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DSGE Models under Imperfect Information: A Dynare-based Toolkit

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DSGE Models under Imperfect Information: A Dynare-based Toolkit*

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Abstract

This paper describes a Dynare-based toolbox which solves, simulates and estimates DSGE rational expectations (RE) models under perfect or imperfect information on the part of agents. The toolbox also delivers tests and conditions for exact and approximate invertibility providing information as to how well VAR residuals map the structural shocks in the RE solution to the DSGE model. Seven working examples come with the package including a version of the Smets and Wouters (2007) model and a standard small-scale New Keyesian (NK) DSGE model. The estimation exercise is conducted on both the NK and Smets-Wouters models. The paper provides alternative estimation results and an assessment for fundamentalness of structural shocks assuming that RE agents do not observe all current state variables (including shock processes) and only have an imperfect information set. Sections of the paper also examine the impulse response functions and unconditional second moments of the estimated model and discuss endogenous persistence.

Keywords: Imperfect Information, DSGE Models, Invertibility, Dynare

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1 Introduction

There is now a growing new literature on the importance of imperfect information (henceforth II) in DSGE models especially with heterogeneous agents. Despite this many (indeed most) such models are still use rational expectations (RE) solutions with the assumption that agents are simply provided with perfect information (henceforth PI) of the current state of the economy, effectively as an endowment. The strong assumption of RE is made even stronger by this informational assumption. For example with exogenous shock processes the assumed information set then includes their current realizations; if agents observe macroeconomic variables with measurement error then these also enter into their information set.

The agents' problem under II is in many respects a standard signal extraction problem, but it will in general feed back, via optimising behaviour, into the behaviour of any endogenous states of the model economy. As a direct result the filtering process itself thus increases the state space relative to the benchmark case of PI and have significant effects for the dynamics of the model, its estimation and the ability of the econometrician to represent its solution as a VAR. These are the issues we pursue in our imperfect information toolbox.

1.1 Literature

II models with heterogenous agents distinguish local (idiosyncratic) information and (aggregate) information e.g. Pearlman and Sargent (2005), Nimark (2008), Angeletos and La'O (2009), Graham and Wright (2010), Nimark (2014), Ilut and Saijo (2018), Rondina and Walker (2018), Huo and Takayama (2018), Angeletos and Huo (2018) and Angeletos and Huo (2020).Angeletos and Lian (2016) provide a recent comprehensive survey.[1]

This paper and toolkit follow a separate II literature where there is no explicit idiosyncratic shocks - see, for example, Collard *et al.* (2009), Neri and Ropele (2012) and Levine *et al.* (2012). Levine *et al.* (2019) (that this toolbox follows closely) show that this class of models can be considered as the limiting case of those cited above with heterogeneous agents and idiosyncratic shocks in the limit as idiosyncratic uncertainly far outweighs aggregate uncertainty a feature that is strongly supported empirically (see, for example, Ilut and Saijo (2018) and Bloom *et al.* (2018)).

1.2 The Toolkit

Turning to the toolkit, it then inputs any DSGE model in either non-linear and linear standard form and carries out the following exercises:

[1]This survey refers to as the incomplete information literature. Here a comment on terminology is called for. Our use of perfect/imperfect Information (PI/II) is widely used in the literature when describing agents' information of the history of play driven by draws by Nature from the distributions of exogenous shocks. In previous papers by the authors, and in dynare, partial rather than imperfect information is used. The complete/incomplete framework of the Angeletos-Lian survey (and other work by these authors) incorporates PI/II, but also refers to agent's beliefs regarding each other's payoffs. In our set-up this informational friction (leading to "Global Games") is absent.

