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Recourse Loans and Ponzi Schemes.

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.

Abstract

Non-recourse borrowing leaves no room for Ponzi schemes, as shown by Araujo, Páscoa and Torres-Martínez (2002). This is not the case with recourse loans, for which, in the event of default and on top of the foreclosure of the collateral, the debtor's estate can be seized or (in a way common in the GE literature) the debtor can suffer utility penalties. We focus on the latter and show that infinite horizon equilibrium with recourse exists in some interesting cases: (i) if utility penalties are low enough and the collateral does not yield utility (for example, when it is a productive asset or a security) or (ii) for a nominal promise backed by real collateral (such as mortgages, whose payments are not tied to a commodity price index).

Keywords: collateral, Ponzi schemes, incomplete markets.

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1 Introduction

The modern general equilibrium literature on default evolved mainly from two seminal contributions, the Dubey, Geanakoplos and Shubik (2005) paper on utility penalties and the Geanakoplos and Zame (1997) work on non-recourse borrowing. In an infinite horizon set-up, non-recourse loans have the appealing feature of being incompatible with Ponzi schemes, at least for time and state separable preferences, as shown by Araujo, Páscoa and Torres-Martínez (2002). Non-recourse is the rule for mortgages in thirteen states in the U.S., but in all the other states and in many countries mortgages are treated as recourse loans. Other types of secured loans tend to be recourse: collateralized borrowing for the purchase of equipment usually requires a default insurance, while in the case of credit for the purchase of securities default triggers personal bankruptcy.

Recourse means that the defaulter's personal estate can be seized by the creditors, either entirely or in proportion to the outstanding debts that were not covered by the value of the collateral at the time the default occurred. There may be other types of default penalties, such as reputational effects, difficulties in applying for credit in the future or even criminal consequences, that we can also see as a form of recourse that ends up affecting the defaulter's welfare. Utility penalties, introduced by Shubik and Wilson (1977) and more recently modelled by Dubey, Geanakoplos and Shubik (2005) in a GE framework, attempt to capture in utility terms the impact of all forms of recourse.

In the presence of utility penalties, collateral may not avoid a Ponzi game. In fact, the penalties may induce agents to repay above the minimum of the promised payment and the collateral value. Then, non-arbitrage cannot rule out that, at the borrowing date, the secured loan would have a negative haircut (the collateral cost would be lower than the loan). The resulting cash flow in an open end setting would give rise to a Ponzi scheme. This is actually what happens when the promise is traded in the examples by Páscoa and Seghir (2009) for utility penalties that make the maximal default prohibitive. However, the argument that also ruled out no-trade outcomes in Páscoa and Seghir (2009) was not correct and, as was pointed out by Martins-da-Rocha and Vailakis (2012a), a no-trade equilibrium could be found by setting the delivery rate at the minimal level, even though such expectation about the delivery rate is not consistent with the harsh penalty. Once the equilibrium is refined, along the lines of the refinement in Dubey, Geanakoplos and Shubik (2005), non-existence of equilibrium prevails.

Our first contribution is to observe that absence of pecuniary Ponzi schemes is not enough for existence of equilibrium. The net gain that the borrower can have at the borrowing moment consists of the loan net of collateral costs plus the utility from consumption of the collateral. Such direct utility effect may allow for an infinite horizon improvement strategy even when collateral costs outweigh the loan (as our Example 1 illustrates). Under non-recourse the whole current benefit had to be non-positive, by non-arbitrage, as at the immediate next nodes new collateral values net of effective repayment were always non-negative.

In our second contribution, we find an upper bound on utility penalty coefficients that make the collateral cost never fall below the promise price and existence of equilibrium is, therefore, guaranteed, under these moderate penalties, provided that the collateral does not give any utility (say it is a durable commodity with no utility yields, as in Fostel and Geanakoplos (2008), a productive asset or a share in it, as in Kubler and Schmedders (2003) or any real security in positive net supply that cannot be short-sold, as in Fostel and Geanakoplos (2015)). Actually, the recourse feature that a utility penalty tries to capture is often observed in collateralized borrowing for the purchase of equipment or securities.

Our third contribution allows for harsher penalties and for utility yields from the consumption of the collateral. Moderate penalties is a strong condition as it makes agents give maximal default (as in the model where utility penalties were absent). However, equilibrium is compatible with partial default or no default, as illustrated in Example 2, where the sum, across next nodes, of the marginal penalty effects is dominated by the sum of the marginal income effects. The problem is that this dominance depends on relative prices and there might be no room to choose relative spot prices if these are already pinned down by market clearing.

There is, nevertheless, an important case where there are degrees of freedom in market clearing prices. It is the case where the promise is nominal but the collateral is a real asset. This case is relevant for mortgages, which should be regarded as loans whose *promised* repayments are not adjusted by commodity price indices, and also for credit for the purchase of shares. In finite horizon economies, there is indeterminacy in equilibrium with respect to the inflation rates. Now, high inflation rates across all the next nodes, devalue the promised payments but not the collateral and, therefore, reduce the real value of default on which the penalty is applied. Our Example 3 illustrates such equilibria with nominal promises.

Example 2 and our two existence results do not collide with the result by Ferreira and Torres-Martinez (2010) on impossibility of recourse. Their result depends on collateral costs being lower than the deflated value of recourse (the repayment in excess of the minimal one) at the next nodes. It is interesting to note that such a condition had to be introduced to make recourse impossible. In Example 2, collateral coefficients do not satisfy such inequality and the haircut is zero (rather than negative as in Ferreira and Torres-Martinez (2010)), which does not allow for a Ponzi scheme (or generalized version of it, since the collateral does not yield utility in this example). In Theorem 2 borrowers' repayment is always the minimal one, while in Theorem 3 it might not be, but the promise is nominal and the result by Ferreira and Torres-Martinez (2010) does not apply.

One may wonder how do our existence results stand in the face of the possibility that trivial no-trade equilibria might be found. Dubey, Geanakoplos and Shubik (2005) showed that for unsecured promises subject to utility penalties on default, a trivial equilibrium always exists by setting promises prices, delivery rates and financial trades equal to zero. Martins-da-Rocha and Vailakis (2012a) found a no-trade incomplete markets equilibrium in an example with secured promises by setting the delivery rate at the minimal level. When the horizon is finite or markets are complete, such no-trade equilibria are trivially found but in infinite horizon incomplete markets that is not always the case, as we illustrate in a companion paper (Páscoa and Seghir (2019). There we also propose a refinement of equilibrium, which is milder than the straightforward extension to secured loans of the one in Dubey, Geanakoplos and Shubik (2005). Even so, the no-trade outcome in Martins-da-Rocha and Vailakis (2012a) still fails to meet this refinement but there exist refined versions for the equilibria we found in the main results of this paper.

The next section presents the model. Section 3 addresses individual optimality. Section 4 presents the existence results. Proofs are presented in the Appendix.

2 The Model

Consumers trade collateralized promises over a countably infinite tree D with finitely many branches at each node. Let $\mathbb{N}_0 = \{0, 1, \dots\}$ be the set of dates and ξ_0 be the root of the tree D. Given a node $\xi \in D$, let $t(\xi) \in \mathbb{N}_0$ be the date of node ξ. We denote by $D(\xi)$ the sub-tree that starts at ξ. We write $\xi' > \xi$ if $\xi' \in D(\xi)$ and $\xi' \neq \xi$. The immediate successors of node ξ constitute the set $\xi^+ \equiv \{\eta > \xi : t(\eta) = t(\xi) + 1\}$ while its immediate predecessor is denoted by ξ^-. We will also use the notation $D_T \equiv \{\xi \in D : t(\xi) = T\}$ and $D^T \equiv \bigcup_{t=0}^{T} D_t$. For any real sequence (a_n), the notation $a \ggg 0$ (or $a \lll 0$) stands for (a_n) being a positive (negative, respectively) sequence uniformly bounded away from zero.

At each node ξ a finite number G of commodities is traded together with a finite set J of one-period promises. Let us start by assuming that promises have real returns. This assumption will be modified later, in section 5.3.

Sales of promises are secured by collateral, which is not necessarily a durable good, but may also be a productive asset or a security in positive net supply that pays real returns and cannot be short sold. This can be accommodated by treating securities formally as durable goods that do not yield utility. In this context, we may have a non-diagonal transformation matrix Y_ξ, of type $G.G$, indicating how commodities of the previous node convert into commodities of the node ξ. If g is a durable good, the only non-null element in column $(Y_\xi)^g$ is $(Y_\xi)_{gg}$, equal to the depreciation factor. If g is a security, $(Y_\xi)_{gg} = 1$ and, its non-negative dividends are given, for $g' \neq g$, by $(Y_\xi)_{g'g}$. We allow also for productive assets (as in Kubler and Schmedders (2003)) which can be treated formally as commodities whose non-null columns in Y_ξ matrices represent their productive returns on other commodities.

Formally, the assumption on promises returns and collateral is the following.

Assumption [R].

(i) Promised returns are real and given by $A_{j\xi} \in \mathbb{R}^G_+$, $\forall j \in J, \xi > \xi_0$.

(ii) At each node ξ, collateral must be posted in at least one $g \in G$ for which the column Y^g_η is non-null at every node $\eta \in \xi^+$. Collateral requirements are given by a $G \times J$ matrix C_ξ .

There are I consumers whose endowments and preferences verify the following assumptions.

Assumption [E]. Endowments of consumer i of commodity g at node ξ, denoted by $\omega^i_{g\xi}$, satisfy

(i) $\exists W \in \mathbb{R}_{++} : \forall i \in I,\ \forall \xi \in D,\ \sum_{g \in G} \omega^i_{g\xi} \leq W.$

(ii) $\omega_{\xi_0} \gg 0$ and, for $\xi > \xi_0$ and any g, $\omega_{g\xi} > 0$ whenever the $g-$th row of Y_ξ is null.

Let $Y_{\xi_0,\xi_n} = Y(\xi_n)Y(\xi_{n-1})\ldots Y(\xi_1)$ for $\xi_{k+1} \in {\xi_k}^+$. The aggregate physical resources available at node ξ are given by $\Omega_\xi = \sum_i W^i_\xi$, where $W^i_\xi = \sum_{\eta \in \{\xi_0,\ldots,\xi^-,\xi\}} Y_{\eta,\xi}\, \omega^i_\eta$.

We say that good g is perishable at node ξ if the g-th column of Y_η is null for any $\eta \in \xi^+$.

Assumption [U]. $\forall i \in I$, preferences over consumption are described by a time and state separable utility U^i with instantaneous utility $v^i_\xi : \mathbb{R}^G_+ \longrightarrow \mathbb{R}_+$ such that

(i) v^i_ξ is monotone and concave,

(ii) v^i_ξ is differentiable on $\mathbb{R}^G_{++}$,

(iii) $\forall \alpha \in \mathbb{R}^G_+$ we have $\sum_{\xi \in D} v^i_\xi(\alpha) < \infty$ and

(iv) $\sum_{\xi \in D} v^i_\xi(\Omega_\xi) < \infty$[2].

