}$$

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[4]His paper actually focuses on a third category, information provided by the news media, and allows for imperfect information in the form of noisy signals, issues which go beyond the scope of our paper.

Internally rational firms can observe their own marginal costs within the period, but in a similar manner to internally rational households, can only observe aggregate inflation with a lag. Note that firms observing their real marginal costs within the period - and households observing their real wage and profits within the period - does not imply that firms and households observe the aggregate price level within the period (as this would be inconsistent with the assumption that aggregate inflation is observed with a lag). As noted in the discussion around the derivation of (42) above, we assume that firms do not know that they are identical. In this case, they observe their own price within the period, and therefore their own real marginal costs and real profits, but not the aggregate price level. Similarly, we assume that households observe prices local to them within the period, and therefore their own real wage. However, they do not realise that the law of one price holds, and therefore do not observe the aggregate price level or aggregate inflation. This is reasonable given the considerable data-gathering costs of observing aggregate macroeconomic variables like inflation, as discussed in Nimark (2014). Note, however, that fully rational agents observe all variables within the period. Also note that we retain the Taylor rule (32) and assume that the central bank observes current inflation and output, thus having the same information advantage as rational agents.

2.2.6 Reduced form

Sections 2.2.1 - 2.2.5 fully describe the New Keynesian model with fixed proportions n of fully rational agents and $(1-n)$ of internally rational agents. Deriving the reduced form is relatively straightforward. First, by rearranging the internally rational household consumption function (36) after substituting in the expectations functions (47) - (50), we find that internally rational households choose their level of consumption such that,

$$r_{n,t} = \pi_{t-1}, \tag{53}$$

in each period. Combining (53) with the monetary policy rule (28), we see that,

$$y_t = -\left(\frac{\theta_\pi}{\theta_y}\right)\pi_t + \left(\frac{1}{\theta_y}\right)\pi_{t-1}, \tag{54}$$

which greatly simplifies the analysis, as we will not need to track output as a separate state variable. In fact, as (54) means that we do not have to separately track the consumption levels of fully rational and internally rational households in the state space form, it is this result that allows us to derive analytical stability conditions in the sequel. Also note that (54) means that the proportion of fully rational households does not affect the equation of

motion for y_t, which allows us to assume that the proportion of fully rational households is equal to the proportion of fully rational firms without loss of generality.

Using the aggregation conditions (45) and (46), and the price setting conditions (41) and (42), we can derive the reduced form New Keynesian Phillips curve with fully rational and internally rational firms,

$$\pi_t = n(\beta \mathbb{E}_t \pi_{t+1} + \kappa y_t) + (1-n)(\delta \beta \pi_{t-1} + \kappa \psi y_t), \tag{55}$$

where the shocks process ms_t is set equal to zero, $mc_t = y_t(1+\phi)/\alpha$, and the composite parameters κ, δ, and ψ are defined in table 1. Finally, by substituting the equation of motion for output (54) into the New Keynesian Phillips curve (55) and rearranging, we arrive at the reduced form model,

$$\mathbb{E}_t \pi_{t+1} = \left(A + \frac{B}{n}\right) \pi_t - \left(C + \frac{D}{n}\right) \pi_{t-1}, \tag{56}$$

where the composite parameters are defined in table 1.

2.3 State space form and stability

The New Keynesian model with fixed proportions n of fully rational agents and $(1-n)$ of internally rational agents, has a reduced form (56) described by a second order forward looking difference equation in inflation. Define the auxiliary variable $z_t = \pi_{t-1}$. Then the state space form of our model is given by,

$$\begin{bmatrix} \mathbb{E}_t \pi_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} A + B/n & -(C + D/n) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \pi_t \\ z_t \end{bmatrix}, \tag{57}$$

where π_t is a jump variable and z_t is a pre-determined variable. We are now in a position to demonstrate our first and second propositions:

Proposition 1: Determinacy and stability in the model described by (57) requires one eigenvalue inside the unit circle and one eigenvalue outside the unit circle (Blanchard and Kahn (1980)). If the monetary policy rule is such that the condition in (34) holds, then a decrease in n does not lead to a loss of determinacy or stability.

Proof: Denote the trace of the system in (57) by $\tau = A + B/n$ and the determinant by $\Delta = C + D/n$. As τ is positive, necessary and sufficient conditions for determinacy and stability in the model described by (57) are,

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1. $\tau - \Delta > 1$,

2. $\tau + \Delta > -1$.

As Δ is positive, condition 1 is the binding condition. Now, suppose that $n = 1$ and condition 1 is satisfied, so the model is determinate and stable. A necessary condition for a decrease in n to render the model indeterminant or unstable is then,

$$\frac{d}{dn}(\tau - \Delta) > 0, \tag{58}$$

as $\tau - \Delta$ would have to pass through unity from above for the model to pass from determinacy and stability to indeterminacy or instability as n decreases.

To prove proposition 1, we first demonstrate that (58) does not hold if (34) holds. From (57), we have,

$$\frac{d}{dn}(\tau - \Delta) = (D - B)n^{-2}.$$

Taking advantage of the definitions of B and D in table 1, after some rearranging we have,

$$\frac{d}{dn}(\tau - \Delta) = \left[\frac{\theta_y(\delta\beta - 1) - \kappa\psi(\theta_\pi - 1)}{\beta\theta_y}\right] n^{-2}. \tag{59}$$

At this point, we assume that (34) holds as a strict inequality. Specifically, we assume that the parameterisation satisfies,

$$\theta_\pi = 1 - \left(\frac{1 - \beta}{\kappa}\theta_y\right) + \epsilon, \tag{60}$$

where ϵ is an arbitrarily small but positive constant. Substituting (60) into (59) and rearranging, we find that,

$$\frac{d}{dn}(\tau - \Delta) = (D - B)n^{-2} = -\left[\frac{\kappa\psi\epsilon}{\beta\theta_y}\right] n^{-2}. \tag{61}$$

At $n = 1$, we therefore have,

$$\frac{d}{dn}(\tau - \Delta)\bigg|_{n=1} = D - B = -\frac{\kappa\psi\epsilon}{\beta\theta_y}.$$
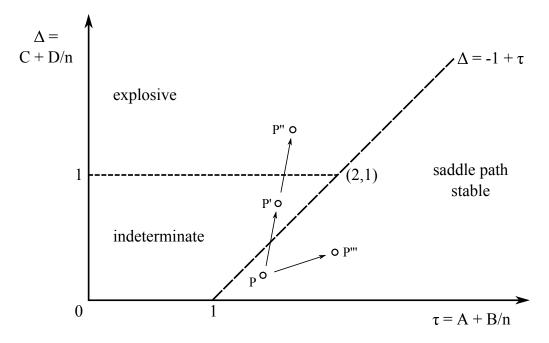
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Figure 1: Graphical illustration of proposition 1, showing a stability plot in the trace τ and determinant Δ for a second order difference equation when $\tau > 0$ and $\Delta > 0$. At point P, which lies within the saddle path stable region (i.e. it satisfies $\tau - \Delta > 1$), a decrease in n moves the model to P' or P'' if $\frac{\partial \tau}{\partial(-n)} < \frac{\partial \Delta}{\partial(-n)}$, or to P''' if $\frac{\partial \tau}{\partial(-n)} > \frac{\partial \Delta}{\partial(-n)}$. Thus a decrease in n moves the model to P' or P'' if $\frac{\partial \tau}{\partial n} > \frac{\partial \Delta}{\partial n}$, or to P''' if $\frac{\partial \tau}{\partial n} < \frac{\partial \Delta}{\partial n}$, as in (58).

As κ and ψ are both positive (see table 1), (61) implies that (58) cannot hold if (34) holds. Finally, as (61) implies $D < B$ when (34) holds, $(\tau - \Delta) \to \infty$ as $n \to 0$ from $n = 1$. Therefore, if the condition in (34) holds, a decrease in n does not lead to a loss of determinacy or stability in the model described by (57). This illustrated graphically in figure 1, which shows the standard stability plot in the trace and determinant for a second order difference equation when both the trace and determinant are positive (see e.g. Hamilton (1994), chapter 1). ∎

Proposition 2: The rational expectations determinacy condition in (34) is identical to the stability condition for (56) when $n = 0$, i.e. when all agents are internally rational.

Proof: From (55), the reduced form model for the case in which $n = 0$ is given by,

$$\pi_t = \delta\beta\pi_{t-1} + \kappa\psi y_t. \tag{62}$$

Substituting out y_t using (54), we have the reduced form,

$$\pi_t = \left(\frac{\delta\beta\theta_y + \psi\kappa}{\theta_y + \psi\kappa\theta_\pi}\right)\pi_{t-1}, \tag{63}$$
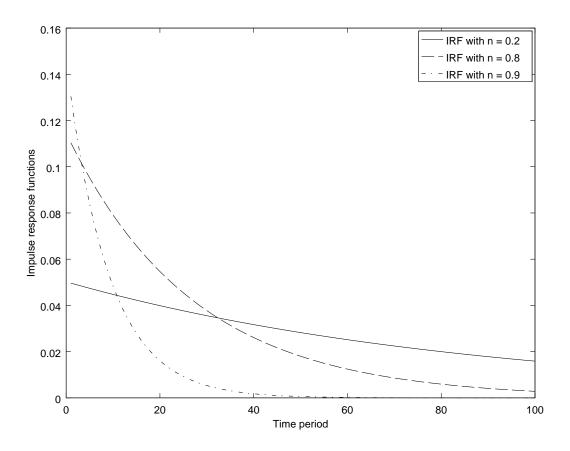
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Figure 2: Impulse response functions of inflation to a positive marginal cost shock, for the model with n fixed, for three different values of n. The remaining parameter values are $\phi = 1$, $\alpha = 0.5$, $\beta = 0.99$, $\xi = 0.7$, $\theta_\pi = 1$, $\theta_y = 0.8$.

which can also be found by rearranging (56). The model in (63) is stable when the coefficient on π_{t-1} is less than one in absolute value. As the coefficient will be positive given the parameter definitions in table 1, this is the case when,

$$\frac{\delta\beta\theta_y + \psi\kappa}{\theta_y + \psi\kappa\theta_\pi} < 1. \tag{64}$$

Rearranging, and taking advantage of the parameter definitions, we arrive at a stability condition identical to (34). Therefore, the rational expectations determinacy condition in (34) is identical to the stability condition for (56) when all agents are internally rational. ∎

2.4 Dynamic behaviour of the model with n fixed

Section 2.3 demonstrates that a decrease in n does not lead to a loss of determinacy and stability if the rational expectations determinacy condition for the monetary rule holds, and

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that the rational expectations determinacy condition is identical to the stability condition for the model in which all agents are internally rational. However, the dynamics of the model will vary with n, as the magnitude of the eigenvalues will change as n changes.