1. A transformation of the Dynare set-up into the Blanchard-Kahn in the form used by Pearlman *et al.* (1986) to solve for the RE solution under PI or II;

2. The stochastic first-order solution as in Pearlman *et al.* (1986) with impulse response functions, unconditional second moments and simulated data suitable for Monte-Carlo exercises;

3. The conditions for invertibility under which imperfect information is equivalent to perfect information as in Levine *et al.* (2019);

4. Multivariate measures of goodness of fit of the innovation residuals to the fundamental shocks, providing information as to how well VAR residuals correspond to the fundamentals in DSGE models;

5. Bayesian first-order estimation of the both the PI and II cases.

1.3 Road-map

The remainder of the paper sets out instructions to demonstrate the working of the software. In what follows, Section 2 first sets out a brief summary of the conversion algorithm, the RE solution under imperfect information and the invertibility tests and measures. Section 3 describes the current implementation including a new novel feature on checking invertibility conditions.[2] Sections 4 and 5 introduce the estimation part of the software and the `.mod` file syntax rules. Section 6 refers to the examples and applications in the literature. Section 7 concludes.

In the appendices of this paper, Appendix A first describes in greater detail the algorithm converting our models to the suitable Blanchard-Kahn form and set out the model equations used in all our examples. It is useful to use artificial data from stochastic simulations of the model to numerically assess the theoretical results of this paper and Appendix B shows how this is done. Appendices C–G set out and describe the models we use as examples to demonstrate the implementation of the toolbox. Appendices H–K plot the empirical autocorrelation and impulse response functions based on the estimated posterior estimates. Appendix L presents the Dynare output produced for the results that we report in Section 6.1 and Tables 2–13. Finally, Appendix M shows instructions for installation.

2 Theoretical Background

In Dynare a non-linear DSGE model can be written as

$$\begin{aligned}\mathbb{E}_t[f(Y_t, Y_{t+1}, Y_{t-1}, \varepsilon_{t+1})] &= 0 \\ \mathbb{E}_t[\varepsilon_{t+1}] &= 0 \\ \mathbb{E}_t[\varepsilon_{t+1}\varepsilon'_{t+1}] &= \Sigma_\varepsilon\end{aligned} \tag{1}$$

[2]To install the software package, we need to make sure that the solution and simulation subroutines are stored as source code in `...\dynare\4.x.y\matlab\partial_information`. See Section 3 for details.

where Y_t is an $n \times 1$ vector of endogenous macroeconomic variables; and ε_t is a $k \times 1$ vector of exogenous Gaussian white noise structural shocks. We assume that the structural shocks are normalized such that their covariance matrix is given by the identity matrix i.e., $\varepsilon_t \sim N(0, I)$. Note that this is quite general in that Y_t can be enlarged to include lagged and forward-looking variables.[3]

Writing $y_t \equiv \log(Y_t/\bar{Y}_t$ where $\bar{Y}_t$ is the long-term deterministic trend and log-linearizing about this trend the general form

$$A_0 y_{t+1,t} + A_1 y_t = A_2 y_{t-1} + \Psi \epsilon_t \tag{2}$$

where $y_{t+1,t}$ denotes $\mathbb{E}_t[y_{t+1}]$ and matrix A_0 may be singular.[4] Note that the user can code the model in either non-linear or linearized form and in the former case dynare carries out the linearization in a first-order perturbation solution. Below we provide examples of both.

We define $y_{t,s} \equiv \mathbb{E}\left[y_t | I_s^A\right]$ where I_t^A is information available at time t to economic agents, given by $I_t^A = \{m_s^A : s \leq t\}$. We assume that all agents have the same information set about some strict subset of the elements of Y_t, hence information is in general imperfect. Similarly, this applies to the $m \times 1$ vector m_t^E, where $m \leq k$, which is the vector of observables available to the econometrician. These vectors of observables available to the econometrician and agents respectively are given by

$$m_t^E = L^E y_t \tag{3}$$

$$m_t^A = L^A y_t \tag{4}$$

Note that measurement errors can be accounted for by including them in the vector ϵ_t. In the special case that agents are endowed with perfect information, $L^A = I$ (the identity matrix).