Consumers take as given prices p for goods, prices q for promises and delivery rates K on the promises. As in Dubey, Geanakoplos and Shubik (2005), these delivery rates are impersonal expectations about the ex post repayment of the promise. In equilibrium, $1 - K_{j\xi}$ is the default rate on promise j in node ξ by the aggregate sellers of that promise (as will be required in item (v) of Definition 1).

A choice variable is a *non-negative* plan $(x, \theta, \varphi, \psi)$ consisting of purchases of goods not for collateral purposes, promises purchases, promises sales and defaults,

[2] When Y is diagonal with elements uniformly bounded away from one, the assumptions that endowments are uniformly bounded and that the utility of a bounded plan is finite are sufficient to ensure $\sum_{\xi \in D} v^i_\xi(\Omega_\xi) < \infty$. (see Páscoa and Seghir (2009)).

respectively. We denote $\tilde{x}^i_\xi = x^i_\xi + C_\xi \varphi^i_\xi$. *Budget constraints* at the initial node or at subsequent nodes $\xi \in D \setminus \{\xi_0\}$, are given, respectively, by:

$$p_{\xi_0}(\tilde{x}^i_{\xi_0} - \omega^i_{\xi_0}) + q_{\xi_0}(\theta^i_{\xi_0} - \varphi^i_{\xi_0}) \leq 0, \tag{1}$$

$$p_\xi\big(\tilde{x}_\xi - \omega^i_\xi - Y_\xi \tilde{x}^i_{\xi^-} - \sum_{j \in J(\xi^-)} A_{j\xi}(K_{j\xi}\theta^i_{j\xi^-} - \varphi^i_{j\xi^-})\big) + q_\xi(\theta^i_\xi - \varphi^i_\xi) \leq \sum_{j \in J(\xi^-)} \psi^i_{j\xi}, \tag{2}$$

To shorten the notations, we define $M_{j\xi} = \min\{p_\xi A_{j\xi}, p_\xi Y_\xi C^j_{\xi^-}\}$, for each node ξ and for each promise $j \in J_{\xi^-}$. The *minimal repayment constraint* requires consumers to repay at least $M_{j\xi}\varphi^i_{j\xi^-}$, that is,

$$\psi^i_{j\xi} \leq (p_\xi A_{j\xi} - M_{j\xi})\varphi^i_{j\xi^-} \tag{3}$$

The right hand side of inequality (3) is the maximal default and the one that would be given under non-recourse. Utility penalties may discourage consumers from defaulting that maximal value. The coefficients of the utility penalty, linear on default, are given by $\tilde{\lambda}^i_{j\xi} = \frac{\lambda^i_{j\xi}}{p_\xi\, b_\xi}$, where $b_\xi \in \mathbb{R}^G_{++}$ is a *reference bundle.* The entire payoff of consumer i is

$$\Pi^i(x^i, \theta^i, \varphi^i, \psi^i; p, q, K) := \sum_{\xi \in D} v^i_\xi(\tilde{x}^i_\xi) - \sum_{\xi \in D \setminus \{\xi_0\}} \sum_{j \in J(\xi^-)} \tilde{\lambda}^i_{j\xi}[\psi^i_{j\xi}]^+$$

where $[a]^+ = \max\{a, 0\}$, for any $a \in I\!R$. Observe that, by the way the penalty is written, there is no need to impose a non-negativity constraint on ψ. Consumer i problem consists in maximizing Π^i subject to (1), (2) and (3) and the following non-negativity constraint

$$x^i, \theta^i, \varphi^i \geq 0 \tag{4}$$

Definition 1. *An equilibrium is a process* $(p, q, K, (x^i, \theta^i, \varphi^i, \psi^i)_{i \in I})$ *such that* $p_\xi > 0$ [3] *at any node* $\xi \in D$ *and verifying:*

(i) $\forall i \in I,\ (x^i, \theta^i, \varphi^i, \psi^i) \in \text{argmax } \Pi^i(x, \theta, \varphi, \psi; p, q, K)$ *subject to (1), (2), (3) and (4).*

(ii) $\sum_{i \in I}[x^i(\xi_0) + C(\xi_0)\varphi^i(\xi_0)] = \sum_{i \in I}\omega^i(\xi_0),$

[3] The reason why we require $p_\xi > 0$ to be an equilibrium condition has to do with the fact that the default penalty coefficient $\tilde{\lambda}^i_{j\xi} \equiv \frac{\lambda^i_{j\xi}}{p_\xi\, b_\xi}$ is only well defined in this case.

(iii) $\sum_{i\in I}[x^i_\xi + C_\xi \varphi^i_\xi] = \sum_{i\in I}[\omega^i_\xi + Y_\xi x^i(\xi^-) + Y_\xi C(\xi^-)\varphi^i(\xi^-)],\ \forall \xi \in D\setminus\{\xi_0\},$

(iv) $\sum_{i\in I}\theta^i = \sum_{i\in I}\varphi^i,$

(v) $\forall j, \xi \in D\setminus\{\xi_0\},\ p_\xi A_{j\xi}(1-K^j_\xi)\sum_{i\in I}\theta^i_j(\xi^-) = \sum_{i\in I}\psi^i_{j\xi}.$

3 Infinite horizon individual optimality.

3.1 Necessary conditions: Euler and transversality conditions

If agent i were optimizing over a finite horizon H, a plan $(\psi^{iH}_{j\xi}, \varphi^{iH}_{j\xi}, \theta^{iH}_{j\xi}, x^{iH}_{g\xi})$ that satisfies (1), (2) and (3) is optimal if and only if it satisfies the Kuhn-Tucker conditions for some non-negative multipliers together with some $d^i_{j\eta} \in [0,1]$ supergradient of the function $\max\{0,\cdot\}$ evaluated at $\psi^i_{j\xi}$. These conditions induce the analogous Euler conditions for the infinite horizon problem, as we report next.

Definition 2. *Given prices $(p, q.K)$ and a plan $Z^i := (x^i, \theta^i, \phi^i, \psi^i)$ that verifies at these prices the constraints (1), (2), (3) and (4),* we say that Z^i satisfies the Euler conditions at (p,q,K) *if there exist supergradients $(d^i_j)_{j\in J}$ of the function* $\max\{0,\cdot\}$ *evaluated at $\psi^i_{j\xi}$ and a non-negative process $(\gamma^i, (\rho^i_j,)_{j\in J})$ of multipliers such that, for any promise $j\in J$ and any node ξ, the following hold*

(i)

$$\tilde{\lambda}^i_{j\xi} d^i_{j\xi} + \rho^i_{j\xi} = \gamma^i_\xi \tag{5}$$

$$\begin{aligned}\gamma^i_\xi\Big(p_\xi C_{j\xi} - q^j_\xi\Big) - v^{i\,\prime}_\xi(\tilde{x}_\xi)C_{j\xi} \quad \geq \quad & \sum_{\eta\in\xi^+}\Big[\gamma^i_\eta\Big(p_\eta Y_\eta C_{j\xi} - M_{j\eta}\Big) \\ & -\tilde{\lambda}^i_{j\eta} d^i_{j\eta}\Big(p_\eta A_{j\eta} - M_{j\eta}\Big)\Big]\end{aligned} \tag{6}$$

$$\gamma^i_\xi q^j_\xi \geq \sum_{\eta\in\xi^+}\gamma^i_\eta K_{j\eta} p_\eta A_{j\eta} \tag{7}$$

$$\forall g\in G, \quad \gamma^i_\xi p_{g\xi} \geq v^{i\prime}_\xi(\tilde{x}_\xi, g) + \sum_{\eta\in\xi^+}\gamma^i_\eta p_\eta (Y_\eta)^g, \tag{8}$$

(ii) equalities in (6), (7) or (8) hold when $\varphi_{j\xi} > 0,\ \theta_{j\xi} > 0$ *or* $x_{g\xi} > 0$*, respectively.*
(iii) $\rho^i_{j\xi}[\psi^i_{j\xi} - (p_\xi A_{j\xi} - M_{j\xi})\varphi^i_{j\xi^-}] = 0$

As in any infinite horizon problem, Euler conditions are not the only necessary conditions for infinite horizon optimality, a transversality condition must also hold. For the problem described in Section 2, we say that a plan $(x^i, \theta^i, \varphi^i, \psi^i)$ satisfies the *transversality condition* at (p, q, K) when for $v^{i\prime}_\xi$ evaluated at $\tilde{x}^i$ we have

$$\limsup_T \sum_{\xi:\, t_\xi = T} \left(\gamma^i_\xi \left[p_\xi \tilde{x}^i_\xi - q_\xi(\theta^i_\xi - \varphi^i_\xi)\right] - v^{i\prime}_\xi \tilde{x}^i_\xi\right) \leq 0 \tag{9}$$

Proposition *Under assumption [U], if the plan* $(x^i, \theta^i, \phi^i, \psi^i)$ *is a maximizer of* $\Pi^i(x, \theta, \varphi, \psi)$ *subject to (1), (2), (3) and (4) at prices* (p, q, K)*, then this plan satisfies the Euler conditions and the transversality condition (9) at* (p, q, K)*.*

Remark 1.

Actually, under (5), (7) and (8), we have that (9) implies the following transversality condition specifically on borrowing,

$$\limsup_T \sum_{\xi:\, t_\xi = T} [\gamma^i_\xi (p_\xi C_\xi - q_\xi) - v^{i\prime}_\xi C_\xi]\varphi^i_\xi \leq 0 \tag{10}$$

The converse, (10) implying (9) might not hold[4]. See Appendix 6.1.

Remark 2.

Notice that any sequence of equilibria $(p^H, q^H, K^H, (x^{iH}, \theta^{iH}, \varphi^{iH}, \psi^{iH})_{i\in I})$ of economies with increasing finite horizon H has a cluster point $(p, q, K, (x^i, \theta^i, \varphi^i, \psi^i)_{i\in I})$ such that $(x^i, \theta^i, \varphi^i, \psi^i)$ satisfies Euler conditions and transversality condition (9) at (p, q, K), for each agent i (see Appendix 6.1).

3.2 A sufficient condition

However, Euler conditions together with the transversality condition (9) usually fail to be sufficient in infinite horizon optimization problems. This is the case for the optimization problem described in Section 2, as we will illustrate in the next section. A sufficient condition can be provided by adding to Euler conditions and

[4]It does if $\theta^i = 0$ and i does not consume any durable good in excess of the collateral bundle.

transversality condition (9) the requirement that all budget feasible plans should satisfy the converse to condition (10). More precisely,

Theorem 1 (Sufficient condition for optimality).
Let $(x^i, \theta^i, \varphi^i, \psi^i)$ be a plan for consumer i that satisfies at (p, q, K) constraints (1), (2), (3) and (4), together with Euler conditions and the transversality condition (9). Suppose that any promises sales trajectory $\hat{\varphi}$ which is part of a plan $(\hat{x}, \hat{\theta}, \hat{\varphi}, \hat{\psi})$ satisfying constraints (1), (2), (3) and (4) at (p, q, K) is such that for ${v^i_\xi}'$ evaluated at $x^i_\xi + C_\xi \varphi^i_\xi$ we have

$$\limsup_T \sum_{\xi:\, t_\xi = T} [{v^i_\xi}' C_\xi - \gamma^i_\xi (p_\xi C_\xi - q_\xi)]\hat{\varphi}_\xi \leq 0, \tag{11}$$

then, under assumptions [U], [E] and [R], the plan $(x^i, \theta^i, \varphi^i, \psi^i)$ is optimal for i at (p, q, K).