This is illustrated in figure 2, which plots impulse response functions of inflation in response to an ms shock with $n = 0.2$, $n = 0.8$, and $n = 0.9$. The remaining parameter values are $\phi = 1$, $\alpha = 0.5$, $\beta = 0.99$, $\xi = 0.7$, $\theta_\pi = 1$, $\theta_y = 0.8$, such that the condition in (34) holds, and the marginal cost shock has no persistence. Although the determinacy and stability properties of the model are unaffected by a reduction in n, given that (34) holds, the response of the model to shocks becomes increasingly persistent as the proportion of fully rational agents decreases.

3   The Model with n Variable

In section 2, we demonstrated our first and second propositions. In this section, we extend the analysis to allow n to vary. Following the literature, we assume that n varies according to a reinforcement learning mechanism laid out in section 3.1. We then derive the reduced form in section 3.2, and consider the state space form and local stability conditions in section 3.3. We establish our third and fourth propositions in this section. First, the rational expectations determinacy condition ensures local determinacy and stability as the cost of being fully rational becomes infinitely negative. Second, if the model starts from a position of indeterminacy, an increase in the fixed cost of being fully rational can lead to the loss of local stability via a Hopf bifurcation. This Hopf bifurcation appears to be super-critical, giving rise to stable limit cycles. As the speed at which agents learn increases, a *rational route to randomness* appears to follow, which we explore with numerical methods in section 3.4.

3.1   Reinforcement learning and predictor fitness

We now extend the model to allow n to vary with the perceived relative forecasting strength of the fully rational and internally rational predictors. Following Branch and McGough (2010) and the literature described in the introduction, denote the fitness of the rational expectations predictor by v_t^{RE}, and the fitness of the internally rational (naive expectations) predictor by v_t^{IR}. Then the proportion of fully rational agents at any point in time is given by,

$$n_t = \frac{\exp[\mu v_t^{RE}]}{\exp[\mu v_t^{RE}] + \exp[\mu v_t^{IR}]}. \tag{65}$$

The parameter μ in (65) is referred to as the intensity of choice parameter, as a higher μ increases the rate at which agents choose strategies with a high fitness level. In this sense, μ governs the speed of learning.

Denote the perceived mean squared error of the internally rational predictor by Φ_t, and define it as follows,

$$\Phi_t = (\pi_t - \mathbb{E}_{t-1}^*[\pi_t])^2 = (\pi_t - \pi_{t-2})^2. \tag{66}$$

If - as we will do in the sequel - we consider a deterministic economy, the mean squared error of the fully rational predictor is zero, as rational expectations is equivalent to perfect foresight in this context. Finally, and in accordance with the literature, we define the fitness measures as follows,

$$v_t^{RE} = -\Upsilon, \tag{67}$$

$$v_t^{IR} = -\Phi_t, \tag{68}$$

where Υ is a fixed cost of using the fully rational predictor. The internally rational predictor is then fit relative to the fully rational predictor when the mean squared error falls below the fixed cost of being fully rational.

3.2   Reduced form

Sections 2.2.1 - 2.2.5, extended to allow n to vary with the equations set out in section 3.1, fully describe the New Keynesian model with fully rational and internally rational agents, where the proportion n of fully rational agents varies over time according to the perceived relative fitness of the two strategies. By substituting (66) - (68) into (65), we find that,

$$n_t = \frac{\exp[-\mu\Upsilon]}{\exp[-\mu\Upsilon] + \exp[-\mu\Phi_t]},$$

$$\Rightarrow n_t = \frac{\exp[-\mu\Upsilon]}{\exp[-\mu\Upsilon] + \exp[-\mu(\pi_t - \pi_{t-2})^2]}. \tag{69}$$

Thus, as the squared difference in inflation, which corresponds to the perceived mean squared error of the internally rational predictor, falls below the fixed cost, Υ, of being fully rational, agents move towards being internally rational and n falls. The speed of this process is determined by the intensity parameter μ. Note that (69) implies,

$$n_t^{-1} = 1 + \exp[-\mu((\pi_t - \pi_{t-2})^2 - \Upsilon)]. \tag{70}$$

As we have changed nothing in the original model other than allowing n to vary, the original reduced form (56) becomes,

$$\mathbb{E}_t \pi_{t+1} = \left(A + \frac{B}{n_t}\right)\pi_t - \left(C + \frac{D}{n_t}\right)\pi_{t-1}, \tag{71}$$

with A, B, C, and D defined as before. Finally, substituting (70) into (71), we arrive at the reduced form New Keynesian model with n variable,

$$\mathbb{E}_t \pi_{t+1} = \left[A + B\left(1 + e^{-\mu((\pi_t - \pi_{t-2})^2 - \Upsilon)}\right)\right]\pi_t - \left[C + D\left(1 + e^{-\mu((\pi_t - \pi_{t-2})^2 - \Upsilon)}\right)\right]\pi_{t-1}. \tag{72}$$

The reduced form (72) is a highly non-linear third order difference equation. The state space form, which we turn to next, simplifies the expression somewhat and allows analytical stability conditions to be derived.

3.3   State space form and stability

As before, define the auxiliary variable $z_t = \pi_{t-1}$, and define a second auxiliary variable $zz_t = z_{t-1} = \pi_{t-2}$. Then the state space form of the model in (72) is given by,

$$\begin{bmatrix} \mathbb{E}_t \pi_{t+1} \\ z_{t+1} \\ zz_{t+1} \end{bmatrix} = \begin{bmatrix} A + B\left(1 + e^{-\mu((\pi_t - zz_t)^2 - \Upsilon)}\right) & -\left[C + D\left(1 + e^{-\mu((\pi_t - zz_t)^2 - \Upsilon)}\right)\right] & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \pi_t \\ z_t \\ zz_t \end{bmatrix},$$

where π_t is a jump variable and z_t and zz_t are pre-determined variables. In the steady state, $\pi_t = z_t = zz_t = 0$, and $n_t^{-1} = (1 + e^{\mu \Upsilon})^{-1}$. Therefore, the Jacobian matrix J evaluated at the steady state is as follows:

$$J|_{\pi_t = z_t = zz_t = 0} = \begin{bmatrix} A + B\left(1 + e^{\mu \Upsilon}\right) & -\left[C + D\left(1 + e^{\mu \Upsilon}\right)\right] & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{73}$$

For local determinacy and stability we require two eigenvalues of the Jacobian matrix (73) inside the unit circle, and one eigenvalue outside. Local indeterminacy occurs when all eigenvalues of the Jacobian matrix (73) are inside the unit circle. If a pair of eigenvalues are complex conjugates, as they pass through the unit circle a Hopf (or Neimark-Sacker) bifurcation occurs (see e.g. Hommes (2013), chapter 3). Proposition 3 considers the case of local determinacy and stability, and proposition 4 considers the case of local indeterminacy and Hopf bifurcation.

Proposition 3: Local determinacy and stability in the model described by (72) requires two eigenvalues inside the unit circle and one eigenvalue outside the unit circle (Blanchard and Kahn (1980)). As the cost of being fully rational becomes infinitely negative, the local determinacy and stability condition is equal to the rational expectations determinacy condition in (34).

Proof: In the steady state, $\pi_t = z_t = zz_t = 0$, and $n_t = \left(1 + e^{\mu \Upsilon}\right)^{-1}$. Therefore, as $\Upsilon \to -\infty$, the steady state proportion of fully rational agents goes to unity. In this case, the condition in (34) ensures local determinacy and stability. ∎

Proposition 4: Local indeterminacy and stability in the model described by (72) requires all eigenvalues inside the unit circle. In this case, an increase in Υ can lead to a loss of local stability via a Hopf bifurcation.

Proof: First, write our system as $x_{t+1} = F(x_t, \varphi)$, $x_t \in \mathbb{R}^n$, and $\varphi \in \mathbb{R}$ is a parameter. Following Iooss et al. (1981) and Gabisch and Lorenz (1987), we have the following theorem:

> **Hopf:** Let the mapping $x_{t+1} = F(x_t, \varphi)$, $x_t \in \mathbb{R}^n$, $\varphi \in \mathbb{R}$, have a fixed point at the origin. If there is a φ_0 such that the Jacobian matrix evaluated at the origin has a pair of complex conjugate eigenvalues $\lambda_{1,2}$ which lie on the unit circle, while the remainder of its spectrum lies at a non-zero distance from the unit circle, and the Hopf transversality condition holds, i.e.
>
> $$\frac{d(\mathrm{mod}\lambda(\varphi))}{d\varphi} > 0,$$
>
> then if $\lambda^n(\varphi_0) \neq \pm 1$ for $n = 1, 2, 3, 4$, there is an invariant closed curve bifurcating from $\varphi = \varphi_0$. So, as a parameter φ is varied, a stable fixed point loses stability as a pair of complex conjugate eigenvalues crosses the unit circle[5].

Denote the trace of the Jacobian in (73) by $\tau = A + B(1 + e^{\mu \Upsilon})$. By inspection, the matrix is non-invertible, so the determinant $\Delta = 0$, and at least one eigenvalue is equal to zero. In fact, the eigenvalues of (73) are given by,

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[5]This wording largely follows Iooss et al. (1981), although it has been altered slightly to fit with the notation of the present paper. Gabisch and Lorenz (1987: 161) considers the case of $x_{t+1} = F(x_t, \varphi)$, $x_t \in \mathbb{R}^2$.
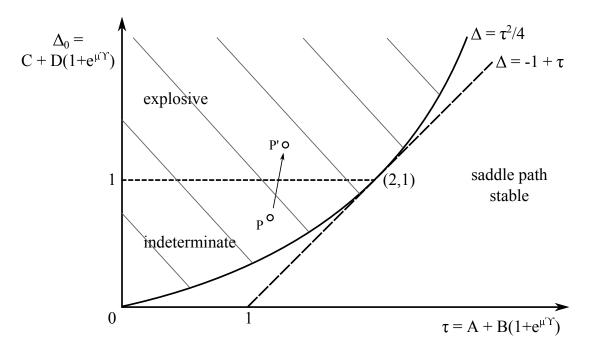
Figure 3: Graphical illustration of proposition 4, showing a stability plot in the trace τ and pseudo-determinant Δ_0 for the model in (72). Note this looks exactly the same as the stability plot in figure 1, as the linearised model is effectively a second order difference equation in π_t and z_t, but we have now shaded the region of complex conjugate eigenvalues with grey lines. As the model moves from points P to P$'$, as Υ is increased, a Hopf bifurcation takes place.

$$\lambda_{1,2} = \tau/2 \pm \sqrt{\tau^2/4 - \Delta_0}, \ \lambda_3 = 0,$$

where $\Delta_0 = C + D(1 + e^{\mu\Upsilon})$ is the pseudo-determinant of (73), i.e. the product of the non-zero eigenvalues. When $\Delta_0 > \tau^2/4$ and the non-zero eigenvalues are complex conjugate, let $\lambda_{1,2} = \beta_1 \pm \beta_2 i$, where $\beta_1 = \tau/2$ and $\beta_2 = \sqrt{\Delta_0 - \tau^2/4}$. The modulus of the complex conjugate eigenvalues is then:

$$\text{mod}(\lambda_{1,2}) = \sqrt{\beta_1^2 + \beta_2^2},$$

from which it follows that $\text{mod}(\lambda_{1,2}) = \sqrt{\Delta_0}$. As the remaining eigenvalue $\lambda_3 = 0$, we require Δ_0 to equal unity for a Hopf bifurcation to occur.