2.1 Conversion to Blanchard-Kahn Form

In order to move seamlessly from the very general class of linear RE models (2) to results that are based on Pearlman *et al.* (1986) - henceforth PCL - we introduce a key result. This form resembles a representative agent model, but from Levine *et al.* (2019) shows (Theorem 2) it also represents a limiting case of a class of heterogenous agent models where idiosyncratic shocks enter as additions to aggregate shocks and the standard deviations of the former dominate the latter.

Theorem 1. *For any information set, (2) can always be converted into the following generalized Blanchard-Kahn form, as used by PCL*

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{5}$$

[3] See Dynare User Guide, chapter 7.

[4] Let x_t be some component of y_t. Then $x_{t,t}$ denoted by $\mathbb{E}_t[x_t]$ (not necessarily equal to x_t under imperfect information) can be incorporated into this set-up by defining a state variable $xL_t \equiv x_{t-1}$ and noting that $x_{t,t} = xL_{t+1,t}$.

$$m_t^A = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \tag{6}$$

where z_t, x_t are vectors of backward and forward-looking variables, respectively. The covariance matrix of shocks is the matrix BB'.

Proof: See Appendix A where a novel comprehensive algorithm is represented that can handle any model in the form (2).

Note that at this stage we focus solely on the agents' informational problem: we specify the properties of $m \times 1$ vector m_t^E where $m \leq k$, the vector of observables available to the econometrician later.

2.2 The Agents' Solution under Perfect Information (API)

Here we assume that agents directly observe all elements of Y_t, hence of (z_t, x_t). Hence $z_{t,t} = z_t$, $x_{t,t} = x_t$, and using the standard BK solution method there is a saddle path satisfying

$$x_t + Nz_t = 0 \quad \text{where} \quad \begin{bmatrix} N & I \end{bmatrix} (G + H) = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{7}$$

where Λ^U is a matrix with unstable eigenvalues. If the number of unstable eigenvalues of $(G + H)$ is the same as the dimension of x_t, then the system will be determinate.[5]

To find N, consider the matrix of eigenvectors W satisfying

$$W(G + H) = \Lambda^U W \tag{8}$$

Then, as for G and H, partitioning W conformably with z_t and x_t, from PCL we have

$$N = W_{22}^{-1} W_{21} \tag{9}$$

From the saddle-path relationship (9), the saddle-path stable RE solution under API is

$$z_t = Az_{t-1} + B\varepsilon_t \qquad x_t = -Nz_t \tag{10}$$

where $A \equiv G_{11} + H_{11} - (G_{12} + H_{12})N$.

2.3 RE Solution under Imperfect Information

PCL propose a general framework for introducing information limitations where agents are not able to perfectly observe states that define the economy at the point agents form expectations. We first briefly outline how the imperfect information setup is solved, and then provide the conditions for invertibility.

Following PCL and Levine *et al.* (2019), we apply the Kalman filter updating given by

$$\begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} = \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} + K \left[m_t^A - \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} - \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \right] \tag{11}$$

[5] Note that in general, as Sims (2002) has pointed out, the dimension of x_t will not match the number of expectational variables in (2), as we see in the algorithm for the proof of Theorem 1 (see Appendix A.2).

where we denote $z_{t,t} \equiv \mathbb{E}_t[z_t]$ and $x_{t,t} \equiv \mathbb{E}_t[x_t]$. The Kalman filter was developed in the context of backward-looking models, but extends as we see here to forward-looking models. The basic idea behind it is that the best estimate of the states $\{z_t, x_t\}$ based on current information is a weighted average of the best estimate using last period's information and the new information m_t^A. Thus the best estimator of the state vector at time $t-1$ is updated by multiple K of the error in the predicted value of the measurement as above, where K (the "Kalman gain") is given by

$$K = \begin{bmatrix} P^A J' \\ -NP^A J' \end{bmatrix} [(M_1 - M_2 N)P^A J']^{-1} \tag{12}$$

and $J \equiv M_1 - M_2 G_{22}^{-1} G_{21}$, $M \equiv [M_1\ M_2]$ is partitioned conformably with $[z_t, x_t]'$, and P^A satisfies the Riccati equation (17) below for the agents' filtering problem. G and H are partitioned conformably with $[z_t, x_t]'$ as in (A.1) in Appendix A.1 and we define F and J below.