To put it in another way, as the horizon truncation goes to infinity, equilibria of finite horizon economies have a cluster point which is actually an equilibrium for the infinite horizon economy if (11) holds for any budget feasible plan.

In the next section we give some intuition on the role of condition (11) and provide an example where it does not hold and a plan satisfying Euler and transversality fails to be individually optimal. It is also an example where a limit of finite horizon equilibria is not an infinite horizon equilibrium.

3.3 Generalized Ponzi schemes.

In the absence of utility penalties, one period non-arbitrage implies that, at each node ξ and for each promise j_ξ, collateral costs $p_\xi C^{j_\xi}_\xi$ cannot be lower than the promise price q^{j_ξ}; this inequality rules out Ponzi schemes and guarantees existence of equilibrium for the infinite horizon economy (see Araujo, Páscoa and Torres-Martínez (2002)). When utility penalties are introduced this inequality does not follow anymore from non-arbitrage and Ponzi schemes may reappear (see Páscoa and Seghir (2009)). Furthermore, as we show now, even if penalties were low enough so that Ponzi schemes could be avoided (as will be the case in subsection 4.1), there might not exist optimal solutions to the consumers' problems.

Let us be more precise. A *Ponzi scheme*, consists in increasing the sale position in promise j at node ξ and then accommodate this by increasing the sale position in another promise at the following nodes. As shown in Páscoa and Seghir (2009), Section 4.1, a Ponzi scheme exists when there is a node $\tilde{\xi}$ such that at all nodes ξ in the sub-tree starting at $\tilde{\xi}$ we have $p_\xi C_\xi^{j_\xi} - q_\xi^{j_\xi} < 0$, for some promise j_ξ.

There may exist nevertheless an extended form of Ponzi schemes, compatible with $p_\xi C_\xi^{j_\xi} - q_\xi^{j_\xi} \geq 0$ holding for any promise j and at any node ξ. This consists in increasing the sale position in some promise j_ξ at nodes ξ where the marginal utility of collateral consumption outweighs the disutility resulting from the haircut $p_\xi C_\xi^{j_\xi} - q_\xi^{j_\xi}$. For such change to be budget feasible, that cost has to be compensated by the reduction in another expenditure, say a decrease in consumption of a perishable good. Now, the utility impact at nodes that immediately follow ξ cancels out the utility gain that occurred at ξ, provided the consumer was already shorting the promise j_ξ (so that the Euler condition on shorting holds as an equality). For this reason, the increase in a sale position in a promise at a certain node does not need to be related to what was the increase in a previous node, it just needs to be affordable by how much the perishable consumption may be cut.

However, for any finite horizon truncation T of such process, we are left with the gain that may occur at date T. In the open end setting, letting $T \to \infty$, there may be a persistent gain (analogous to the limiting gains occurring in Ponzi schemes done in the case of unsecured unbounded promises or in the case of secured recourse promises with a negative haircut).

By definition, the improvement consisting in a generalized Ponzi scheme is done in spite of the absence of finite horizon arbitrage opportunities (ruled out by Euler conditions) and, therefore, the set of nodes where short positions are being increased must be an infinite set. When a generalized Ponzi scheme can be done upon a cluster point of finite horizon equilibria, the latter is not an infinite horizon equilibrium. Let us give an example.

Example 1. Two consumers trade one promise in a deterministic setting. The promise pays in a perishable good, fruit, and is secured by a productive asset, fruit tree. The former is the numeraire and also the reference good in the real default penalty. We assume that the first consumer just cares about fruit, with linear preferences, while the second consumer has quasi-linear preferences, linear in fruit and strictly concave in the shade provided by the fruit tree. Formally, $U^{(1)}(x) = \sum_{t=1}^{\infty} \beta_1^t x_t$ and $U^{(2)}(x,z) = \sum_{t=1}^{\infty} \beta_2^t (x_t + n_t(z_t))$, where $n_t(.)$ is a strictly concave function to be specified below. The utility that agent 2 gets at time t

from the shade is $v_t(z_t) = \beta_2^t n_t(z_t)$. We have $\gamma_t^{(1)} = \beta_1^t$ when $x_t^{(1)} > 0$ and, for $(x_t^{(2)}, z_t^{(2)}) >> 0$, we have also $\gamma_t^{(2)} = \beta_2^t$. We construct finite horizon equilibria where both agents consume fruit at every date in spite of the linearity of preferences in fruit.

Fruit trees just last from one date to the next. Trees take one period to yield fruit, at a rate $y < 1$ that is constant over time, and then die. The transformation matrix Y has a first row given by $[0y]$ and a null second row. At each date, new fruit trees are born in the orchards of each consumer, in the amounts ω_t^i. *We assume that, for both consumers, the sequence $(\omega_t^i)_t$ converges and $\omega^i \ggg 0$.* Let p_t and q_t be the tree and the promise prices, at date t.

We assume $\beta_1 > \beta_2$ and that the default penalty coefficient σ of agent 2 is lower than one (implying $\tilde{\lambda}_t^{(2)} < \gamma_t^{(2)}$), so that this agent would always give maximal default when selling the promise. We suppose each unit of a promise traded at date t has a fruit yield $A_{t+1} = \eta y C_t$, with $\eta > 1$, so that the minimal delivery of the promise becomes $M_{t+1} = yC_t$. Let $K_{t+1} = yC_t/A_{t+1} = 1/\eta$.

We set the promise price equal to the willingness to pay of agent 1, $q_t = \beta_1 y C_t$, and also to the reservation price of agent 2 as a seller of the promise, $q_t = p_t C_t - n_t' C_t + \beta_2 \sigma_{t+1} d_{t+1}(A_{t+1} - yC_t)$. If the promise is actually traded, agent 1 will buy it and agent 2 will sell it. The price of a tree must satisfy $p_t \geq \beta_1 y$ and $p_t \geq \beta_2 y + n_t'(z_t^{(2)})$, holding with equalities if the collateral constraint $(z_t^{(i)} \geq C_t \varphi_t^{(i)})$ has null shadow value for the respective agent. Let us look for an equilibrium where the promise is traded and these shadow values are positive for agent 1 and null for agent 2. This is compatible with $z^{(1)}$ and $\varphi^{(1)}$ both zero, while agent 2 could be consuming trees in excess of the collateral requirement but actually will not, as we will see. Then, $\beta_1 y < p_t = \beta_2 y + n_t'(z_t^{(2)})$. This implies $p_t C_t - q_t = [n_t' - (\beta_1 - \beta_2)y]C_t > 0$.

Market clearing requires $\theta_t^{(1)} = \varphi_t^{(2)}$, $C_t \varphi_t^{(2)} = \omega_t^{(1)} + \omega_t^{(2)}$ and $x_t^{(1)} + x_t^{(2)} = (\omega_{t-1}^{(1)} + \omega_{t-1}^{(2)})y$. The promise short position is then given by $\varphi_t^{(2)} = (\omega_t^{(1)} + \omega_t^{(2)})/C_t$.

Our specification of C_t and n_t will allow for agent 2 to construct a generalized Ponzi scheme upon the limit $(p, q, K, \theta, \varphi, x)$ of finite horizon equilibrium plans. Suppose agent 2 increases the sale of the promise by α_t at each date and gives maximal default on α_t at the next date. The extra expenditure $(p_t C_t - q_t)\alpha_t$ is accommodated by decreasing the consumption of numeraire.

Claim 1(i): if $x^{(2)} \ggg 0$, then a bounded budget feasible sequence α of sales increases makes consumer 2 improve upon $(x^{(2)}, \varphi^{(2)})$ provided that the following condition holds

$$\limsup_T [v_T' C_T - \gamma_T^{(2)}(p_T C_T - q_T)]\alpha_T > 0 \tag{12}$$

In fact, the net utility gain at date t of the budget feasible increase in φ_t by α_t (accommodated by decreasing $x^{(2)t}$) has a first order approximation given by $[v_t' C_t - \gamma_t^{(2)}(p_t C_t - q_t)]\alpha_t \equiv B_t$, where v_t' is evaluated at $C_t\varphi_t^{(2)}$. Now, since agent 2 has the Euler condition on sales holding with equality, this gain will cancel out with the utility impact (including penalty impact) that such date t changes will have on date $t+1$ utility, $[\gamma_{t+1}^{(2)}(yC_t - M_t) - \tilde{\lambda}_t^{(2)} d_t(A_{t+1} - M_{t+1})]\alpha_t \equiv \tilde{B}_{t+1}$.

Then, up to date T the accumulated utility gain has a first order approximation given by $[v_T' C_T - \gamma_T^{(2)}(p_T C_T - q_T)]\alpha_T$. The choice for α is determined by how much the consumption of the numeraire can be reduced. If $x_t^{(2)} \ggg 0$, we can reduce it by $sx_t^{(2)}$ at each date, for any $s \in (0,1)$ and make $\alpha_t = sx_t^{(2)}/[p_t C_t - q_t]$. Hence, as $T \to \infty$, the first order estimate of the utility gain remains positive if (12) holds.

To be more precise, let $\hat{U}^{(2)}(\varphi) \equiv U^{(2)}(x(\varphi), C\varphi)$, where $x_t(\varphi) = p_t\omega_t^{(2)} - (p_t C_t - q_t)\varphi_t + (yC_{t-1} - M_t)\varphi_{t-1}$. The above estimate of the utility gain is actually the right-hand-side directional derivative $\delta^+\hat{U}^{(2)}(\varphi^{(2)}; \alpha)$ of $\hat{U}^{(2)}$ along the direction α if α is a bounded sequence[5]. Agent 2 can improve upon if $\delta^+\hat{U}^{(2)}(\varphi^{(2)}; \alpha) > 0$. □

Claim 1(ii): *α is a bounded sequence if $C_t = 1/\beta_2^t$ and $n_t(z_t) = (\beta_1 - \beta_2)yz_t +$* $\beta_2^t\sqrt{z_t}$ (that is, the instantaneous utility from fruit trees is becoming less strictly concave as times goes by).