Now, as $\Delta_0 = C + D(1 + e^{\mu\Upsilon})$, $\text{mod}(\lambda_{1,2}) = 1$ when,

$$C + D(1 + e^{\mu\Upsilon}) = 1. \tag{74}$$

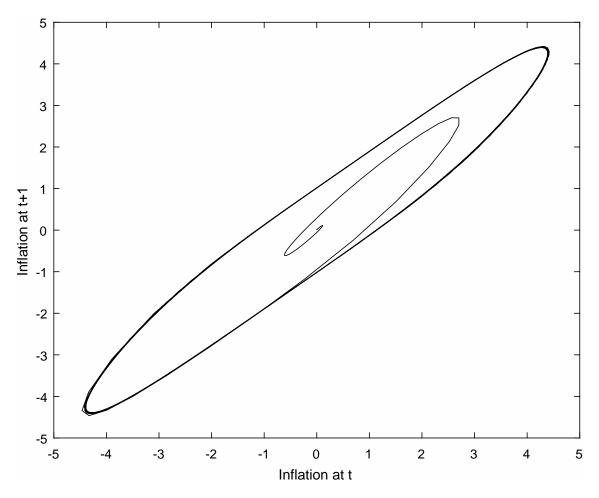Taking advantage of the parameter definitions in table 1 and re-arranging, this condition reduces to,

Figure 4: Phase plot of inflation with n variable, illustrating a stable limit cycle. The parameter values are $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.9$, $\theta_y = 1$, $\mu = 0.1$, $\Upsilon = -10$.

$$\mu\Upsilon = \ln\left[\frac{\beta\theta_y - \kappa}{\delta\beta\theta_y + \kappa\psi}\right]. \tag{75}$$

As the right hand side of (75) is finite, as $\Upsilon \to \infty$, Δ_0 will pass through unity from below if it starts from a parameterisation in which $\Delta_0 < 1$. Precisely, we have,

$$\frac{d(\mathrm{mod}\lambda_{2,3}(\Upsilon))}{d\Upsilon} = \frac{d\sqrt{\Delta_0}}{d\Upsilon} > 0.$$

Therefore, if the non-zero eigenvalues are complex conjugate as Δ_0 passes through unity, then the model undergoes a Hopf bifurcation. This is illustrated graphically in figure 3, which presents the same stability plot as in figure 1, as the model in (72) linearised is effectively a second order difference equation, but with the region of complex conjugate eigenvalues highlighted. ∎
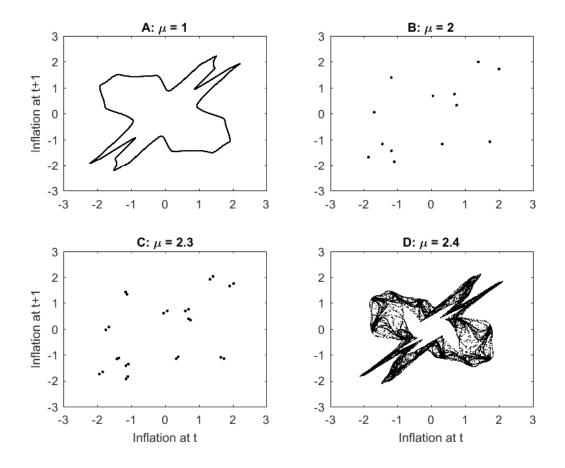
Figure 5: Simulated trajectories for various values of μ, illustrating the *rational route to randomness*. The remaining parameter values are $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.3$, $\theta_y = 1$, $\Upsilon = 0.1$.

3.4 Rational route to randomness

Section 3.3 demonstrates that an increase in the fixed cost of being fully rational can lead to the loss of local stability via a Hopf bifurcation if the model starts from a position of local indeterminacy. The existence of limit cycles therefore depends on the underlying parameterisation, and in particular the choice of Υ. Figure 4 presents a plot of a single simulated trajectory of the model in (72), numerically demonstrating the existence of a stable limit cycle in the inflation rate. The underlying parameterisation is the same parameterisation used in the rest of the paper, and is a fairly standard prior for the basic New Keynesian model.

The existence of a Hopf bifurcation and stable limit cycles indicate the possibility of a *rational route to randomness*. Following Brock and Hommes (1997), this is a bifurcation route to instability, cycles, and chaos as the intensity of choice parameter (speed of learning) increases. Mathematically, this route to chaos is associated with the emergence of a homoclinic loop, as the equilibrium becomes a saddle-focus with one stable and two unstable eigenvalues after the Hopf bifurcation, associated with a one dimensional stable manifold
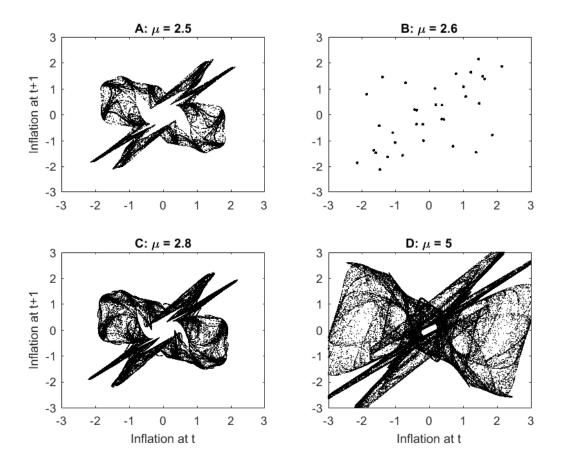
Figure 6: Simulated trajectories for various values of μ, illustrating the *rational route to randomness*. The remaining parameter values are $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.3$, $\theta_y = 1$, $\Upsilon = 0.1$.

and a two dimensional unstable manifold, respectively.

Retaining the same underlying parameterisation, and setting $\Upsilon = 0.1$, figures 5 and 6 plot several trajectories as μ increases. As is evident from the plots, the stable limit cycle quickly loses its smoothness as μ increases, and then varies between periodic attractors and strange attractors. This evolution is not dissimilar to the evolution in the Henon-like map discussed in Gonchenko et al. (2014), in which simple Shilnikov scenarios in three dimensional maps are discussed in some detail. Finally, figure 7 plots a bifurcation diagram as μ is increased, and the simulated largest Lyapunov exponents for the model over the same range of μ. Both panels in figure 7 are plotted using the software *E&F Chaos* - see Diks et al. (2008).

The bifurcation diagram is constructed by simulating the model for T periods, k times for k different values of μ equally spaced between 1 and 3. For each of the k values of μ, this yields T different simulated values of inflation which are plotted on the vertical axis (although a long burn-in period for each simulation ensures that the simulated values of inflation constitute the fixed point(s) for the system). The Lyapunov exponents are simulated, and measure the average rate of separation of a trajectory before and after a small perturbation. As a positive Lyapunov exponent is an important indicator of chaos, we can
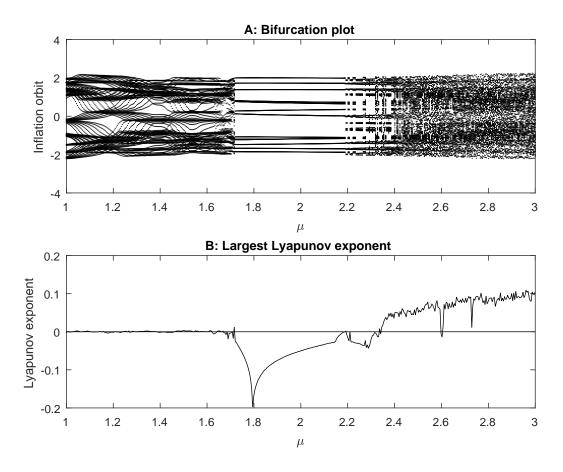
Figure 7: Panel A: Bifurcation plot of the orbit of inflation against μ. Panel B: Largest Lyapunov exponent against μ. The parameter values are $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.3$, $\theta_y = 1$, $\Upsilon = 0.1$.

state with some confidence that the model in (72) displays a *rational route to randomness.*

Finally, although an analytical proof of the existence of a homoclinic loop is not forthcoming, there exist parameterisations in which near-homoclinic trajectories are particularly apparent in numerical simulation. Figure 8 presents an example of this, and plots the phase diagram in two dimensions and three dimensions. The plotted trajectory starts very close to the steady state, and spirals away from it across the unstable manifold. Throughout this process the proportion of internally rational agents fluctuates with the fluctuations in inflation. As the trajectory gets further from the steady state, it becomes increasingly difficult to forecast, leading to agents shifting away from the internally rational predictor towards the rational expectations predictor for longer periods of time. At this point the model stabilises, and re-approaches the steady state down the stable manifold. The corresponding time series of inflation and n, the proportion of rational firms, are plotted in figure 9, which illustrates this dynamic from a different perspective. This dynamic is common to models of this form, in which agents shift between destabilising bounded rational predictors and stabilising fully rational predictors, following Brock and Hommes (1997).
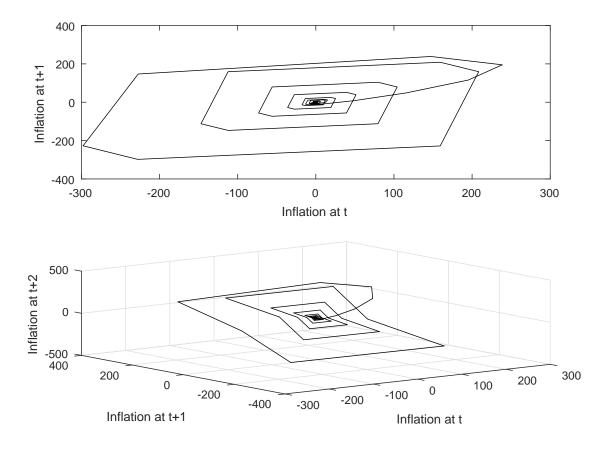
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Figure 8: Trajectories in two and three dimensions, respectively, of the first 103 iterations of the model in which $\mu = 1$ and $\Upsilon = 0$. The remaining parameterisation is $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.3$, $\theta_y = 1$.