Using the Kalman filter, the agents' solution under imperfect information (henceforth AII) as derived by Pearlman *et al.* (1986) is given by the following processes describing the pre-determined and non-predetermined variables $z_t = \tilde{z}_t + z_{t,t-1}$ and x_t, and a process describing the prediction errors $\tilde{z}_t \equiv z_t - z_{t,t-1}$

$$\begin{aligned}
\text{Predictions}: \quad z_{t+1,t} &= Az_t + A\left[P^A J'(JP^A J')^{-1} J - I\right]\tilde{z}_t && (13)\\
\text{Non-predetermined}: \quad x_t &= -Nz_t + (G_{22}^{-1}G_{21} - N)\left[P^A J'(JP^A J')^{-1} J - I\right]\tilde{z}_t && (14)\\
\text{Prediction Errors}: \quad \tilde{z}_t &= F[I - P^A J'(JP^A J')^{-1} J]\tilde{z}_{t-1} + B\varepsilon_t && (15)\\
\text{Measurement Equation:} \quad m_t^A &= Ez_t + E\left[P^A J'(JP^A J')^{-1} J - I\right]\tilde{z}_t && (16)
\end{aligned}$$

where $A \equiv G_{11} + H_{11} - (G_{12} + H_{12})N$, $F \equiv G_{11} - G_{12}G_{22}^{-1}G_{21}$, $J \equiv M_1 - M_2 G_{22}^{-1} G_{21}$ and $E \equiv M_1 + M_3 - (M_2 + M_4)N$. The matrix A is the autoregressive matrix of the states z_t in the agents' solution under perfect information (henceforth API); B captures the direct (but unobservable) impact of the structural shocks ϵ_t and $P^A = \mathbb{E}[\tilde{z}_t \tilde{z}_t']$ is the solution of a Riccati equation given by

$$P^A = Q^A P^A Q^{A'} + BB' \qquad \text{where } \ Q^A = F\left[I - P^A J'(JP^A J')^{-1} J\right] \tag{17}$$

We can see that the solution procedure above is a generalization of the Blanchard-Kahn solution for perfect information and that the determinacy of the system is independent of the information set.[6,7]

[6]Full details of the algorithm for converting the state space under partial information to the Blanchard-Kahn form (the PCL solver) and its simulation implementation can be found in Appendix A.

[7]Under perfect information we have that $M_1 = I$ and $M_2 = 0$ so $x_t = -Nz_t$ is also observed. Then $J = I$, but then the this information set is in general of higher dimension than the shocks, so we pick a linear combination $\bar{J}$ of the information set such that $\bar{J}B$ is invertible and $\bar{Q}^A = F(I - B(\bar{J}B)^{-1}\bar{J})$ has stable eigenvalues (which is possible if (F, B) is controllable). From (17) it follows that $P^A = BB'$, the covariance matrix of the structural shocks, and $\bar{Q}^A$ is as above. Hence $\bar{Q}^A B = 0$ and thefefore $\tilde{z}_t = B\epsilon_t$. Finally, adding $\tilde{z}_{t+1}$ to both sides of (13) yields the result for PI. More details of the solution under agents' perfect information can be found in Levine *et al.* (2019).