In fact, α is bounded if $(p_t C_t - q_t)_t \ggg 0$, that is, if $((n_t' - (\beta_1 - \beta_2)y)C_t)_t \ggg 0$. Now, $p_t C_t - q_t = \beta_2^t C_t/[2\sqrt{\omega_t^{(1)} + \omega_t^{(2)}}]$, where ω^i are bounded sequences. □

[5] $U^{(2)}$, being (l^∞, l^1) Mackey continuous, has at $(x^{(2)}, C\varphi^{(2)}) \ggg 0$ a Gateaux derivative in l^1 (see Lemma 1 in Araujo, Novinski and Pascoa (2011)) and, by composition with a linear map, $\hat{U}^{(2)}$ has a Gateaux derivative $D\hat{U}^{(2)}(\varphi^{(2)}) \in l^1$ at $\varphi^{(2)}$. If $\alpha \in l^\infty$, then $\delta^+\hat{U}^{(2)}(\varphi^{(2)}; \alpha) \equiv \lim_{h\to 0+}[\hat{U}^{(2)}(\varphi^{(2)} + h\alpha) - \hat{U}^{(2)}(\varphi^{(2)})]/h$ exists and is equal to $D\hat{U}^{(2)}(\varphi^{(2)})\alpha = \sum_{t=1}^{\infty}(B_t + \tilde{B}_{t+1}) = \lim_T [v_T' C_T - \gamma_T^2(p_T C_T - q_T)]\alpha_T$.

Claim 1(iii): *inequality (12) holds, for* $x^{(2)} \ggg 0$ *and under the assumptions in 1.(ii)* (where the limsup is actually the limit, due to 1(ii)).

This requires $\lim_{t\to\infty} \frac{s x_t^{(2)}(\beta_1-\beta_2)y\beta_2^t}{n_t'-(\beta_1-\beta_2)y} > 0$, that is, $\lim_{t\to\infty} s x_t^{(2)}(\beta_1-\beta_2)y2\sqrt{\omega_t^{(1)}+\omega_t^{(2)}} > 0$, which holds if $\omega^{(i)} \ggg 0$ for $i = 1, 2$ and $x^{(2)} \ggg 0$. □

Claim 1(iv): $x^{(2)} \ggg 0$*, under the assumptions in 1.(ii).*

In fact, $x_t^{(2)} = p_t\omega_t^{(2)} - (p_tC_t - q_t)\varphi_t^{(2)}$ where p_t tends to $\beta_1 y$, while φ_t goes to zero and $p_tC_t - q_t = 1/[2\sqrt{\omega_t^{(1)}+\omega_t^{(2)}}]$, where $\omega_t^{(i)} \ggg 0$. □

Actually, the example illustrates more than a failure of cluster points of finite horizon equilibria to become infinite horizon equilibria. It illustrates that a plan satisfying Euler and transversality conditions may fail to be individually optimal.

In fact, in Example 1, the portfolio $\varphi^{(2)} + \alpha$ together with the perishable good consumption $x^{(2)}(1 - s)$ satisfy budget constraints at prices (p, q) and $K_t = 1/\eta$. Notice that $\varphi^{(2)}$ satisfies the transversality condition $\lim_{t\to\infty}[v_t'C_t - \gamma_t^{(2)}(p_tC_t - q_t)]\varphi_t^{(2)} = 0$, since $v_t'C_t - \gamma_t^{(2)}(p_tC_t - q_t) = (\beta_1 - \beta_2)y$ and $\lim_{t\to\infty} \varphi_t^{(2)} = 0$. Then (11) fails for $\hat{\varphi} = \varphi^{(2)} + \alpha$.

Examples 1 and 2 in Páscoa and Seghir (2009) illustrated why finite horizon equilibra may fail to induce infinite horizon equilibria for another reason. Utility penalties were high enough to discourage maximal default (there is no default in the former and default below the maximal one in the latter). An equality in (6) for the short, made the haircut $p_tC_t - q_t$ become negative in finite horizon equilibria with trade (as shown in steps I and A of those examples, respectively). That allowed for a Ponzi scheme upon a cluster of finite horizon equilibria *with trade.* The novelty in the example we just described is that a negative haircut is not necessary to allow the short to improve upon such cluster point.

If utility penalties were absent, generalized Ponzi schemes could never be done. Constituting collateral and short-selling generates in this case non-negative returns $(\sum_{\eta\in\xi^+} \gamma_\eta^i\Big(p_\eta Y_\eta C_\eta^j - M_{j\eta}\Big))$ which, by non-arbitrage (see (6)), must induce a non-negative promise cost $\gamma^i_\xi\big(p_\xi C^{j_\xi}_\xi - q^{j_\xi}_\xi\big) - {v^i_\xi}' C^{j_\xi}_\xi$ (net of utility gains).

In Sections 5 we present existence results in contexts where (11) holds.

4 Existence results

4.1 Moderate penalties.

Let $r^i_\xi(b_\xi)$ be the minimum of the derivative $(v^i_\xi)'(z)b_\xi$, of v^i_ξ along the direction of the reference bundle b_ξ (used in the definition of penalties), taken over all feasible bundles z. This minimum is well defined, by Lemma 3 in Appendix 6.2.

Theorem 2 (Moderate penalties). *Under assumptions [R], [E] and [U], if for every promise $j \in J$ we have (a) $\lambda^i_{j\xi} < r^i_\xi$, then $p_\xi C_\xi \geq q_\xi$ in equilibrium of finite-horizon economies and Ponzi schemes, in stricto sensu, are avoided. Equilibrium for the infinite-horizon economy exists if, in addition, (b) the collateral does not yield utility (say, it is a productive asset or a security in positive net supply that cannot be short sold).*

Our moderate penalties assumption is in marginal terms (compares penalty coefficients and marginal utilities), whereas the moderation assumption contemplated in Páscoa and Seghir (2009) was in total terms: for each node ξ and each agent i, it assumed (1) $\tilde{\lambda}^i_{j\eta}[p_\eta A_{j\eta} - M_{j\eta}]\varphi^i_{j\xi} < v^i_\eta(\omega^i_\eta),\ \forall \eta \in \xi^+$, whenever (2) $C^j_\xi \varphi^i_{j\xi} \leq \sum_i W^i_\xi$. That total terms condition only suffices to get existence of equilibrium in infinite horizon economies, if promises sales plans are required to satisfy (2) as a borrowing constraint (alternatively, (1) alone should be imposed). In fact, condition (11) holds in this case since $\limsup_T \sum_{\xi:\, t_\xi = T} \Big({v^i_\xi}'(\overline{Z}^i) C_\xi - \gamma^i_\xi \left(p_\xi C_\xi - q_\xi\right)\Big)\varphi_\xi < \limsup_T \sum_{\xi:\, t_\xi = T} v^i(\omega^i)$, which is zero since $U^i(\omega^i) < \infty$.

4.2 Equilibrium without maximal default.

However, the above low penalties, implying maximal default when the promises are traded, are not necessary for equilibrium existence. Partial default or even full repayment are compatible with equilibrium and may occur under higher penalty coefficients. In fact, generalized Ponzi schemes are obviously avoided when, for all i and all j, we have $\gamma^i_\xi(p_\xi C_{j\xi} - q^j_\xi) - {v^i_\xi}' \cdot C_{j\xi} \geq 0,\ \forall i$, at all nodes far away in the event tree (as this implies inequality (11) in Theorem 1. By (6) it suffices to have, for all i, all ξ and all j, the following

$$\sum_{\eta\in\xi^+}\gamma_\eta^i(p_\eta Y_\eta C_{j\xi}-M_{j\eta})-\sum_{\eta\in\xi^+}\tilde{\lambda}_{j\eta}^i d_{j\eta}^i(p_\eta A_{j\eta}-M_{j\eta})\geq 0,\ \forall i,\ \forall\xi, \tag{13}$$

where $d_{j\eta}^i$ satisfies (5). We can state condition (5) in a more suggestive way. In market parlance, *the consumer's home equity* is the difference between the collateral liquidation value and the repayment due on the loan. Let $EQ_\eta^j := p_\eta Y_\eta C_{j\xi} - p_\eta A_{j\eta}$. Home equity is linear on φ_j^i with an impersonal coefficient EQ_η^j that determines what the sign of home equity will be. We name EQ_η^j *the equity per unit of promise j at node η.*

Condition (13) can be equivalently written as follows

$$\sum_{\eta\in\xi^+}\gamma_\eta^i[EQ_\xi^j]^+\geq\sum_{\eta\in\xi^+}\tilde{\lambda}_{j\eta}^i d_{j\eta}^i[EQ_\xi^j]^- \tag{14}$$

which says that, for each promise j and summing over all immediate successors of node ξ, the marginal utility gains from positive per unit equity should outweigh default penalties on negative per unit equity. In the non-recourse case, the former occurred exclusively and were responsible for $p_\xi C_\xi^j$ never being below than $q_{j\xi}$, which ruled out Ponzi schemes.

Moreover, by (5), we see that (13) holds if (but not only if)

$$\sum_{\eta\in\xi^+}\gamma_\eta^i EQ_\xi^j\geq 0,\ \forall i,\ \forall\xi \tag{15}$$

is satisfied for all i and for all ξ. Moreover, when the collateral does not yield utility gains, it is enough to have the inequality in (13) (or in (15)) satisfied, at each node ξ, for some agent i_ξ, as this implies $p_\xi C_{j\xi}\geq q_\xi^j$.

The difficulty is that condition (15) depends on relative spot prices p_η and on the marginal utilities of income γ_η^i and, in general, it is not possible to guarantee that the market clearing spot prices (and the induced multiplier γ^i) are such that (15) is satisfied, for an arbitrary combination of returns (A^j) and collateral yields $Y\,C^j$. Let us give, nevertheless, an example where (15) holds for arbitrary penalty coefficients. This example will motivate our next result.

Example 2 (partial default or full repayment) There are two infinite-lived agents, the event-tree has two branches at each node ξ (up (u_ξ) and down (d_ξ)). There is one consumption good and preferences are given by $U^i(Z) = \sum_{\xi \in D} \beta^{t_\xi} \nu^i_\xi Z_\xi$, where $\nu^i_{u_\xi} + \nu^i_{d_\xi} = \nu^i_\xi$, $\sum_{\xi : t_\xi = t} \nu^i_\xi = 1, \ \forall t$. There is one promise paying in the consumption good and using as collateral a real security (or a productive asset) that is short-lived but is issued (or endowed) at each node. Formally, this collateral instrument can be treated as a second commodity that transforms into the consumption good at the next date and then disappears. Denote by a_ξ the promised returns and by y_ξ the collateral yields. The collateral coefficient is $C_\xi = 1$ and we take the perishable cumption good ($g = 1$) as the numeraire.

Let the reference bundle in the penalty function be $b_\xi = (1, 0)$. Each agent i has a penalty coefficient σ^i_ξ to be specified below. The penalty is then given by $\sum_{\xi \in D} \beta^{t_\xi} \sigma^i_\xi \nu_\xi [a_\xi \, \varphi_{\xi^-} - \Delta_\xi]^+$, where Δ_ξ stands for the delivery ($\Delta_\xi = a_\xi \varphi_{\xi^-} - \psi_\xi$).