4   Robustness to Monetary Policy Rules with Persistence

In sections 2 and 3, we demonstrate four propositions. First, with n fixed, we demonstrate that a decrease in the proportion of fully rational agents does not destabilise the system if the rational expectations determinacy conditions holds. Second, the rational expectations determinacy condition is identical to the stability condition for the model in which all agents are internally rational in this model. Third, with n variable, that the rational expectations determinacy condition ensures local determinacy and stability as the cost of being fully rational becomes infinitely negative. Fourth, if this model starts from a position of indeterminacy, an increase in the fixed cost of being fully rational can lead to the loss of local stability via a Hopf bifurcation.

These are analytical results, which have been achieved by imposing some relatively restrictive assumptions. Arguably the most important of these is the lack of persistence is the policy rule (28). In this final section, we relax this assumption in order to check the robustness of our results in sections 2 and 3, and suggest a fruitful line of future enquiry.

Figure 9: Trajectories, respectively, of the first iterations 10 to 140 of the model in which $\mu = 1$ and $\Upsilon = 0$. The remaining parameterisation is $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.3$, $\theta_y = 1$.

Specifically, we generalise the monetary policy rule to the standard rule with persistence,

$$r_{n,t} = \rho_r r_{n,t-1} + (1 - \rho_r)(\theta_\pi \pi_t + \theta_y y_t), \tag{76}$$

where $\rho_r \in (0, 1]$.

For the case of pure rational expectations, with n fixed and equal to 1, the policy space for the rule in (76) is given by,

$$\theta_\pi + \frac{1 - \beta}{\kappa} \theta_y > 1 - \rho_r, \tag{77}$$

which is a result obtained in Woodford (2003), appendix C.

For the case of pure internal rationality, with n fixed and equal to 0, using the monetary policy rule (76) leads to a second order generalisation of the model in (63),

$$\pi_t = \left[\frac{\psi\kappa + (1-\rho_r)\delta\beta\theta_y}{(1-\rho_r)(\theta_y + \psi\kappa\theta_\pi)}\right]\pi_{t-1} - \left[\frac{\rho_r}{(1-\rho_r)(\theta_y + \psi\kappa\theta_\pi)}\right]\pi_{t-2}. \tag{78}$$

Using $z_t = \pi_{t-1}$ as before, we can re-write the model in (78) as,

$$\begin{bmatrix} \pi_t \\ z_t \end{bmatrix} = \begin{bmatrix} \dfrac{\psi\kappa + (1-\rho_r)\delta\beta\theta_y}{(1-\rho_r)(\theta_y + \psi\kappa\theta_\pi)} & -\dfrac{\rho_r}{(1-\rho_r)(\theta_y + \psi\kappa\theta_\pi)} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ z_{t-1} \end{bmatrix}. \tag{79}$$

Denoting the trace of the model in (79) by τ and the determinant by Δ, necessary and sufficient conditions for stability are (also see Section 2.1.6),

1. $\Delta < 1$,

2. $1 - \tau + \Delta > 0$,

3. $1 + \tau + \Delta > 0$.

As τ and Δ are both positive the third condition is not binding, and for $\rho_r < 1$ condition 3 yields the familiar condition $\theta_\pi + \frac{1-\beta}{\kappa}\theta_y > 1$. But condition 1 adds a further restriction on persistence in the monetary policy rule, given by,

$$\rho_r < \frac{\theta_\pi\psi\kappa}{\theta_y + \psi\kappa(1 + \theta_\pi)}. \tag{80}$$

Thus we have our fifth and final result:

Proposition 5: With persistence in the interest rate, the policy space (θ_π, θ_y) under rational expectations is increased to $\theta_\pi + \frac{1-\beta}{\kappa}\theta_y > 1 - \rho_r$. Under internal rationality the policy space remains as $\theta_\pi + \frac{1-\beta}{\kappa}\theta_y > 1$ and persistence is constrained by (80).

By considering the limiting case of $\theta_y = 0$, one can see that (80) restricts the stability region of the model with $n = 0$ quite substantially. This is further illustrated by considering the limiting case of $\rho_r = 1$. By re-parameterising the rule as,

$$r_{n,t} = r_{n,t-1} + \alpha_\pi\pi_t + \alpha_y y_t, \tag{81}$$

then the case $\alpha_y = 0$ gives $\Delta r_{n,t} = \theta_\pi\Delta p_t$, where $\pi_t = p_t - p_{t-1}$ and p_t is the price level. Thus $r_{n,t} = \theta_\pi p_t$, and (81) is a *price level rule*. Putting $\alpha_\pi = (1-\rho_r)\theta_\pi$ and $\alpha_y = (1-\rho_r)\theta_y$ into the previous result and letting $\rho_r \to 1$, the policy space (α_π, α_y) under rational expectations is $\alpha_\pi + \frac{1-\beta}{\kappa}\alpha_y > 0$ and the policy space under internal rationality is $\alpha_\pi + \frac{(1-\beta\xi)}{\kappa}\alpha_y > 1$.

Hence under rational expectations and $\rho_r = 1$, at least one slightly positive feedback from inflation and output is necessary and sufficient to result in saddle-path stability. Under internal rationality and $\rho_r = 1$, the policy space is considerably reduced for plausible values of ξ. Thus proposition 5 qualifies proposition 2, and implies that the latter is not robust to changes in the monetary policy rule. In turn, this suggests that proposition 1 and proposition 3 may be qualified when alternative monetary policy rules are considered. Although we do not consider the general model with n fixed or the model with n variable when the monetary policy rule incorporates interest persistence, as these cases induce a considerable increase in complexity, proposition 5 suggests that a fruitful line of future enquiry would be to examine the interplay between bounded rationality, learning, and the choice of monetary policy rule.

5   Concluding Remarks

This paper constructs and explores the monetary policy consequences of the workhorse New Keynesian model with internal rationality and heterogeneous agents. First, we derive the model with a fixed proportion n of agents fully rational and a fixed proportion $(1 - n)$ of agents internally rational, in a similar manner to Massaro (2013). We then extend the model to include predictor selection along the lines of Branch and McGough (2010).

In the model with n fixed, we establish two propositions. First, a decrease in the proportion of fully rational agents does not destabilise the system if the rational expectations determinacy condition holds. Second, the rational expectations determinacy condition is identical to the stability condition for the model in which all agents are internally rational. In the model with n variable, we establish two further propositions. First, the rational expectations determinacy condition ensures local determinacy and stability as the cost of being fully rational becomes infinitely negative. Second, if the model starts from a position of indeterminacy, an increase in the fixed cost of being fully rational can lead to the loss of local stability via a Hopf bifurcation.

After the Hopf bifurcation, a *rational route to randomness* appears exists as the speed of learning increases. The model readily exhibits chaotic behaviour over wide ranges of the parameter space, which we have illustrated numerically. Taken together, these results indicate that complex dynamics in the internally rational New Keynesian model are closely associated with monetary policy failures. Finally, we consider the robustness of our results to alternative monetary policy rules, and demonstrate that the rational expectations determinacy condition is not identical to the stability condition for the model in which all agents are internally rational for a monetary policy rule with interest rate persistence.

The major contributions of the present paper is the reworking of the model presented in Branch and McGough (2010) using the more plausible learning equilibrium concept of internal rationality rather than Euler learning. The result is a model similar to that presented in Massaro (2013), with strategy switching between internal rationality and full rationality.

Moreover, we demonstrate our main results analytically, which we feel is particularly useful given the reliance of the existing literature on numerical results. Our results suggest that a fruitful line of enquiry would be to examine the interplay between bounded rationality, learning, and the choice of monetary policy rule, which we leave to future work.

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Appendices

A Non-Linear Foundations of the New Keynesian Model

In this appendix we make explicit the micro-foundations of the New Keynesian model. We proceed from rational expectations to internal rationality in stages.

A.1 The Rational Expectations Model

A.1.1 Households

Household j chooses between work and leisure. Let $C_t(j)$ be consumption and $H_t(j)$ be hours worked. The within-period utility function is,

$$U_t(j) = U(C_t(j), H_t(j)) = \log(C_t(j)) - \kappa \frac{H_t(j)^{1+\phi}}{1+\phi}, \tag{A.1}$$

and the value function of the representative household at time t is,

$$V_t(j) = V_t(B_{t-1}(j)) = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s U(C_{t+s}(j), H_{t+s}(j)) \right]. \tag{A.2}$$

The household's problem at time t is to choose paths for consumption $\{C_t(j)\}$, labour supply $\{H_t(j)\}$, and holdings of financial savings to maximize $V_t(j)$ given by (A.2) given its flow budget constraint in period t,

$$B_t(j) = R_t B_{t-1}(j) + W_t H_t(j) + \Gamma_t - C_t(j), \tag{A.3}$$

where $B_t(j)$ is the given net stock of financial assets at the end of period t, W_t is the wage rate and R_t is the ex post real interest rate paid on assets held at the beginning of period t. The ex post real interest rate is given by,

$$R_t = \frac{R_{n,t-1}}{\Pi_t}, \tag{A.4}$$

where $R_{n,t}$ and Π_t are the nominal interest and inflation rates respectively and Γ_t are profits from wholesale and retail firms owned by households. W_t, R_t, and Γ_t are all exogenous to household j. As usual all variables are expressed in real terms relative to the price of final output.

The first order conditions are,

$$U_{C,t}(j) = \beta \mathbb{E}_t \left[R_{t+1} U_{C,t+1}(j) \right], \tag{A.5}$$

$$\frac{U_{L,t}(j)}{U_{C,t}(j)} = W_t. \tag{A.6}$$

An equivalent representation of the Euler consumption equation (A.5), which will be useful when we consider the behaviour of firms, is,

$$1 = \mathbb{E}_t \left[\Lambda_{t,t+1}(j) R_{t+1} \right], \tag{A.7}$$

where $\Lambda_{t,t+1}(j) \equiv \beta \frac{U_{C,t+1}(j)}{U_{C,t}(j)}$ is the real stochastic discount factor for household j, over the interval $[t, t+1]$.

For our choice of utility function, $U_{C,t} = \frac{1}{C_t}$ and $U_{H,t} = -\kappa H_t^{\phi}$, so the household's first order conditions become,

$$\frac{1}{C_t(j)} = \beta \mathbb{E}_t \left[\frac{R_{t+1}}{C_{t+1}(j)} \right], \tag{A.8}$$

$$\kappa C_t(j) H_t(j)^{\phi} = W_t \Rightarrow H_t = \left(\frac{W_t}{\kappa C_t(j)} \right)^{\frac{1}{\phi}}. \tag{A.9}$$

In a symmetric equilibrium of identical households, $C_t(j) = C_t$, aggregate per household consumption, and $H_t(j) = H_t$, average hours worked.