2.4 A-Invertibility and E-Invertibility

First we have two definitions and a lemma for E-invertibility under API:

Definition 1. *A-Invertibility. The system in (2) is A-invertible if agents can infer the true values of the shocks* ϵ_t *from the history of their observables,* $\{m_s^A : s \leq t\}$*, or equivalently, if the number of observables equals the number of shocks* $(m = k)$ *and* $P^A = BB'$ *is a stable fixed point of the agents' Ricatti equation, (17).*

Corresponding to A-invertibility we now define the corresponding concept from the viewpoint of the econometrician:

Definition 2. *E-Invertibility. The system in (2) is E-invertible if the values of the shocks* ϵ_t *can be deduced from the history of the econometrician's observables,* $\{m_s^E : s \leq t\}$*.*

Lemma 1. *If agents have perfect information, the conditions for E-invertibility (as in Definition 2) are: the square matrix* EB *is of full rank and* $A(I - B(EB)^{-1}E)$ *is a stable matrix.*

We now pose the question: given the econometrician's information set, under what conditions do the RE solutions under agents' different information sets actually differ? When can the econometrician infer the full state vector, including shocks? We now consider the more general case of E-invertibility when agents have imperfect information, Levine *et al.* (2019) then show the following result that generalizes the "Poor Man's Invertibility Condition" (PMIC) of Fernandez-Villaverde *et al.* (2007):

Theorem 2. *Assume that the number of observables equals the number of shocks* $(m = k)$. *Assume further that the PMIC conditions in Lemma 1 hold (so the system would be E-invertible under API) but agents do not have perfect information. Then each of the following conditions is necessary and sufficient for each of the other two (i.e., the three conditions are equivalent):*

a) The RE solution where agents have imperfect information is E-invertible (see Definition 2);

b) The square matrix JB *is of full rank, and* $Q_A = F(I - B(JB)^{-1}J)$ *is a stable matrix;*

c) The RE solution where agents have imperfect information is A-invertible (see Definition 1).

This is a new result in the literature, which says that given the econometrician's observations m_t^E, if the RE solution to a model under agents' perfect information is invertible, it does not follow that with the same information set the RE solution to a model under agents' imperfect information is also invertible.[8]

[8] Fernandez-Villaverde *et al.* (2007) and Baxter *et al.* (2011) evaluate the invertibility of rational expectations models. The former have done this within the context of a general form of the saddle-path solution of a rational expectations model which is equivalent to a set of vector VARMA processes. This can encompass perfect or imperfect information for agents, but the authors focus only on general conditions. The only information set mentioned in Fernandez-Villaverde *et al.* (2007) is that which is available to the econometrician, but their general results are applicable no matter what is the information set of private agents.

2.5 Measures of Fundamentalness

A key issue in estimation is to be able to generate the theoretical responses to a fundamental shock. Levine *et al.* (2019) also examine measures of approximate fundamentalness when invertibility fails for both perfect and imperfect information cases. More recently, Beaudry *et al.* (2016) and Forni *et al.* (2017) have suggested ways of addressing whether close approximations to the fundamental shocks can be retrieved from the VARs. The latter paper suggests a regression of the fundamental shocks on the residuals from the VAR suitable for non-square systems where agents observe with noisy observations of news shocks. Levine *et al.* (2019) provide a generalisation of Forni *et al.* (2017) and develop measures of approximate fundamentalness for both perfect and imperfect information cases based on the following measure of goodness of fit

$$\mathbb{F}_i^{PI} = cov(\varepsilon_{i,t}) - cov(\varepsilon_{i,t}, \hat{\varepsilon}_t)cov(\hat{\varepsilon}_t)^{-1}cov(\hat{\varepsilon}_t, \varepsilon_{i,t}) = 1 - (EB)_i'(EP^E E')^{-1}(EB)_i \tag{18}$$