Given endowments $\omega^i_\xi = \left(\omega^i_{1\xi}, \omega^i_{2\xi}\right)$ of the consumption good and the collateral instrument, we write consumers' constraints as usual, denoting by p_ξ the collateral price and by q_ξ the promise price. Suppose $\nu^{(1)}_{u_\xi} = \nu^{(1)}_{d_\xi} = \frac{1}{2}^{t_\xi + 1}$, whereas $\nu^{(2)}_{u_\xi} = \frac{2}{3} \nu^{(2)}_\xi$ and $\nu^{(2)}_{d_\xi} = \frac{1}{3} \nu^{(2)}_\xi$. We will construct equilibria where both agents consume the perishable good at every node, which implies that $\gamma^i_\xi = \beta^{t(\xi)} \nu^i_\xi$

Claim 2(i): If $a_{u_\xi} = 2, \ a_{d_\xi} = 1, \ y_{u_\xi} = 1$ and $y_{d_\xi} = 2, \ \forall \xi$, then a cluster point of finite horizon equilibria is an infinite horizon equilibrium.

In fact, the equity at each node is such that (15) holds with equality for agent 1 (and therefore (13) holds with equality for this agent, for any penalty coefficients $\sigma^{(1)}_\xi$). As the collateral does not yield utility, (11) holds for both agents. □

Let us look for equilibrium prices and delivery rates for some possible configurations of default penalties of the two agents. For agent 2, we assume $\sigma^{(2)}_\xi \geq 1$ (i.e.: $\lambda^{(2)}_\xi \geq \gamma^{(2)}_\xi, \ \forall \xi$) and, for both agents, we take $\rho^{(i)}_\xi = 0$, implying that (5) holds for $d^{(i)}_\xi = \frac{1}{\sigma^{(i)}_\xi}$. Observe that if $\beta \leq 2/3$ then (8) holds, for both agents.

Suppose first that $\sigma^{(1)}_\xi = 1$ (i.e.: $\lambda^{(1)}_\xi = \gamma^{(1)}_\xi$) $\forall \xi$. Then $K_{u_\xi} = 0.9, \ p_\xi = q_\xi = \frac{4.6}{3} \beta$ and $K_{d_\xi} = 1$ satisfy Euler conditions (5) through (7), with agent 1 on-the-verge of selling and agent 2 on-the-verge of buying.

If $\sigma_\xi^{(1)} > 1$ instead (i.e.: $\lambda_\xi^{(1)} > \gamma_\xi^{(1)}$) $\forall\xi$, we see that $K_{u_\xi} = K_{d_\xi} = 1$ and $p_\xi = q_\xi = \frac{5}{3}\beta$ satisfy Euler conditions (5) through (8), with agent 1 on-the-verge of selling and agent 2 on-the-verge of buying.

It remains to specify agents' endowments and construct the equilibrium allocation of consumption plans and portfolios. Taking $\omega_\xi^{(1)} = (1,0)$ and $\omega_\xi^{(2)} = (1,w)$, $\forall\xi$, let $\theta_\xi^{(2)} = w$, $\varphi_\xi^{(1)} = w$, $\theta_\xi^{(i)}\varphi_\xi^{(i)} = 0$ and $x_{2\,\xi}^{(i)} = 0$ (no purchase of commodity 2 beyond what might be used as collateral).

Claim 2(ii): if $w < 1$, we can accommodate both $\sigma_\xi^{(1)} = 1$ and $\sigma_\xi^{(1)} > 1$ in equilibrium.

In the first case, where $\sigma_\xi^{(1)} = 1$, we obtain $\Delta_{u_\xi}^{(1)} = 0.9a_{u_\xi}\,w = 1.8w$, $\Delta_{d_\xi}^{(1)} = a_{d_\xi}\,w = w$. Take $x_{1\xi}^{(i)} = \omega_{1\xi}^{(i)} + Y_\xi C_{\xi^-}\varphi_{\xi^-}^i - \Delta_\xi^{(i)} + K_\xi a_\xi\,\theta_{\xi^-}^i$. Then, $x_{1\,u_\xi}^{(1)} = 1 - 0.8w$, $x_{1\,d_\xi}^{(1)} = 1 + w$, $x_{1\,u_\xi}^{(2)} = 1 + 1.8w$, $x_{1\,d_\xi}^{(2)} = 1 + w$. Market clearing follows ($\sum\limits_i x_{1\xi}^i = \sum\limits_i \omega_{1\xi}^i + y_\xi\,w$) and we assume $w < 1.25$ to obtain an equilibrium.

In the second case, where $\sigma_\xi^{(1)} > 1$, $\forall\xi$ (that is $\lambda_\xi^{(1)} = \gamma_\xi^{(1)}$ $\forall\xi$), the equilibrium allocation is given by the same promise allocation, $\Delta_{u_\xi}^{(1)} = 2w$, $\Delta_{d_\xi}^{(1)} = w$, $x_{1\,u_\xi}^{(1)} = 1 - w$, $x_{1\,d_\xi}^{(1)} = 1 + w$, $x_{1\,u_\xi}^{(2)} = 1 + 2w$, $x_{1\,d_\xi}^{(2)} = 1 + w$. □

Remark *on the result by Ferreira and Torres-Martinez (2010) on impossibility of recourse.*

Example 2 shows that recourse can actually occur in infinite horizon equilibria. The borrower (agent 1) repays more than the minimum between the promise and the collateral value. But recourse does not open up room for Ponzi schemes: the haircut does not become negative, it is just zero.

This seems to collide with the claim by Ferreira and Torres-Martinez (2010) that recourse is impossible in infinite horizon equilibria but a closer look shows that their assumption on collateral bounds is not satisfied.

As in Ferreira and Torres-Martinez (2010), we write $K_{j\xi} = M_{j\xi} + Q_{j\xi}(p_\xi A_{j\xi} - Y_\xi C_{j\xi^-})^+$, where $Q_{j\xi} \in [0,1]$ measures the repayment above the minimal one (that is, the degree of recourse). Ferreira and Torres-Martinez (2010) showed that the haircut $p_\xi C_{j\xi} - q_{j\xi}$ becomes negative if $\sum_g C_{jg\xi} << \sum_{\eta\in\xi^+} Q_{j\xi}\underline{\pi}_\eta A_{j\eta}/\overline{\pi}_\eta \ \equiv \Psi_\xi$, where $\underline{\pi}_{g\xi}$ and $\overline{\pi}_\xi$ are lower and upper (over all goods) bounds, respectively, for $\gamma^i_\xi p_{g\xi}$.

In Example 2, there is just one promise and two cases. In case 1 (for $\sigma^{(1)} = 1$) we have $Q_{u\xi} = 0.8$ while $Q_{d\xi}$ takes any value in $[0, 1]$. In case 2 (when $\sigma^{(1)} > 1$) we have $Q_{u\xi} = 1$ while $Q_{d\xi}$ takes any value in $[0, 1]$. We actually know what $\gamma^i_\xi p_{g\xi}$ is. For the numeraire (the perishable good), it is $\beta^{t(\xi)}\nu^i_\xi$ while for the commodity that serves as collateral it is $\beta^{t(\xi)}\nu^i_\xi p_\xi$. Let $\Psi^i_\xi \equiv \sum_{\eta\in\xi^+} Q_{j\xi}\gamma^i_\eta A_\eta/\gamma^i_\xi p_\xi \geq \Psi_\xi$. Even for $Q_{d\xi} = 1$, we see that in case 1, $\Psi^{(2)}_\xi = 4.2/4.6$ while in case 2, $\Psi^{(2)}_\xi = 1$. As $C = 1$, it follows that the condition in Ferreira and Torres-Martinez (2010) is not satisfied. The haircut is actually zero and Ponzi schemes cannot be done (neither can generalized ones as the collateral does not yield any utility).

4.3 Nominal contracts.

The above example where both the promise and the collateral are numeraire assets, could be redone with both being nominal assets (say, the promise is a loan, with exogenous yields, whose purpose is the purchase of a bond). This leads us to study what happens when this promise or the collateral are nominal assets. In both cases, collateralized borrowing is not inflation proof.

Formally, we replace assumption [R] by

Assumption [N].
We allow for nominal promises or nominal collateral. A promise j not satisfying items (i) and (ii) of [R] is such that

(i) its returns are nominal given by $b^j_\eta \in \mathbb{R}_+$ at $\eta \in \xi^+$, for $j \in J$.

(ii) its collateral may be real as in [R (ii)] or nominal. In the latter, the collateral requirement at $\eta \in \xi^+$ is $c^j_\eta \in \mathbb{R}_{++}$ and the collateral has an exogenous nominal return $\tilde{y}^j_\eta \in \mathbb{R}_{++}$ at any node $\eta \in \xi^+$.

As usual, given a promise with nominal returns b^{j_ξ}, we let $A_{j\xi} = \frac{b^j_\xi}{S_\xi}\, \mathbb{1}$ where S_ξ stands for $\|p_\xi\|_1$ and $\mathbb{1} = (1, ..., 1)$. Analogously, in the case of nominal collateral, we make $C_{j\xi} = \frac{\tilde{y}^j_\xi}{S_\xi}\, \mathbb{1}$. Equilibrium is still given by Definition 1.

Recall that for unsecured nominal assets, we had a homogeneity of commodity demand with respect to $(S_\eta)_{\eta\in\xi^+}$: if we multiply S_η by $\alpha > 0$, $\forall\eta \in \xi^+$, and adjust the portfolio (multiplying by α) and asset prices (dividing by α), we can maintain the original bundle at the same relative spot prices. However, that homogeneity does not hold for promises secured by exogenous collateral requirements.

The indeterminacy with respect to inflation rates[6], at finite horizon equilibria with nominal promises or nominal collateral, may allow us to pick an equilibrium where marginal penalty effects $\left(\tilde{\lambda}^i_{j\eta} d^i_{j\eta}[EQ^j_\eta]^-\right)$ may become dominated by marginal income effects $\left(\gamma^i_\eta [EQ^j_\eta]^+\right)$. If that is the case, (13) holds (and, therefore, (11) holds at the cluster point)[7].

Theorem 3. *Let $J^\star$ be the set of promises for which assumptions (a) or (b) of Theorem 2 fail. Under assumptions [E] and [U], equilibrium exists, if every $j \in J^*$ is a nominal promise backed by real collateral, as in [N].*

For a nominal promise j backed by real collateral, (13) holds if:

$$\sum_{\eta\in\xi^+} S_\eta^{-1} \max\{\lambda^i_{j\eta}, \gamma^i_\eta\}\, b^j_\eta \le \sum_{\eta\in\xi^+} \min\{\lambda^i_{j\eta}, \gamma^i_\eta\}\, p_\eta Y_\eta C^j_\eta,\ \forall i,\ \forall\xi \tag{16}$$

Theorem 3 allows for direct utility gains from collateral in the case of nominal promises backed by real collateral, by showing that (16) holds for every agent. If the inequality in (16) held for just one agent and there were no utility gains from collateral, then the condition in Theorem 1 would still be verified and there would exist an equilibrium for the infinite horizon economy. The next example illustrates this case, actually in an economy where the nominal promise/real collateral contract coexists with a nominal promise/nominal collateral contract with endogenous margins.