A.1.2   Firms in the Wholesale

Wholesale firms employ a Cobb-Douglas production function to produce a homogeneous output,

$$Y_t^W = F(A_t, H_t) = A_t H_t^{\alpha}, \tag{A.10}$$

where A_t is total factor productivity. Profit-maximizing demand for labour results in the first order condition,

$$W_t = \frac{P_t^W}{P_t} F_{H,t} = \alpha \frac{P_t^W}{P_t} \frac{Y_t^W}{H_t}. \tag{A.11}$$

A.1.3 Firms in the Retail Sector

The retail sector uses a homogeneous wholesale good to produce a basket of differentiated goods for aggregate consumption,

$$
C_t = \left(\int_0^1 C_t(m)^{(\zeta-1)/\zeta} dm \right)^{\zeta/(\zeta-1)}, \tag{A.12}
$$

where ζ is the elasticity of substitution. For each m, the consumer chooses $C_t(m)$ at a price $P_t(m)$ to maximize (A.12) given total expenditure $\int_0^1 P_t(m)C_t(m)dm$. This results in a set of consumption demand equations for each differentiated good m with price $P_t(m)$ of the form,

$$
C_t(m) = \left(\frac{P_t(m)}{P_t} \right)^{-\zeta} C_t \Rightarrow Y_t(m) = \left(\frac{P_t(m)}{P_t} \right)^{-\zeta} Y_t, \tag{A.13}
$$

where $P_t = \left[\int_0^1 P_t(m)^{1-\zeta} dm \right]^{\frac{1}{1-\zeta}}$. P_t is the aggregate price index. C_t and P_t are Dixit-Stigliz aggregates - see Dixit and Stiglitz (1977).

For each variety m the retail good is produced costlessly from wholesale production according to

$$
Y_t(m) = Y_t^W = A_t H_t(m)^\alpha. \tag{A.14}
$$

Following Calvo (1983), we now assume that there is a probability of $1 - \xi$ at each period that the price of each retail good m is set optimally to $P_t^0(m)$. If the price is not re-optimized, then it is held fixed.[6] For each retail producer m, given its real marginal cost MC_t, the objective is at time t to choose $\{P_t^0(m)\}$ to maximize discounted profits,

$$
\mathbb{E}_t \sum_{k=0}^\infty \xi^k \Lambda_{t,t+k} Y_{t+k}(m) \left[P_t^0(m) - P_{t+k} MC_{t+k} \right], \tag{A.15}
$$

subject to (A.13), where $\Lambda_{t,t+k} \equiv \beta^k \frac{U_{C,t+k}/P_{t+k}}{U_{C,t}/P_t}$ is now the *nominal* stochastic discount factor over the interval $[t, t+k]$. The solution to this is,

$$
\mathbb{E}_t \sum_{k=0}^\infty \xi^k \Lambda_{t,t+k} Y_{t+k}(m) \left[P_t^0(m) - \frac{1}{(1 - 1/\zeta)} P_{t+k} MC_{t+k} \right] = 0, \tag{A.16}
$$

and by the law of large numbers the evolution of the price index is given by,

---

[6]Thus we can interpret $\frac{1}{1-\xi}$ as the average duration for which prices are left unchanged.

$$P_{t+1}^{1-\zeta} = \xi P_t^{1-\zeta} + (1-\xi)(P_{t+1}^0)^{1-\zeta}. \tag{A.17}$$

In order to set up the model in non-linear form as a set of difference equations, we need to represent the price dynamics as difference equations. First define k period ahead inflation as,

$$\Pi_{t,t+k} \equiv \frac{P_{t+k}}{P_t} = \frac{P_{t+1}}{P_t}\frac{P_{t+2}}{P_{t+1}}\cdot\cdot\frac{P_{t+k-1}}{P_{t+k}} = \Pi_{t,t+1}\Pi_{t+1,t+2}\cdot\cdot\Pi_{t+k-1,t+k},$$

noting that $\Pi_{t,t+1} = \Pi_{t+1}$ and $\Pi_{t,t} = 1$.

Next, using (A.13) with $P_{t+k}(m) = P_0(m)$, the price set at time t which survives with probability ξ, we have that,

$$\Lambda_{t,t+k}Y_{t+k}(m) = \beta^k\frac{U_{C,t+k}}{U_{C,t}}\frac{P_t}{P_{t+k}}\left(\frac{P_0(m)}{P_{t+k}}\right)^{-\zeta}Y_{t+k} = \beta^k\frac{U_{C,t+k}}{U_{C,t}}\Pi_{t,t+k}^{\zeta-1}\left(\frac{P_0(m)}{P_t}\right)^{-\zeta}Y_{t+k}.$$

Hence, cancelling out $\left(\frac{P_0(m)}{P_t}\right)^{-\zeta}$ and multiplying by $\frac{U_{C,t}}{P_t}$, we can write (A.16) as,

$$E_t\sum_{k=0}^{\infty}(\xi\beta)^k U_{C,t+k}\Pi_{t,t+k}^{\zeta-1}Y_{t+k}\left[\frac{P_t^0(m)}{P_t} - \Pi_{t+k}MC_{t+k}MS_{t+k}\right] = 0. \tag{A.18}$$

We seek a symmetric equilibrium where firms who are either re-setting their prices or are locked into a contract are identical. In such an equilibrium, the price dynamics can be written as difference equations as follows:

$$\frac{P_t^0}{P_t} = \frac{J_t}{JJ_t}, \tag{A.19}$$

$$JJ_t - \xi\mathbb{E}_t\left[\Pi_{t+1}^{\zeta-1}JJ_{t+1}\Lambda_{t,t+1}\right] = Y_t, \tag{A.20}$$

$$J_t - \xi\mathbb{E}_t\left[\Pi_{t+1}^{\zeta}J_{t+1}\Lambda_{t,t+1}\right] = \left(\frac{1}{1-\frac{1}{\zeta}}\right)Y_tMC_tMS_t, \tag{A.21}$$

$$1 = \xi\Pi_t^{\zeta-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{1-\zeta}, \tag{A.22}$$

$$\Delta_t = \xi\Pi_t^{\frac{\zeta}{\alpha}}\Delta_{t-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{\frac{\zeta}{\alpha}}, \tag{A.23}$$

$$\tag{A.24}$$

$$MC_t = \frac{P_t^W}{P_t} = \frac{W_t}{F_{H,t}}, \tag{A.25}$$

where (A.34) uses (A.11). Note that we have introduced a mark-up shock MS_t, and that the real marginal cost, MC_t, is variable.

Price dispersion lowers aggregate output as follows. Market clearing in the labour market gives,

$$H_t = \sum_{m=1}^{n} H_t(m) = \sum_{m=1}^{n} \left(\frac{Y_t(m)}{A_t} \right)^{\frac{1}{\alpha}} = \left(\frac{Y_t}{A_t} \right)^{\frac{1}{\alpha}} \sum_{m=1}^{n} \left(\frac{P_t(m)}{P_t} \right)^{-\frac{\zeta}{\alpha}}, \tag{A.26}$$

using (A.13). Hence equilibrium for good m gives,

$$Y_t = \frac{Y_t^W}{\Delta_t^{\alpha}}, \tag{A.27}$$

where price dispersion is defined by,

$$\Delta_t \equiv \left(\sum_{m=1}^{n} \left(\frac{P_t(m)}{P_t} \right)^{-\frac{\zeta}{\alpha}} \right). \tag{A.28}$$

Price dispersion is linked to inflation as follows. Assuming as before that the number of firms is large, we obtain the following dynamic relationship:

$$\Delta_t = \xi \Pi_t^{\frac{\zeta}{\alpha}} \Delta_{t-1} + (1 - \xi) \left(\frac{J_t}{JJ_t} \right)^{-\frac{\zeta}{\alpha}}. \tag{A.29}$$

A.1.4 Profits

To close the model in a manner that will be useful when we come to consider internal rationality, we require total profits from retail and wholesale firms, Γ_t, remitted to households. This is given in real terms by,

$$\Gamma_t = \underbrace{Y_t - \frac{P_t^W}{P_t} Y_t^W}_{\text{retail}} + \underbrace{\frac{P_t^W}{P_t} Y_t^W - W_t H_t}_{\text{Wholesale}} = Y_t - \alpha \frac{P_t^W}{P_t} Y_t^W, \tag{A.30}$$

using the first-order condition (A.11).

A.1.5 Closing the Model

The model is closed with a resource constraint,

$$Y_t = C_t, \tag{A.31}$$

and a monetary policy rule for the nominal interest rate given by the following Taylor-type rule,

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1 - \rho_r)\left(\theta_\theta \log\left(\frac{\Pi_t}{\Pi}\right) + \theta_y \log\left(\frac{Y_t}{Y}\right)\right). \tag{A.32}$$

Finally, there is an exogenous AR1 shock process to marginal cost (e.g. a mark-up shock):

$$\log MS_t - \log MS = \rho_{MS}(\log MS_{t-1} - \log MS) + \epsilon_{MS,t}. \tag{A.33}$$

A.1.6 Summary of Model

Households:

$$
\begin{aligned}
U_t &= U(C_t, H_t) = \log C_t - \kappa \frac{H_t^{1+\phi}}{1+\phi} \\
U_{C,t} &= \beta \mathbb{E}_t \left[R_{t+1} U_{C,t+1} \right] \\
R_t &= \frac{R_{n,t-1}}{\Pi_t} \\
U_{C,t} &= \frac{1}{C_t} \\
U_{H,t} &= -\kappa H_t^\phi \\
\frac{U_{L,t}}{U_{C,t}} &= W_t
\end{aligned}
$$

Firms:

$$
\begin{aligned}
Y_t^W &= F(A_t, H_t) = A_t H_t^\alpha \\
Y_t &= \frac{Y_t^W}{\Delta_t^\alpha} \\
\frac{P_t^W}{P_t} F_{H,t} &= \frac{P_t^W}{P_t} \frac{\alpha Y_t^W}{H_t} = W_t \\
\frac{P_t^0}{P_t} &= \frac{J_t}{JJ_t} \\
JJ_t &= \xi \mathbb{E}_t \left[\Pi_{t+1}^{\zeta-1} JJ_{t+1} \Lambda_{t,t+1} \right] + Y_t \\
J_t &= \xi \mathbb{E}_t \left[\Pi_{t+1}^{\zeta} J_{t+1} \Lambda_{t,t+1} \right] + \left(\frac{1}{1-\frac{1}{\zeta}} \right) Y_t MC_t MS_t \\
1 &= \xi \Pi_t^{\zeta-1} + (1-\xi) \left(\frac{J_t}{JJ_t} \right)^{1-\zeta} \\
\Delta_t &= \xi \Pi_t^{\frac{\zeta}{\alpha}} \Delta_{t-1} + (1-\xi) \left(\frac{J_t}{JJ_t} \right)^{\frac{\zeta}{\alpha}} \\
MC_t &= \frac{P_t^W}{P_t} = \frac{W_t}{F_{H,t}}
\end{aligned}
$$