$\mathbb{F}_i$ corresponds to a measure of goodness of fit of the innovations residuals to the fundamental shocks. In addition, the maximum eigenvalue of $\mathbb{F}_i$ then provides a measure of overall non-fundamentalness obtained from the models. If $m = k$, and if $\mathbb{F}_i = 0$ for all i, then since $\mathbb{F}^{PI}$ is by definition a positive definite matrix, it must be identically equal to 0. The more of the eigenvalues of $\mathbb{F}$ that are close to 0, the more one can trust that at least some of the residuals are good approximations to the fundamental shocks.[9]

$$\mathbb{F}^{PI} = I - B'E'(EP^E E')^{-1}EB \tag{19}$$
$$\mathbb{F}^{II} = I - B'J'(JP^A J')^{-1}JP^A E'(EZE')^{-1}EP^A J'(JP^A J')^{-1}JB \tag{20}$$

where the diagonal terms then correspond to the terms $\mathbb{F}_i$ of (18). In (19) we note that $EP^E E' = cov(\hat{\varepsilon}_t)$, and $(EB)_i = cov(\hat{\varepsilon}_t, \varepsilon_{i,t})$. Analogously to the perfect information case, $EZE' = cov(\hat{\varepsilon}_t)$, with $EP^A J'(JP^A J')^{-1}JB = cov(\hat{\varepsilon}_t, \varepsilon_t)$. Z satisfies the Riccati solution corresponding to $(A, P^A J'(JP^A J')^{-1}JP^A, E)$

$$Z = AZA' - AZE'(EZE')^{-1}EZA' + P^A J'(JP^A J')^{-1}JP^A \tag{21}$$

m_t^E is a $m \times 1$ is vector of observables for the econometrician[10] and P^E the Riccati equation for the problem for the econometrician when estimating the parameters of the system. The econometrician's innovations representation follows the Fernandez-Villaverde *et al.* (2007)'s ABCD state-space form and the Riccati solution for the econometrician's problem in also given in Section 3 of Levine *et al.* (2019). Analogously, we can apply these measures of fundamentalness to the case when all variables are lagged

$$\mathbb{F}^{II,lagged} = cov(\varepsilon_t) - cov(\varepsilon_t, \hat{\varepsilon}_{t-1})cov(\hat{\varepsilon}_{t-1})^{-1}cov(\hat{\varepsilon}_{t-1}, \varepsilon_t) \tag{22}$$

[9] This provides how well the VAR residuals correspond to the fundamentals. See Levine *et al.* (2019), Theorem 5.

[10] Later in Appendix we distinguish between m_t^E and the vector of observations by the economic agents in the model, m_t^A.

$cov(\hat{\varepsilon}_{t-1})$ is of course equal to $cov(\hat{\varepsilon}_t) = EZE'$, so the only change is to $cov(\hat{\varepsilon}_{t-1}, \varepsilon_t)$, which after a little effort can be derived as

$$
\begin{aligned}
cov(\hat{\varepsilon}_{t-1}, \varepsilon_t) = {} & EAP^A J'(JP^A J')^{-1} JB - EAZE'(EZE')^{-1} EP^A J'(JP^A J')^{-1} JB \\
& + EP^A J'(JP^A J')^{-1} JFB - EP^A J'(JP^A J')^{-1} JFP^A J'(JP^A J')^{-1} JB \qquad (23)
\end{aligned}
$$

Then the goodness of fit $\mathbb{F}_i^{II,lagged}$ to the ith shock is just given by the ith main diagonal term of $\mathbb{F}^{II,lagged}$. By construction, these measures of approximate fundamentalness when invertibility fails for both perfect and imperfect information cases can be applied to possible non-square systems, i.e., when considering models with the number of observables $\leq$ the number of shocks (i.e., $m \leq k$, k is the row dimension of the structural shocks). In the case when $m = k$, the software below also reports $\mathbb{B}^{PI} = EP^E E' - EBB'E'$ and $\mathbb{B}^{II} = EZE' - EBB'E'$, the Beaudry *et al.* (2016) measures, which are abbreviated to the difference between the variances of the innovations and the fundamentals as in (19), (20) and (23). In particular, the theoretical values of $\mathbb{F}_i$ (and $\mathbb{B}_i$) and details of deriving these fundamentalness measures are explained in Levine *et al.* (2019).