For a nominal promise backed by a nominal collateral, (13) holds if:

$$\sum_{\eta\in\xi^+} S_\eta^{-1} \max\{\lambda^i_{j\eta}, \gamma^i_\eta\}\, b^j_\eta \le \sum_{\eta\in\xi^+} S_\eta^{-1} \min\{\lambda^i_{j\eta}, \gamma^i_\eta\}\, \tilde{y}^j_\eta\, C_{j\xi},\ \forall i,\ \forall\xi \tag{17}$$

[6] We are not interested in checking whether the degree of freedom in the choice of inflation rates implies real indeterminacy of equilibria.

[7] Given a nominal promise b^{j_ξ}, the condition that ruled out recourse in Ferreira and Torres-Martinez (2010) will not hold for the real returns analog $A_{j\xi} = \frac{b^j_\xi}{S_\xi}\, \mathbb{1}$, for an appropriate choice of inflation rates $S_\xi = \|p_\xi\|_1$.

Example 3 For the economy of Example 2 take agent (1) and the pair of contracts: one nominal-real with $b^1_{u_\xi} = 1,\ b^1_{d_\xi} = 2$ and $Y_{u_\xi} = (1,0),\ Y_{d_\xi} = (1,0),\ C^1 = (0,1)$ and another nominal-nominal with $b^2_{u_\xi} = 1,\ b^2_{d_\xi} = 3$ and $\tilde{y}_{u_\xi} = \tilde{y}_{d_\xi} = 1,\ c^2$ to be determined. For $\lambda^{(1)}_{j\eta} = \beta^{t_\eta}(\frac{1}{2})^{t_\eta}\sigma^{(1)\,j}_\eta$, let $\sigma^{(1)\,j}_{u_\xi} = 2$ and $\sigma^{(1)\,j}_\eta = 1$ otherwise $(j = 1,\ 2)$.

Claim 3(i): conditions (16) and (17) hold (for $j = 1$ and $j = 2$ respectively), with an exogenous collateral requirement for $j = 1$ and endogenous ones for $j = 2$.

These conditions are:

$$2S^{-1}_{u_\xi}b^1_{u_\xi} + S^{-1}_{d_\xi}b^1_{d_\xi} \leq Y_{u_\xi} + Y_{d_\xi},$$

$$2S^{-1}_{u_\xi}b^2_{u_\xi} + S^{-1}_{d_\xi}b^2_{d_\xi} \leq (S^{-1}_{u_\xi}\,\tilde{y}_{u_\xi} + S^{-1}_{d_\xi}\,\tilde{y}_{d_\xi})\,c^2.$$

Holding as equalities for $S^{-1}_{u_\xi} = S^{-1}_{d_\xi} = 0.5$ and $c^2 = 2.5$, implying that at u_ξ, both promises are above collateral values (with opportunity for default, which will not be used as $\lambda^{(1)j}_{u_\xi} > \gamma^{(1)}_{u_\xi}$) while at d_ξ the first promise matches the collateral values whereas the second one falls below it. □

5 Concluding remarks

Non-recourse loans have the beauty of eliminating Ponzi schemes and, therefore, the infinite horizon economy has an equilibrium under the same costless assumptions that made the finite horizon economy avoid the well-known Hart's problem. There are however many credit contracts that are recourse and arguments that may explain why for some particular contracts recourse is more appealing, but we should ask ourselves why do they hold on in an open end framework. Our paper addresses this question.

Apart from the cases of mortgages in Europe and in most (37) of the U.S. states, there are other examples of recourse collateralized loans. The most important are the security financing transactions (SFT), which take either the form of repo or security lending. In the former, a security serves as collateral for a cash loan (possibly for the purchase of the security itself), whereas in the latter a security is being lent against a collateral that can be either cash or another security. In both types of SFT, failure to redeliver the lent object constitutes an event of default and triggers bankruptcy. More precisely, the lender of cash in repo or the lender of the security who are both holding the collateral can dispose of it (there is no automatic stay) and then the remaining value of the loan (not covered by the current collateral value) will be claimed from the defaulter's liquidated estate. There was only a brief exception to this, when the Fed allowed repo to be non-recourse in a short period in the aftermath of Lehman Brothers bankruptcy.

Our model is quite general and abstracts from institutional details that different recourse loans may have[8] . It is however a first step towards understanding why is recourse borrowing compatible with open end equilibrium (where either successive term loans become chained or an open end loan is present). We focus on the case of recourse due to the presence of a utility penalty on default. This case tries to capture non-explicit pecuniary, reputational or credit access penalties and is sometimes regarded as an approximate proxy for more elaborate forms of recourse. Our results show that while open end equilibrium does not exist in the same straightforward way as it did in non-recourse, there are nevertheless interesting cases, relevant for observed recourse contracts, where equilibrium exists, such as the case of moderate utility penalties combined with non-consumed collateral (as in the above STFs or in loans for the purchase of equipment) and the case of nominal promises backed by real collateral (as in most mortgages, where payments are not indexed to commodity prices).

Moreover, in our work, recourse is not just an ex ante scenario. We illustrate (in Example 2) how harsh utility penalties induce actual recourse (debts repayments above the minimal repayment) while still allowing for non-negative haircuts, as desired to avoid Ponzi schemes, and equilibrium is shown to exist. We illustrate also (in Example 1) why absence of such schemes is not enough to ensure infinite horizon equilibrium when there are direct utility gains from the consumption of the collateral.

[8] For finite horizon general equilibrium models that capture these institutional details, see Araujo and Páscoa (2002) on recourse and unsecured loans, and Poblete-Cazenave and Torres-Martinez (2013) on limited recourse and secured loans.

6 Appendix

6.1 On Section 3

Proof of Proposition 1:

For each node ξ we define the Lagrangian function for agent i as:

$$L^i_\xi(Z_\xi, Z_{\xi^-}, \gamma, p, q, K) = v^i_\xi(\tilde{x}_\xi) - \sum_j \tilde{\lambda}^i_{j\xi}[\psi^i_{j\xi}]^+$$

$$-\gamma_\xi[p_\xi(\tilde{x}^i_\xi - \omega^i_\xi - Y_\xi \tilde{x}^i_{\xi^-}) + q_\xi(\theta^i_\xi - \varphi^i_\xi) - \sum_j \psi^i_{j\xi} - \sum_j p_\xi A_{j\xi}(K_{j\xi}\theta^i_{j\xi^-} - \varphi^i_{j\xi^-})]$$

$$-\sum_j \rho^j_\xi[\psi^i_{j\xi} - (p_\xi A_{j\xi} - M_{j\xi})\varphi^i_{j\xi^-}].$$

For $L^i_{\xi_0}$ to be well defined, we set $Z_{\xi_0^-} = (0,0,0,0)$.

(1) The claim in Proposition 1 on *Euler conditions* is as in Páscoa and Seghir (2009) and can be proven using the Kuhn-Tucker conditions of finite horizon truncated problems and making the horizon go to infinity (as in Araujo, A., M.R. Páscoa and J.P. Torres-Martínez, 2011). In fact, for each node, the sequence of Kuhn-Tucker multipliers has a cluster point, as the next lemma establishes.

A finite horizon H *truncated problem* is defined by imposing on the optimization problem described in Section 2 the additional constraints $(\theta_\xi, \varphi_\xi) = 0$ for $t(\xi) \geq H$ and $(x_\xi, \psi_\xi) = 0$ for $t(\xi) > H$.

Let us start by recalling the saddle point property (see Rockafellar (1997), Theorem 38.3). For any finite horizon H truncated problem, at an optimal plan (Z^{iH}, p, q, K) and for any non-negative plan $(Z_\xi)_{\xi\in D^H}$ we have

$$\sum_{\xi\in D^H} L^i_\xi(Z_\xi, Z_{\xi^-}, \gamma^{iH}, p^H, q^H, K^H) \leq \Pi^i(Z^{iH}; p, q, K) \tag{18}$$

By appropriately choosing the plan $(Z_\xi)_{\xi\in D^H}$ we get the following result.

Lemma 1. *For each node $\xi \in D$ and for any economy with finite horizon $H \geq t_\xi$, one has:* $0 \leq \gamma^{iH}_\xi < \frac{U^i(\Omega)}{\underline{W}^i_\xi \|p_\xi\|_1}$.

Proof: For $t \leq H$, let $Z = (Z_\xi)_{\xi \in D^H}$ be such that $Z_\xi = \left(\mathcal{W}^i_\xi, 0, 0, 0\right)$ if $\xi \in D^{t-1}$ and $Z_\xi = 0$ otherwise. By (18) we get

$$\sum_{\xi \in D^t} L_{\xi^i}(Z^{iH}_\xi, Z^{iH}_{\xi^-}, \gamma^{iH}_\xi, p^H, q^H, K^H) \leq \sum_{\xi \in D^H} v^i_\xi(\tilde{x}^{iH}_\xi). \tag{19}$$

Hence, $\sum\limits_{\xi \in D_t} \gamma^{iH}_\xi p^H_\xi \mathcal{W}^i_\xi \leq \sum\limits_{\xi \in D^H} v^i_\xi(\tilde{x}^{iH}_\xi)$, where $v^i_\xi(\tilde{x}^{iH}_\xi) \leq v^i_\xi(\Omega_\xi)$. □

It follows from Lemma 1 that multipliers γ^{iH}_ξ have upper bounds $\overline{\gamma}^i_\xi$ that are independent of the terminal horizon H of the economy, since $\underline{W}^i_\xi > 0$ by Assumption [E]. Moreover, it follows from equation (5) that ρ^{iH}_ξ also has an upper bound independent of H. Letting $H \to \infty$, we can find cluster points, for the product topology of the countable event tree, of the sequences $(\gamma^{iH}_\xi, \rho^{iH}_\xi, (Z^{iH}_\xi)_i)$. Denote these cluster points by $(\gamma^i_\xi, \rho^i_\xi, (Z^i_\xi)_i))$. We still have $\gamma^i_\xi < \frac{U^i(\Omega)}{\underline{W}^i_\xi \|p_\xi\|_1}$.

The fulfillment of Euler conditions at an optimal plan for the infinite horizon problem can then we established as in Araujo, A., M.R. Páscoa and J.P. Torres-Martínez (2011), proof of item (i) of Proposition 1.

(2) The fulfillment of *the transversality condition* (9) follows from the saddle point property:

Claim: $\sum\limits_{\xi:\, t_\xi = T} \sum\limits_{\eta \in \xi^+} L^i_{2\,\eta}(\overline{Z}^i)\overline{Z}^i_\xi \leq \sum\limits_{\xi \in D \setminus D^{T-1}} v^i_\xi(\overline{Z}^i_\xi)$.