Closure:

$$
\begin{aligned}
Y_t &= C_t \\
\log \left(\frac{R_{n,t}}{R_n} \right) &= \rho_r \log \left(\frac{R_{n,t-1}}{R_n} \right) + (1-\rho_r) \left(\theta_\theta \log \left(\frac{\Pi_t}{\Pi} \right) + \theta_y \log \left(\frac{Y_t}{Y} \right) \right) + \epsilon_{M,t} \\
\log MS_t - \log MS &= \rho_{MS} (\log MS_{t-1} - \log MS) + \epsilon_{MS,t}
\end{aligned}
$$

A.1.7 Steady State

In recursive form the zero-growth zero-inflation ($\Pi = 1$) steady state of can be written,

$$
\begin{aligned}
R &= \frac{1}{\beta} \\
\Lambda &= \beta \\
\frac{P^W}{P} &= 1 - \frac{1}{\zeta} \\
\frac{C}{Y} &= 1 \\
H &= \left(\frac{\alpha}{\kappa}\right)^{\frac{1}{1+\phi}} \\
Y^W &= (AH)^\alpha \\
Y &= Y^W \\
W &= \alpha \frac{P^W}{P} \frac{Y^W}{H} \\
J &= JJ = \frac{Y}{1 - \beta\xi} \\
\Delta &= 1
\end{aligned}
$$

using $P^W Y^W = PY$ by the free entry condition.

For a particular steady state inflation rate $\Pi > 1$ the New Keynesian features of the steady state become,

$$
\begin{aligned}
\frac{J}{JJ} &= \left(\frac{1 - \xi\Pi^{\zeta-1}}{1 - \xi}\right)^{\frac{1}{1-\zeta}} \\
MC = \frac{P^W}{P} &= \left(1 - \frac{1}{\zeta}\right) \frac{J(1 - \beta\xi\Pi^\zeta)}{JJ(1 - \beta\xi\Pi^{\zeta-1})} \\
\Delta &= \frac{(1-\xi)^\alpha \left(\frac{J}{JJ}\right)^{-\zeta}}{1 - \xi\Pi^\zeta}
\end{aligned}
$$

Then $P^W Y^W / PY = MC\Delta \neq 1$.

A.2 Exogenous Point Expectations

As a first step towards internal rationality we now formulate the consumption and pricing decision of the household and firms respectively in terms of current and expected future aggregate variables exogenous these agents.

A.2.1 Households

For households, solving (A.3) forward in time and imposing the transversality condition on debt we can write,

$$B_{t-1}(j) = \text{PV}_t(C_t(j)) - \text{PV}_t(W_t H_t(j)) - \text{PV}_t(\Gamma_t), \tag{A.34}$$

where the present (expected) value of a series $\{X_{t+i}\}_{i=0}^{\infty}$ at time t is defined by,

$$\text{PV}_t(X_t) \equiv \mathbb{E}_t \sum_{i=0}^{\infty} \frac{X_{t+i}}{R_{t,t+i}} = \frac{X_t}{R_t} + \frac{1}{R_t}\text{PV}_{t+1}(X_{t+1}), \tag{A.35}$$

where $R_{t,t+1} \equiv R_t R_{t+1} R_{t+2} \cdots R_{t+i}$ is the real interest rate over the interval $[t, t+i]$.

The forward-looking budget constraint (A.34) holds for the representative household. In aggregate there is no net debt so $B_{t-1} = 0$. Then in a symmetric equilibrium, substituting for H_t from (A.9) we have,

$$\text{PV}_t(C_t) = \frac{1}{\kappa^{\frac{1}{\phi}}}\text{PV}_t\left(\frac{W_t^{1+\frac{1}{\phi}}}{C_t^{\frac{1}{\phi}}}\right) + \text{PV}_t(\Gamma_t). \tag{A.36}$$

Solving (A.8) forward in time we have, for $i \geq 1$,

$$\frac{1}{C_t} = \beta^i \mathbb{E}_t\left[\frac{R_{t+1,t+i}}{C_{t+i}}\right]. \tag{A.37}$$

The internally rational solution to the household optimization problem seeks a solution to its decision functions for C_t and H_t that are functions of *non-rational point expectations* $\{\mathbb{E}_t^* W_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$ and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$, treated as exogenous processes given at time t as opposed to rational model-consistent expectations $\{\mathbb{E}_t W_{t+i}\}_{i=0}^{\infty}$, etc[7]. With point expectations we use (A.37) to obtain,

$$\mathbb{E}_t^* C_{t+i} = C_t \beta^i \mathbb{E}_t^* R_{t+1,t+i} ; i \geq 1, \tag{A.38}$$

$$\mathbb{E}_t^*(W_{t+i} H_{t+i}) = \frac{1}{\kappa^{\frac{1}{\phi}}}\frac{(\mathbb{E}_t^* W_{t+i})^{1+\frac{1}{\phi}}}{(\mathbb{E}_t^* C_{t+i})^{\frac{1}{\phi}}}. \tag{A.39}$$

Substituting (A.38) and (A.39) into the forward-looking household budget constraint, and using $\sum_{i=0}^{\infty} \beta^i = \frac{1}{1-\beta}$, we arrive at,

$$\frac{C_t}{R_t(1-\beta)} = \frac{1}{R_t(\kappa C_t)^{\frac{1}{\phi}}}\left(W_t^{1+\frac{1}{\phi}} + \sum_{i=1}^{\infty}(\beta^{\frac{1}{\phi}})^{-i}\left(\frac{\mathbb{E}_t^* W_{t+i}}{\mathbb{E}_t^* R_{t+1,t+i}}\right)^{1+\frac{1}{\phi}}\right) + \sum_{i=0}^{\infty}\frac{\mathbb{E}_t^* \Gamma_{t+i}}{\mathbb{E}_t^* R_{t,t+i}}, \tag{A.40}$$

---

[7]With point expectations agents treat $\mathbb{E}_t^*(\cdot)$ as certain, although the environment is stochastic (see Evans and Honkapohja (2001), page 61). Since $\mathbb{E}_t f(X_t) \approx f(\mathbb{E}_t(X_t))$ and $\mathbb{E}_t f(X_t Y_t)) \approx f(\mathbb{E}_t(X_t Y_t))$ up to a first-order Taylor-series expansion, assuming point expectations is equivalent to using a linear approximation of (A.36) and (A.37) as is usually done in the literature.

$$H_t = \left(\frac{W_t}{\kappa C_t} \right)^{\frac{1}{\phi}}. \tag{A.41}$$

(A.40) and (A.41) constitute the consumption and hours decision rules given point expectations of $\{\mathbb{E}_t^* W_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$, and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$.

A.2.2 Retail Firms

Turning next to price-setting by retail firms, write (A.20) and (A.21) as,

$$J_t = \left(\frac{1}{1 - \frac{1}{\zeta}} \right) Y_t MC_t MS_t + \mathbb{E}_t \sum_{k=1}^{\infty} \xi^k \Lambda_{t,t+k} \Pi_{t,t+k}^{\zeta} Y_{t+k} MC_{t+k} MS_{t+k}, \tag{A.42}$$

$$JJ_t = Y_t + \mathbb{E}_t \sum_{k=1}^{\infty} \xi^k \Lambda_{t,t+k} \Pi_{t,t+k}^{\zeta-1} Y_{t+k}. \tag{A.43}$$

Assuming point expectations, as for households, we have,

$$J_t = \left(\frac{1}{1 - \frac{1}{\zeta}} \right) \left(Y_t MC_t MS_t + \sum_{k=1}^{\infty} \xi^k \mathbb{E}_t^* \Lambda_{t,t+k} (\mathbb{E}_t^* \Pi_{t,t+k})^{\zeta} \mathbb{E}_t^* Y_{t+k} \mathbb{E}_t^* MC_{t+k} \mathbb{E}_t^* MS_{t+k} \right)$$

$$= \left(\frac{1}{1 - \frac{1}{\zeta}} \right) \left(Y_t MC_t MS_t + \Omega_{3,t} \right), \tag{A.44}$$

$$JJ_t = Y_t + \sum_{k=1}^{\infty} \xi^k \mathbb{E}_t^* \Lambda_{t,t+k} (\mathbb{E}_t^* \Pi_{t,t+k})^{\zeta-1} \mathbb{E}_t^* Y_{t+k}$$

$$= Y_t + \Omega_{4,t}, \tag{A.45}$$

where, noting that $\mathbb{E}_t^* \Lambda_{t,t+1} = \frac{1}{\mathbb{E}_t^* R_{t+1}}$ and $\Pi_{t,t+1} = \Pi_{t+1}$, we have,

$$\Omega_{3,t} = \xi \frac{(\mathbb{E}_t^* \Pi_{t+1})^{\zeta} \mathbb{E}_t^* Y_{t+1} \mathbb{E}_t^* MC_{t+1} \mathbb{E}_t^* MS_{t+1}}{\mathbb{E}_t^* R_{t+1}} + \xi \frac{\mathbb{E}_t^* \Pi_{t+1}^{\zeta}}{\mathbb{E}_t^* R_{t+1}} \Omega_{3,t+1}, \tag{A.46}$$

$$\Omega_{4,t} = \xi \left(\frac{(\mathbb{E}_t^* \Pi_{t+1})^{\zeta-1} \mathbb{E}_t^* Y_{t+1}}{\mathbb{E}_t^* R_{t+1}} + \xi \frac{\mathbb{E}_t^* \Pi_{t+1}^{\zeta-1}}{\mathbb{E}_t^* R_{t+1}} \Omega_{4,t+1}. \tag{A.47}$$

Recalling that the optimal price re-setting decision rule is given by $\frac{P_t^O}{P_t} = \frac{J_t}{JJ_t}$, (A.44) and (A.45) now give us the this rule given exogenous expectations of $\{\mathbb{E}_t^* \Pi_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* Y_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* MC_{t+i}\}_{i=0}^{\infty}$, and $\{\mathbb{E}_t^* MS_{t+i}\}_{i=0}^{\infty}$.

A.3 Internal rationality in the NK Model

The final step to complete the IR equilibrium is to choose the learning rule for $\{\mathbb{E}_t^* W_{t+i}\}_{i=0}^\infty$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^\infty$ and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^\infty$ for households and $\{\mathbb{E}_t^* \Pi_{t+i}\}_{i=0}^\infty$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^\infty$, $\{\mathbb{E}_t^* Y_{t+i}\}_{i=0}^\infty$, $\{\mathbb{E}_t^* MC_{t+i}\}_{i=0}^\infty$ and $\{\mathbb{E}_t^* MS_{t+i}\}_{i=0}^\infty$ for retail firms.

We assume general bounded rational expectations rules so that,

$$\mathbb{E}_t^*[W_{t+i}] = \mathbb{E}_t^*[W_{t+1}] \text{ for } i \geq 1, \tag{A.48}$$

and similarly for $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^\infty$, $\{\mathbb{E}_t^* \Pi_{t+i}\}_{i=0}^\infty$, $\{\mathbb{E}_t^* Y_{t+i}\}_{i=0}^\infty$, $\{\mathbb{E}_t^* MC_{t+i}\}_{i=0}^\infty$ and $\{\mathbb{E}_t^* MS_{t+i}\}_{i=0}^\infty$, whilst,

$$\mathbb{E}_t^* R_{t,t+i} = R_t \frac{R_{n,t}}{\mathbb{E}_t^* \Pi_{t+1}} (\mathbb{E}_t^* R_{t+1})^{i-1}, \tag{A.49}$$

which takes into account the observation of $R_{n,t}$ at time t. One-period ahead forecasts are given in the main body of the text.

With adaptive expectations, (A.40) becomes,

$$\frac{C_t}{R_t(1-\beta)} = \frac{1}{R_t(\kappa C_t)^{\frac{1}{\phi}}} \left(W_t^{1+\frac{1}{\phi}} + \frac{(\mathbb{E}_t^* W_{t+1})^{1+\frac{1}{\phi}}}{\beta^{\frac{1}{\phi}} (\mathbb{E}_t^* R_{t+1})^{1+\frac{1}{\phi}} - 1} \right) + \frac{\mathbb{E}_t^* \Gamma_{t+1}}{\mathbb{E}_t^* R_{t+1} - 1},$$

whilst (A.46) and (A.47) now become,

$$\Omega_{3,t} = \frac{\xi(\mathbb{E}_t^* \Pi_{t+1})^\zeta \mathbb{E}_t^* Y_{t+1} \mathbb{E}_t^* MC_{t+1} \mathbb{E}_t^* MS_{t+1}}{\mathbb{E}_t^* R_{t+1} - \xi(\Pi_{t+1})^\zeta}$$

$$\Omega_{4,t} = \frac{\xi(\mathbb{E}_t^* \Pi_{t+1})^{\zeta-1} \mathbb{E}_t^* Y_{t+1}}{\mathbb{E}_t^* R_{t+1} - \xi(\mathbb{E}_t^* \Pi_{t+1})^{\zeta-1}}.$$

This completes the internally rational equilibrium with point adaptive expectations.

A.4 Proof of Lemma

In the first order conditions for Calvo contracts and expressions for value functions we are confronted with expected discounted sums of the general form,

$$\Omega_t = \mathbb{E}_t \left[\sum_{k=0}^\infty \beta^k X_{t,t+k} Y_{t+k} \right], \tag{A.50}$$

47

where $X_{t,t+k}$ has the property $X_{t,t+k} = X_{t,t+1}X_{t+1,t+k}$ and $X_{t,t} = 1$ (for example an inflation, interest or discount rate over the interval $[t, t+k]$).

Lemma

Ω_t can be expressed as,

$$\Omega_t = Y_t + \beta\mathbb{E}_t\left[X_{t,t+1}\Omega_{t+1}\right]. \tag{A.51}$$

Proof

$$
\begin{aligned}
\Omega_t &= X_{t,t}Y_t + \mathbb{E}_t\left[\sum_{k=1}^{\infty}\beta^k X_{t,t+k}Y_{t+k}\right] \\
&= Y_t + \mathbb{E}_t\left[\sum_{k'=0}^{\infty}\beta^{k'+1} X_{t,t+k'+1}Y_{t+k'+1}\right] \\
&= Y_t + \beta\mathbb{E}_t\left[\sum_{k'=0}^{\infty}\beta^{k'} X_{t,t+1}X_{t+1,t+k'+1}Y_{t+k'+1}\right] \\
&= Y_t + \beta\mathbb{E}_t\left[X_{t,t+1}\Omega_{t+1}\right]. \qquad \square
\end{aligned}
$$

A.5 Proof of Equation A.29

In the next period, ξ of these firms will keep their old prices, and $(1-\xi)$ will change their prices to P_{t+1}^O. By the law of large numbers, we assume that the distribution of prices among those firms that do not change their prices is the same as the overall distribution in period t. It follows that we may write,

$$
\begin{aligned}
\Delta_{t+1} &= \xi\sum_{j_{no\ change}}\left(\frac{P_t(j)}{P_{t+1}}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta} \\
&= \xi\left(\frac{P_t}{P_{t+1}}\right)^{-\zeta}\sum_{j_{no\ change}}\left(\frac{P_t(j)}{P_t}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta} \\
&= \xi\left(\frac{P_t}{P_{t+1}}\right)^{-\zeta}\sum_{j}\left(\frac{P_t(j)}{P_t}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta} \\
&= \xi\Pi_{t+1}^{\zeta}\Delta_t + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta}. \qquad \square
\end{aligned}
$$

B Linearization

B.1 Households

The Euler equation and choice of hours supplied,

$$U_{C,t} = \beta \mathbb{E}_t^* \left[R_{t+1} U_{C,t+1} \right], \tag{B.1}$$

$$-\frac{U_{H,t}}{U_{C,t}} = W_t, \tag{B.2}$$

which with choice of utility function,

$$U_t = U(C_t, H_t^s) = \log(C_t) - \kappa \frac{(H_t^s)^{1+\phi}}{1+\phi}, \tag{B.3}$$

gives,

$$\frac{1}{C_t} = \beta \mathbb{E}_t^* \left[\frac{R_{t+1}}{C_{t+1}} \right], \tag{B.4}$$

$$H_t^s = \left(\frac{W_t}{\kappa C_t} \right)^{\frac{1}{\phi}}. \tag{B.5}$$

Let $c_t \equiv \log(C_t/C)$ and $r_t \equiv \log(R_t/R)$. Then the log-linearization of (B.4) (B.5) and the Fischer equation gives,

$$c_t = \mathbb{E}_t^* \left[c_{t+1} - r_{t+1} \right],$$
$$h_t^s = \frac{1}{\phi}(w_t - c_t),$$
$$r_t = r_{n,t-1} - \pi_t.$$

The forward-looking consumption equation under perfect foresight (or assuming point expectations) is,

$$\frac{C_t}{R_t(1-\beta)} = \frac{1}{R_t(\kappa C_t)^{\frac{1}{\phi}}} \left(W_t^{1+\frac{1}{\phi}} + \Omega_{1,t} \right) + \Omega_{2,t}, \tag{B.6}$$

$$\Omega_{1,t} \equiv \sum_{i=1}^{\infty} (\beta^{\frac{1}{\phi}})^{-i} \left(\frac{\mathbb{E}_t^* W_{t+i}}{\mathbb{E}_t^* R_{t+1,t+i}} \right)^{1+\frac{1}{\phi}},$$

$$\Omega_{2,t} \equiv \sum_{i=0}^{\infty} \frac{\mathbb{E}_t^* \Gamma_{t+i}}{\mathbb{E}_t^* R_{t,t+i}},$$

$$H_t = \left(\frac{W_t}{\kappa C_t} \right)^{\frac{1}{\phi}}. \tag{B.7}$$

Hence,

$$\Omega_{1,t} = \frac{1}{\beta^{\frac{1}{\phi}}} \left(\left(\frac{\mathbb{E}_t^* W_{t+1}}{\mathbb{E}_t^* R_{t+1}} \right)^{1+\frac{1}{\phi}} + \frac{\mathbb{E}_t^* \Omega_{1,t+1}}{\mathbb{E}_t^* R_{t+1}^{1+\frac{1}{\phi}}} \right), \tag{B.8}$$

$$\Omega_{2,t} = \frac{1}{R_t} \left(\Gamma_t + \mathbb{E}_t^* \Omega_{2,t+1} \right). \tag{B.9}$$

(B.6) and (B.7) constitute the consumption and hours decision rules given expectations $\{\mathbb{E}_t^* W_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$, and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$.

Let $c_t \equiv \log(C_t/C)$, $w_t \equiv \log(W_t/W)$, $r_t \equiv \log(R_t/R)$, $\gamma_t \equiv \log(\Gamma_t/\Gamma)$, $h_t \equiv \log(H_t/H)$, $\omega_{1,t} \equiv \log(\Omega_{1,t}/\Omega_1)$, and $\omega_{2,t} \equiv \log(\Omega_{2,t}/\Omega_2)$. Then the log-linearization of (B.6) and (B.7) gives,

$$\alpha_1 c_t = \alpha_2 w_t + \alpha_3 (\omega_{2,t} + r_t) + \alpha_4 \omega_{1,t}, \tag{B.10}$$

$$\omega_{1,t} = \alpha_5 \mathbb{E}_t^* w_{t+1} - \alpha_6 \mathbb{E}_t^* r_{t+1} + \beta \mathbb{E}_t^* \omega_{1,t+1}, \tag{B.11}$$

$$\omega_{2,t} = (1-\beta)(\gamma_t - r_t) - \beta r_t + \beta \mathbb{E}_t^* \omega_{2,t+1}, \tag{B.12}$$

$$\gamma_t = \frac{c_y}{\gamma_y} c_t - \frac{\alpha}{\gamma_y}(w_t + h_t), \tag{B.13}$$

where the (positive) coefficients are given by,

$$\alpha_1 \equiv 1 + \frac{\alpha}{\phi c_y},$$

$$\alpha_2 \equiv (1-\beta)\left(1 + \frac{1}{\phi}\right)\frac{\alpha}{c_y},$$

$$\alpha_3 \equiv \frac{\gamma_y}{c_y},$$

$$\alpha_4 \equiv \frac{\beta \alpha}{c_y},$$

$$\alpha_5 \equiv (1-\beta)\left(1 + \frac{1}{\phi}\right),$$

$$\alpha_6 \equiv 1 + \frac{1}{\phi}$$

where $c_y = 1$ and $\gamma_y = 1 - \alpha$

B.2   Firms

The non-linear price dynamics are given by,

50

$$JJ_t - \xi\beta\mathbb{E}_t[\Pi_{t+1}^{\zeta-1}JJ_{t+1}] = Y_tU_{C,t}, \tag{B.14}$$

$$J_t - \xi\beta\mathbb{E}_t[\Pi_{t+1}^{\zeta}J_{t+1}] = \left(\frac{1}{1-\frac{1}{\zeta}}\right)Y_tU_{C,t}(MC_t + MS_t), \tag{B.15}$$

$$1 = \xi\Pi_t^{\zeta-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{1-\zeta}. \tag{B.16}$$