This claim can be established using (18) where for $T \leq H$ we let $(Z_\xi)_{\xi \in D^T}$ be such that $Z_\xi = \overline{Z}^{iH}_\xi \chi_{D^{T-1}}(\xi)$. Then $\sum\limits_{\xi:\, t_\xi = T} \sum\limits_{\eta \in \xi^+} L^i_{2\,\eta}(\overline{Z}^{iH})\overline{Z}^{iH}_\xi + \sum\limits_{\xi \in D^H \setminus D^{T-1}} \gamma^{iH}_\xi p_\xi \omega^i_\xi \leq \sum\limits_{\xi \in D^H \setminus D^{T-1}} (v^i_\xi(\overline{Z}^{iH}_\xi) - v^i_\xi(0))$. We let $H \to \infty$ and get the claimed inequality.

The claim implies that

$$\limsup_T \sum_{\xi:\, t_\xi = T} \sum_{\eta \in \xi^+} L^i_{2\,\eta}(\overline{Z}^i)\overline{Z}^i_\xi \leq 0. \tag{20}$$

Claim: (9) holds if and only if (20) holds.

In fact, $\sum\limits_{\eta \in \xi^+} L^i_{2\,\eta}(\overline{Z}^i)\overline{Z}^i_\xi = -L^i_{1\,\eta}(\overline{Z}^i)\overline{Z}^i_\xi = -[L^i_{1x\,\eta}(\overline{Z}^i)\overline{x}^i_\xi + L^i_{1\theta\,\eta}(\overline{Z}^i)\overline{\theta}^i_\xi + L^i_{1\varphi\,\eta}(\overline{Z}^i)\overline{\varphi}^i_\xi + L^i_{1\psi\,\eta}(\overline{Z}^i)\overline{\psi}^i_\xi]$. Now, $L^i_{1\psi\,\eta}(\overline{\psi}^i)\overline{\psi}^i_\xi = 0$ by (5). Then, (20) is equivalent to (9).

This concludes the proof of Proposition 1.

On Remark 1:

Condition (10) holds if and only if $\limsup_T \sum_{\xi:\, t_\xi = T} L^i_{1\varphi\,\eta}(\overline{Z}^i)\overline{\varphi}^i_\xi \leq 0$.

Now, notice that $\limsup_T \sum_{\xi:\, t_\xi = T} (-L^i_{1\varphi\,\eta}(\overline{Z}^i)\overline{\varphi}^i_\xi) \leq \limsup_T \sum_{\xi:\, t_\xi = T} (-L^i_{1\,\eta}(\overline{Z}^i)\overline{Z}^i_\xi$, since $-L^i_{1x\,\eta}(\overline{Z}^i)\overline{x}^i_\xi$ and $-L^i_{1\theta\,\eta}(\overline{Z}^i)\overline{\theta}^i_\xi$ are both non-negative (as $L^i_{2x\,\eta}(\overline{Z}^i)\overline{x}^i_\xi$ and $L^i_{2\theta\,\eta}(\overline{Z}^i)\overline{\theta}^i_\xi$ are non-negative). This establishes that (9) implies (10).

Let us next see that (10) implies (9) when $\theta^i = 0$ and consumer i does not consumer more of any durable good than the collateral bundle. These two assumptions ensure that $-L^i_{1x\,\eta}(\overline{Z}^i)\overline{x}^i_\xi$ and $-L^i_{1\theta\,\eta}(\overline{Z}^i)\overline{\theta}^i_\xi$ are both zero. Hence, $\limsup_T \sum_{\xi:\, t_\xi = T} (-L^i_{1\varphi\,\eta}(\overline{Z}^i)\overline{\varphi}^i_\xi) = \limsup_T \sum_{\xi:\, t_\xi = T} L^i_{1\varphi\,\eta}(\overline{Z}^i)\overline{\varphi}^i_\xi$.

On Remark 2:

The existence of a cluster point follows from the fact that equilibrium allocations $Z^{iH} \equiv (x^{iH}, \theta^{iH}, \varphi^{iH}, \psi^{iH})$ of finite horizon economies have upper bounds, uniformly on the horizon H (for portfolios this follows from the collateral requirements and assumption [E]). Actually, equilibrium prices and the associated equilibrium multipliers also have uniform upper bounds: we normalize prices by placing (p_ξ, q_ξ) in the $G + J_\xi - 1$ dimensional simplex and multipliers $(\gamma^i_\xi, \rho_{j\xi})$ have upper bounds that are independent of prices and of the terminal horizon T, as established in Remark A.1 in the Appendix. So, node by node, equilibrium variables (prices, delivery rates, allocations, multipliers and the above supergradients) of all finite horizon economies have common upper bounds.

Then the sequence $\left(p^H, q^H, K^H, (Z^{iH}, \gamma^{iH}, \rho^{iH}, d^{iH})_i\right)$ of equilibrium prices, allocations, multipliers and supergradients of the functions $\max\{0, \cdot\}$ verifying the Kuhn-Tucker conditions, for the truncated economies. This sequence has, node by node, a cluster point $\left(p, q, K, (Z^i, \gamma^i, \rho^i, d^i)_i\right)$ satisfying Euler conditions. Observe that at the price cluster point p the payoff functions are well defined, as $p_\xi > 0$ at any node ξ, by the following lemma.

Lemma 2. *At each node, the sum of spot prices is bounded away from zero, uniformly in the finite horizon H and, therefore, also bounded away from zero in the infinite horizon economy.*

This follows from Lemma 3 as in part (b) of Lemma A.2 in Páscoa and Seghir (2009). We have $\frac{q^j_\xi}{\sum_g p_{\xi g}} \leq \overline{C}^j_\xi + \frac{1}{r^i_\xi(I\!I)} \sum_{\eta\in\xi^+} \frac{\tilde\lambda_{j\eta}\overline{A}^j_\eta}{\underline{b}_\eta} \equiv m^j_\xi$ and $\sum_g p_{\xi g} \geq (1+\sum_j m^j_\xi)^{-1}$. □

Observation 1: if C, A and λ are uniformly bounded on the event tree, $\underline{b} \ggg 0$ and the instantaneous utility v^i_ξ is node-invariant, then the positive lower bound refereed to in Lemma 2 is uniform across nodes, denoted by $\underline{p} \in \mathbb{R}_+ +$.

Observation 2: under the conditions in Observation 1, γ^i_ξ has a uniform upper bound on the event tree, provided that $\underline{\mathcal{W}}^i_\xi \ggg 0$.

Proof of Theorem 1. To shorten the notation, we omit in this proof the dependence of the Lagrangean on prices (as these are fixed in this proof) and write $L^i_\xi(Z) \equiv L^i_\xi(Z_\xi, Z_{\xi^-}, \gamma, p, q, K)$. Let the vectors $L^i_{1\,\xi}$ and $L^i_{2\,\xi}$ be partial supergradients of $L^i_\xi(Z)$ with respect to the current and past decision variables, respectively, verifying Euler conditions. These conditions can be written as:

$$L^i_{1\,\xi}(\overline{Z}^i) + \sum_{\eta\in\xi^+} L^i_{2\,\eta}(\overline{Z}^i) \leq 0, \tag{21}$$

$$\Big(L^i_{1\,\xi}(\overline{Z}^i) + \sum_{\eta\in\xi^+} L^i_{2\,\eta}(\overline{Z}^i)\Big)\overline{Z}^i_\xi = 0. \tag{22}$$

Let $\Pi^{iT}(x^i, \theta^i, \varphi^i, \psi^i) := \sum_{\xi\in D^T} v^i_\xi(\tilde{x}^i_\xi) - \sum_{\xi\in D^T\setminus\{xi_0\}} \sum_{j\in J(\xi^-)} \tilde\lambda^i_{j\xi}[\psi^i_{j\xi}]^+$.

Then, $\Pi^{iT}(Z) - \Pi^{iT}(\overline{Z}^i) \leq \sum_{\xi:\, t_\xi \leq T} \Big(L^i_\xi(Z) - L^i_\xi(\overline{Z}^i)\Big) \leq$

$$\sum_{\xi:\, t_\xi\leq T} \Big(L^i_{1\,\xi}(\overline{Z}^i)(Z_\xi - \overline{Z}^i_\xi) + L^i_{2\,\xi}(\overline{Z}^i)(Z_{\xi^-}) - \overline{Z}^i_{\xi^-})\Big)$$

$$= \sum_{\xi:\, t_\xi<T} \Big(L^i_{1\,\xi}(\overline{Z}^i) + \sum_{\eta\in\xi^+} L^i_{2\,\eta}(\overline{Z}^i)\Big) Z_\xi + \sum_{\xi:\, t_\xi=T} L^i_{1\,\xi}(\overline{Z}^i) Z_\xi - \sum_{\xi:\, t_\xi=T} L^i_{1\,\xi}(\overline{Z}^i)\overline{Z}^i_\xi.$$

Now, the transversality condition (9) implies $\limsup_T \sum_{\xi:\, t_\xi=T} \sum_{\eta\in\xi^+} L^i_{2\,\eta}(\overline{Z}^i)\overline{Z}^i_\xi \leq 0$. Thus, by (21) and (22), $\limsup_T \Big(\Pi^{iT}(Z) - \Pi^{iT}(\overline{Z}^i)\Big) \leq \limsup_T \sum_{\xi:\, t_\xi=T} L^i_{1\,\xi}(\overline{Z}^i) Z_\xi$.

Now, $L^i_{1\,\xi}(\overline{Z})Z_\xi = \Big(v^{i\,\prime}_\xi(\overline{Z}) - \gamma^i_\xi p_\xi\Big)x_\xi - \gamma^i_\xi q_\xi \theta_\xi + \Big(v^{i\,\prime}_\xi(\overline{Z})C_\xi - \gamma^i_\xi(p_\xi C_\xi - q_\xi\Big)\varphi_\xi - \Big(\tilde{\lambda}^i_{j\xi} d^i_{j\xi} + \rho_{j\xi} - \gamma^i_\xi\Big)\psi^i_{j\xi}$. Here, $v^{i\,\prime}_\xi(\overline{Z}) - \gamma^i_\xi p_\xi \leq -\sum_{\eta\in\xi^+} \gamma^i_\eta p_\eta Y_\eta \leq 0$ and $\tilde{\lambda}^i_{j\xi} d^i_{j\xi} + \rho_{j\xi} - \gamma^i_\xi = 0$. □

6.2 On section 4

Lemma 3. *Let* $\mathcal{W}_\xi = \sum_i \mathcal{W}^i_\xi$. *Given any bundle* $\kappa_\xi \in \mathbb{R}^G_{++}$*, the directional derivative* $(v^i_\xi)'(.)\kappa_\xi$ *has a positive lower bound* $r^i(\kappa_\xi)$ *on the set of bundles* $\{z \in \mathbb{R}^G_{++} : z \leq \mathcal{W}_\xi\}$.