The zero growth and positive inflation rate steady state, Π, for the NK features are,

$$J(1 - \beta\xi\Pi^{\zeta}) = \frac{YU_CMC}{\left(1-\frac{1}{\zeta}\right)}, \tag{B.17}$$

$$JJ(1 - \beta\xi\Pi^{\zeta-1}) = YU_C, \tag{B.18}$$

$$\frac{J}{JJ} = \left(\frac{1 - \xi\Pi^{\zeta-1}}{1-\xi}\right)^{\frac{1}{1-\zeta}}, \tag{B.19}$$

$$MC = \left(1 - \frac{1}{\zeta}\right)\frac{J(1 - \beta\xi\Pi^{\zeta})}{JJ(1 - \beta\xi\Pi^{\zeta-1})}, \tag{B.20}$$

$$\Delta = \frac{(1-\xi)^{\frac{1}{1-\zeta}}(1 - \xi\Pi^{\zeta-1})^{\frac{-\zeta}{1-\zeta}}}{1 - \xi\Pi^{\zeta}}. \tag{B.21}$$

For a zero-inflation steady state $\Pi = 1$, we arrive $\frac{J}{JJ} = \Delta = 1$ and $MC = \left(1 - \frac{1}{\zeta}\right)$, but in general there is a long-run inflation-output trade-off in the choice of the steady-state inflation rate. The implications of introducing a non-zero inflation steady state into the standard New Keynesian model are explored by Ascari and Ropele (2007).

Expanding (B.14) as a Taylor series yields,

$$JJ + JJ_t - JJ - \xi\beta\mathbb{E}_t[\Pi^{\zeta-1}JJ + (\zeta - 1)\Pi^{\zeta-2}JJ(\Pi_{t+1} - \Pi)$$
$$+ \Pi^{\zeta-1}(JJ_{t+1} - JJ)] = YU_C + U_C(Y_t - Y) + Y(U_{C,t} - U_C). \tag{B.22}$$

Cancelling out the constants on both sides, putting $\Pi = 1$ and dividing by JJ, we have,

$$jj_t \equiv \frac{JJ_t - JJ}{JJ} = \xi\beta E_t\left[(\zeta - 1)\pi_{t+1} + jj_{t+1}\right] + \frac{YU_C}{JJ}(y_t + u_{C,t}). \tag{B.23}$$

Similarly linearizing (B.15), we arrive at,

$$j_t \equiv \frac{J_t - J}{J} = \xi\beta E_t\left[\zeta\pi_{t+1} + j_{t+1}\right] + \frac{YU_CMC}{J\left(1-\frac{1}{\zeta}\right)}(y_t + u_{C,t} + mc_t + ms_t). \tag{B.24}$$

Next we linearize (B.16) and put $\Pi = 1$ to obtain,

$$\xi(\zeta - 1)\pi_t + (1 - \xi)(1 - \zeta)\left(\frac{J}{JJ}\right)^{-\zeta}(j_t - jj_t) = 0. \tag{B.25}$$

Using the steady state relationships with $\Pi = 1$, we have that $\frac{YU_C}{JJ} = \frac{YU_C MC}{J} = 1 - \beta\xi$, and (B.24) and (B.25) give,

$$
\begin{align}
jj_t &= \xi\beta E_{t+1}\left[(\zeta - 1)\pi_{t+1} + jj_{t+1}\right] + (1 - \beta\xi)(y_t + u_{C,t}), \tag{B.26}\\
j_t &= \xi\beta E_{t+1}\left[\zeta\pi_{t+1} + j_{t+1}\right] + y_t + (1 - \beta\xi)(u_{C,t} + mc_t + ms_t), \tag{B.27}\\
\xi\pi_t &= (1 - \xi)(j_t - jj_t). \tag{B.28}
\end{align}
$$

Finally, subtracting (B.27) from (B.26), and using (B.28), we arrive at the *linear NK Phillips Curve*,

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \frac{(1 - \xi)(1 - \beta\xi)}{\xi}mc_t. \tag{B.29}$$

This can be solved forward in time to give,

$$\pi_t = \frac{(1 - \xi)(1 - \beta\xi)}{\xi}\sum_{i=0}^{\infty}\beta^i mc_{t+i}, \tag{B.30}$$

telling us that in the NK model in proportional deviation terms about the steady state, inflation is proportional to the discounted sum of expected future deviations of marginal costs.

The rest of the supply sides consists of a first-order demand for hours and a Cobb-Douglas production function:

$$
\begin{align}
W_t &= \alpha\frac{P_t^W}{P_t}\frac{Y_t^W}{H_t^d}, \\
Y_t &= Y_t^W = A_t(H_t^d)^\alpha, \\
MC &\equiv \frac{P_t^W}{P_t},
\end{align}
$$

from which we arrive at the log-linearization,

$$y_t = a_t + \alpha h_t^d mc_t = w_t - y_t + h_t^d. \tag{B.31}$$

B.3 Monetary Rule, equilibrium, and shock process

We consider a monetary policy rules for the nominal interest rate given by the following Taylor-type rule:

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\left(\theta_\theta \log\left(\frac{\Pi_t}{\Pi}\right) + \theta_y \log\left(\frac{Y_t}{Y}\right)\right), \tag{B.32}$$

The log-linear form is,

$$r_{n,t} = \rho_r r_{n,t-1} + (1-\rho_r)(\theta_\pi \pi_t + \theta_y y_t). \tag{B.33}$$

Equilibria in the output and labour markets are given by,

$$Y_t = C_t, \tag{B.34}$$
$$H_t^s = H_t^d = H_t, \tag{B.35}$$

which have the log-linear forms,

$$y_t = c_t, \tag{B.36}$$
$$h_t^s = h_t^d = h_t. \tag{B.37}$$

Finally, the AR1 shock process us already in log-linear form if $ms_t \equiv \log MS_t - \log MS = \log MS_t/MS$:

$$\log MS_t - \log MS = \rho_{MS}(\log MS_{t-1} - \log MS) + \epsilon_{MS,t}. \tag{B.38}$$

B.4 Summary of Linearized RE-IR Model

B.4.1 RE Model

To summarise, the linearised rational expectations model is given by,

$$
\begin{aligned}
c_t &= \mathbb{E}_t\left[c_{t+1} - r_{t+1}\right] \\
\textbf{or } \alpha_1 c_t &= \alpha_2 w_t + \alpha_3(\omega_{1,t} + r_t) + \alpha_4 \omega_{2,t} \\
\omega_{1,t} &= \alpha_5 \mathbb{E}_t w_{t+1} - \alpha_6 \mathbb{E}_t r_{t+1} + \beta \mathbb{E}_t \omega_{1,t+1} \\
\omega_{2,t} &= (1-\beta)(\gamma_t - r_t) - \beta r_t + \beta \mathbb{E}_t \omega_{2,t+1} \\
\gamma_t &= \frac{c_y}{\gamma_y} c_t - \frac{\alpha}{\gamma_y}(w_t + h_t^s) \\
h_t^s &= \frac{1}{\phi}(w_t - c_t)
\end{aligned}
$$

$$
\begin{aligned}
r_t &= r_{n,t-1} - \pi_t \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi}(mc_t + ms_t) \\
y_t &= a_t + \alpha h_t^d \\
mc_t &= w_t - y_t + h_t^d \\
y_t &= c_t \\
h_t^s &= h_t^d
\end{aligned}
$$

plus a policy rule,

$$
r_{n,t} = \rho_r r_{n,t-1} + (1-\rho_r)(\theta_\pi \pi_t + \theta_y y_t) + mps_t \tag{B.39}
$$

giving 9 (or 11) equations in c_t (or c_t, $\omega_{1,t}$, $\omega_{2,t}$), y_t, h_t^s, h_t^d, w_t, r_t, $r_{n,t}$, π_t and mc_t given the AR1 exogenous process for ms_t.

B.4.2 IR Model

The linearised model with internal rationality is given by,

$$
\begin{aligned}
\alpha_1 c_t &= \alpha_2 w_t + \alpha_3(\omega_{1,t} + r_t) + \alpha_4 \omega_{2,t} \\
\gamma_t &= \frac{c_y}{\gamma_y} c_t - \frac{\alpha}{\gamma_y}(w_t + h_t^s) \\
\omega_{1,t} &= \frac{1}{1-\beta}\left[\alpha_5 \mathbb{E}_t^* w_{t+1} + \alpha_6 \mathbb{E}_{h,t}^* \pi_{t+1}\right] - \alpha_6\left(r_{n,t} + \frac{\beta}{1-\beta}\mathbb{E}_t^* r_{n,t+1}\right) \\
\omega_{2,t} &= (1-\beta)\gamma_t + \beta \mathbb{E}_t^* \gamma_{t+1} - \left(r_{n,t-1} + \beta r_{n,t} + \frac{\beta^2}{1-\beta}\mathbb{E}_t^* r_{n,t+1}\right) + \pi_t + \frac{\beta}{1-\beta}\mathbb{E}_{h,t}^* \pi_{t+1} \\
h_t^s &= \frac{1}{\phi}(w_t - c_t)
\end{aligned}
$$

$$\pi_t = \frac{(1-\xi)}{\xi}(p_t^o - p_t)$$

$$= \frac{(1-\xi)}{\xi}\left(\frac{1}{1-\beta\xi}\mathbb{E}_{f,t}^*\pi_{t+1} + (1-\beta\xi)\left((mc_t + ms_t) + \frac{\beta\xi}{1-\beta\xi}\mathbb{E}_t^*(mc_{t+1} + ms_{t+1})\right)\right)$$

with point expectations given in the main body of the text.

B.4.3 Composite RE-IR Model

$$h_t^d = n_t\,(h_t^s)^{RE} + (1-n_t)\,(h_t^s)^{IR} = h_t$$

$$y_t = a_t + \alpha h_t$$

$$c_t = n_t\,(c_t)^{RE} + (1-n_t)\,(c_t)^{IR} = y_t$$

$$p_t^o = n_t\,(p_t^o)^{RE} + (1-n_t)\,(p_t^o)^{IR}$$

$$\pi_t = \frac{(1-\xi)}{\xi}(p_t^o - p_t)$$