We can take $\kappa_\xi = \mathbb{1}$ or $\kappa_\xi = b_\xi$ or even κ_ξ being the $\hat{g}$-th canonical vector.

Proof: Denote by $S(0,\alpha)$ the sphere with center 0 and radius α and let $B(0,\alpha)$ be the open ball bounded by $S(0,\alpha)$. For any $\epsilon > 0$ let $\tilde{S}$ be the translation of $S(0,\mathcal{W}_\xi) \cap \mathbb{R}^G_+$ by the vector $\varepsilon\kappa_\xi \in \mathbb{R}^G_{++}$, that is, $\tilde{S} := S(0,\mathcal{W}_\xi) \cap \mathbb{R}^G_+ + \varepsilon\kappa_\xi$, which is a compact set.

For $y \in S(0,\mathcal{W}_\xi) \cap \mathbb{R}^G_+$, let T_y be the affine set that runs through y in the direction of κ_ξ, that is, $T_y = \{z \in \mathbb{R}^G : \ z = a\kappa_\xi + y, \text{ for some } a\}$. Let z_y be the point where the line T_y hits the spherical sector $\tilde{S}$. Actually, $\tilde{S}$ is the set of such points z_y.

Now, the concavity of v^i implies the monotonicity of the directional derivative $(v^i_\xi)'(.)\kappa_\xi$ on T_y. Hence, for any $z \in T_y \cap B(0,\mathcal{W}_\xi) \cap \mathbb{R}^G_{++}$, we have $(v^i_\xi)'(z)\kappa_\xi \geq (v^i_\xi)'(z_y)\kappa_\xi$. We know that $(v^i_\xi)'(z_y)\kappa_\xi > 0$ by monotonicity of v^i. Concavity implies the continuity of directional derivatives and, therefore, we can say that the set $\hat{S} := \{(v^i_\xi)'(z_y)\kappa_\xi, \text{ for some } y\}$ is a continuous image of the compact set $\tilde{S}$. Then, $\hat{S}$ is a compact set of positive real numbers, hence it has a positive lower bound, which we denote by $r^i_\xi(\kappa_\xi)$. □

Observation 3: If $\mathcal{W}_{g\xi}$ has a uniform upper bound $\overline{\mathcal{W}}_g$ for each good g and the instantaneous utility v^i_ξ is node-invariant, then, given a bundle $\kappa \in \mathbb{R}^G_{++}$, the positive lower bound referred to in Lemma A.1 is uniform across nodes, denoted by $r^i(\kappa)$. This follows by adapting the proof of the lemma using $\overline{\mathcal{W}}$ instead of $\mathcal{W}_\xi$.

Proof of Theorem 2. Both along the sequence of finite economies equilibrium and at the limit point of the relevant subsequence, we have, by (8), that $\lambda^i_{j\xi} < r^i_\xi(b_\xi)$ implies $\tilde{\lambda}^i_{j\xi} < \gamma^i_\xi$. It follows, by (5), that $\rho^i_{j\xi} > 0$ and, therefore, the delivery is $M_{j\xi}\varphi^i_{j\xi^-}$. Suppose that for any agent (6) cannot hold with $d_{j\eta} = 0$, $\forall \eta \in \xi^+$ (otherwise we get immediately $p_\xi C_{j\xi} \geq q^j_\xi$, by (6)).

If promise j is traded at ξ, we get $K_{j\eta} = \frac{M_{j\eta}}{p_\eta A_{j\eta}}$ for $\eta \in \xi^+$ (along that subsequence and at its limit point) and (7) holds as equality for some agent. Combining with (8), we get $p_\xi C_{j\xi} \geq q^j_\xi$, as for *this* agent we have:

$$\gamma^i_\xi\Big(p_\xi C_{j\xi} - q^j_\xi\Big) \geq v'_\xi(\overline{x}^i_\xi)C_{j\xi} + \sum_{\eta\in\xi^+}\gamma^i_\eta\Big(p_\eta Y_\eta C_{j\xi} - M_{j\eta}\Big) \geq 0, \qquad (23)$$

If promise j is not traded, but was traded along a subsequence (of the above subsequence), the above argument still applies. Otherwise, we can re-set $K_{j\eta} = \frac{M_{j\eta}}{p_\xi A_{j\xi}}$ (in fact, (7) remains true as we just lower the right hand side). Now, if (7) holds with strict inequality for every agent, with $K_{j\eta} = \frac{M_{j\eta}}{p_\xi A_{j\xi}}$, we lower q^j_ξ, until $q^j_\xi = \max_i \sum_{\eta\in\xi^+} \frac{\gamma^i_\eta}{\gamma^i_\xi} M_{j\eta}$ (notice that (6) still holds, as we just raise the left-hand side). The agent(s) for whom this maximum occurs will have (23) satisfied and, therefore, $p_\xi C_{j\xi} \geq q^j_\xi$.

Actually the above resetting of q^j_ξ, $K_{j\eta}$ and $d_{j\eta}$ (for $\eta \in \xi^+$) when asset j is not traded at node ξ, along any subsequence of truncated economies equilibria, could be done already along the relevant converging subsequence (rather than by modifying the limit point), so we are back in the exact setting addressed by Theorem 1, knowing that $p_\xi C_{j\xi} \geq q^j_\xi$.

When the collateral does not yield any utility gains, condition (11) holds. □

It can be seen from the proof of Theorem 2 that agents who have (7) holding with equality, for every promise, beyond some node ξ, will have (23) satisfied at these nodes for all promises and, therefore, have no opportunities for doing generalized Ponzi schemes. So, only agents that do not purchase some promise at each node would have an opportunity to do a generalized Ponzi scheme.

Proof of Theorem 3.

For the finite horizon economy, we adapt the proof of Theorem 1 in Dubey, Geanakoplos and Shubik (2005). As in their proof the relative prices set is $P_\xi = \Big\{(p_\xi, q_\xi) : \sum_g p(\xi, g) = 1,\ p(\xi, g) \geq s,\ q^j_\xi \in [0, 1/s]\Big\}$. The correspondence that picks at each node the *relative* prices is defined by

$$\tau^0_s \equiv \arg\max_{\prod_{\xi\in D} P_\xi} \Big\{ \sum_{\xi\in D} \Big(p_\xi \cdot \sum_i (x^i_\xi + \sum_{j\in J} C_{j\xi}\varphi^i_{j\xi} - W^i_\xi) + q_\xi \cdot \sum_i (\theta^i_\xi - \varphi^i_\xi) \Big) \Big\} \quad (24)$$

Step 1. Now, we select the outcome that makes marginal penalty effects be dominated by marginal income effects (actually we make (16) hold as an equality). We do this by creating a correspondence that defines what the inverse ι_η of the *absolute* spot commodity prices sum S_η should be,

$$\tau^\iota_{s\xi} = \arg\min_{(\iota_\eta)\in E} \Big[\sum_{\eta\in\xi^+} \iota_\eta \max_{i,j} \max\{\lambda^i_{j\eta}, \gamma^i_\eta\} b^j_\eta - \sum_{\eta\in\xi^+} \min_{i,j} \min\{\lambda^i_{j\eta}, \gamma^i_\eta\} p_\eta Y_\eta C^j_\eta \Big]^2 \quad (25)$$

where $E = \{(\iota_\eta)_{\eta\in\xi^+} : \iota_\eta \in [0, \chi_\xi]\}$ and $\chi_\xi = \frac{\max_j \sum_{\eta\in\xi^+} \min_i \min\{\lambda^i_{j\eta},\gamma^i_\eta\} p_\eta Y_\eta C_{j\xi}}{\min_j \sum_{\eta\in\xi^+} \max_i \max\{\lambda^i_{j\eta},\gamma^i_\eta\} b^j_\eta}$.

Step 2. We accommodate nominal promises in the real promises framework using the function $(\iota_\xi, b^j_\xi) \longmapsto A^j_{\xi\, g} = \iota_\xi b^j_\xi$.

The correspondence that picks the repayment rates is defined as

$$K_s \equiv \operatorname{argmin} \Big\{ \sum_{\eta\in\xi^+} \Big((\sum_i \theta^i_\xi)(1 - K^j_{s\eta}) p_\eta A_{j\eta} - \sum_i \psi^i_{j\eta} \Big)^2 :\ K^j_{s\eta} \in [0,1],\ \forall \eta \in \xi^+ \Big\} \quad (26)$$

Step 3. Consumers have the standard constrained demand correspondence $\tau^h_s = \operatorname{argmax}_{Z_\xi} \Big\{\Pi^i(Z) :\ (1), (2) \text{ and } (3) \text{ hold at } (p, q, K), \text{ given } A, \text{ for } Z = (x, \theta, \varphi, \psi) \text{ such that } x_\xi \leq \mathcal{W}_\xi(1{+}e),\ \varphi^i_{j\xi} \leq \frac{\mathcal{W}(1+e)}{\max_g C^j_{g\xi}} \equiv L^j_\xi,\ \theta^i_{j\xi} \leq (\#I) L^j_\xi,\ \psi^i_{j\xi} \leq (\max_g A^j_{g\xi}) L^j_\xi, \text{ for some } e \text{ relatively small}\Big\}$.

Step 4. Lagrange multipliers are constructed through correspondence $\mathbb{L} = \prod_{(i,\xi)} \mathbb{L}^i_\xi$ where $\mathbb{L}^i_\xi = \operatorname{argmin}_{(\gamma^i_\xi, \rho_{j\xi})} \Big\{ L^i_\xi\big(Z^i_\xi, Z^i_{\xi^-}, p_\xi, q_\xi, K_\xi, \gamma^i_\xi, \rho_\xi) :\ \gamma^i_\xi,\ \rho_{j\xi} \leq \overline{\gamma}^i_\xi \Big\}$.

Final step. For each $s > 0$, a fixed point of $\tau^0_s \times K_s \times \tau^\iota_s \times A \times \mathbb{L} \times (\prod_h \tau^h)$ exists, as $\tau^\iota_{s\xi}$ is nonempty valued (take $\iota_\eta = \iota,\ \forall \eta \in \xi^+$, with $\iota \leq \frac{\min_{j\in J^*} \sum_{\eta\in\xi^+} \min_i \min\{\lambda^i_{j\eta},\gamma^i_\eta\} p_\eta Y_\eta C_{j\xi}}{\max_{j\in J^*} \sum_{\eta\in\xi^+} \max_i \max\{\lambda^i_{j\eta},\gamma^i_\eta\} b^j_\eta}$) and upper hemicontinuous.

As in the proof of Theorem 1 in Dubey, Geanakoplos and Shubik (2005), when $s \longrightarrow 0$, aggregate excess demand goes to zero, $p(\eta, g)$ does not go to zero and q_ξ stays bounded. Moreover, ι_η stays both bounded from above and bounded away from zero. Passing to subsequences, we obtain a limit point which is an equilibrium for the finite horizon economy and satisfying condition (13), for any $(\eta, j) \in D \times J^*$, since at $S_\eta = \lim \iota_\eta^{-1}$ we have (16) satisfied.

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