3 Current Implementation and Use

The software so far is designed to solve the model, perform simulation and generate impulse response functions (IRFs) in Dynare. To use the package, download the zip-file latest Dynare version 4.6.2 from click to download the latest Dynare and extract its content. Also download the zip-file for this package from a GitHub/Dynare source repository (click to download the Toolbox)[11] and extract the folder which contains a sub-folder called `partial_information`, this paper in pdf format and seven example files that replicate the results reported in this paper and its appendices.

The code is organised such that, for installation, the user simply copies and moves the files from `partial_information` to Dynare's `partial_information` subfolder, overwriting its content.[12] When this is done, the user simply runs the `.mod` file in Dynare as usual. In order to do this, there are a number of syntax rules that the user is required to adopt when writing the `.mod` file.

Section 4 describes estimation under imperfect information using Dynare. Bayesian maximum likelihood estimation proceeds in the usual way for Dynare;[13] the only change is

[11] Users can also view and download the Toolkit from this link to a Dropbox folder.

[12] The stable Dynare version of the source code is located in `...\dynare\4.x.y\matlab\partial_information`

[13] However, there is an area where the software differs from the standard Dynare software; in the latter, one occasionally encounters an error message that the Hessian evaluated at the mode is not positive definite. This is almost invariably the consequence of rounding error or ill-conditioning when computing the second derivatives numerically. To resolve this, when the estimation software, in `dynare_estimation_1.m`, encounters such a problem (which turns out to be more prevalent under imperfect information because there are considerably more stages involved in the computations and therefore increased scope for numerical problems), the software uses the approximate Hessian (e.g. `chol(inv(hessian_csminwel))`) from the optimization subroutines – not however from the Nelder-Mead simplex algorithm as this does not utilize the Hessian.

the additional `varobs` command and the command `options_.usePartInfo=1;` specified at the beginning of the `.mod` file. Output has the identical format, including marginal likelihood results.

Appendix M sets out the details for the installation instructions. The user only needs to call this version of Dynare setup with the toolbox for all simulations and estimations with perfect and imperfect information. The only differences are in the syntax rules explained in Sections 3.1 and 5.

3.1 Dynare Syntax

The only changes that are required from standard Dynare syntax are (i) to declare in a `varobs` command those variables that are observed – in the example below `pi`, `y` and `r` can represent inflation, output and the nominal interest rate, respectively. If the `varobs` statement is not present then all endogenous variables are assumed to be observed too (identical to the case of perfect information); (ii) the inclusion of `partial_information` as an option in the `stoch_simul` command. Thus the final two lines of the program are:

```
varobs pi;
stoch_simul(partial_information, irf=20) pi y r;
```

This option instructs `stoch_simul` to use partial information (PCL) solver and produces all the second order imperfect information statistics conditional on the observed `pi` that would normally be produced by Dynare, with one exception: since the covariance matrix of the variables is a nonlinear function of the covariance matrix of the shocks, it is impossible to generate a variance decomposition.

3.2 Invertibility (Rank) Condition

Before the computation of first and second order theoretical moments (variance decompositions are omitted), the PCL solver also checks and reports a sufficient condition for imperfect information to be equivalent to perfect information when number of observables = number of shocks (a necessary condition). From (16) it requires $EP^AJ'(JP^AJ')^{-1}$ to be of full rank (necessary condition) and that J is of full row rank (sufficient condition), then imperfect information is equivalent to perfect information, and the system is then invertible.[14] If only the latter is rank-deficient, the Dynare output automatically generates a message

```
THE INVERTIBILITY CONDITION IS NOT SATISFIED:
no. of measurements = no. of shocks, but cannot mimic perfect information.
```

[14]In addition, the Riccati equation (17) is solved using the subroutine `dare.m`, that is located in MATLAB's Control System Toolbox.

This implies that the VARMA RE solution of the model is not invertible and no longer has a VAR reduced-form representation. For systems that are otherwise invertible under imperfect information, the simulation output in Dynare generates:

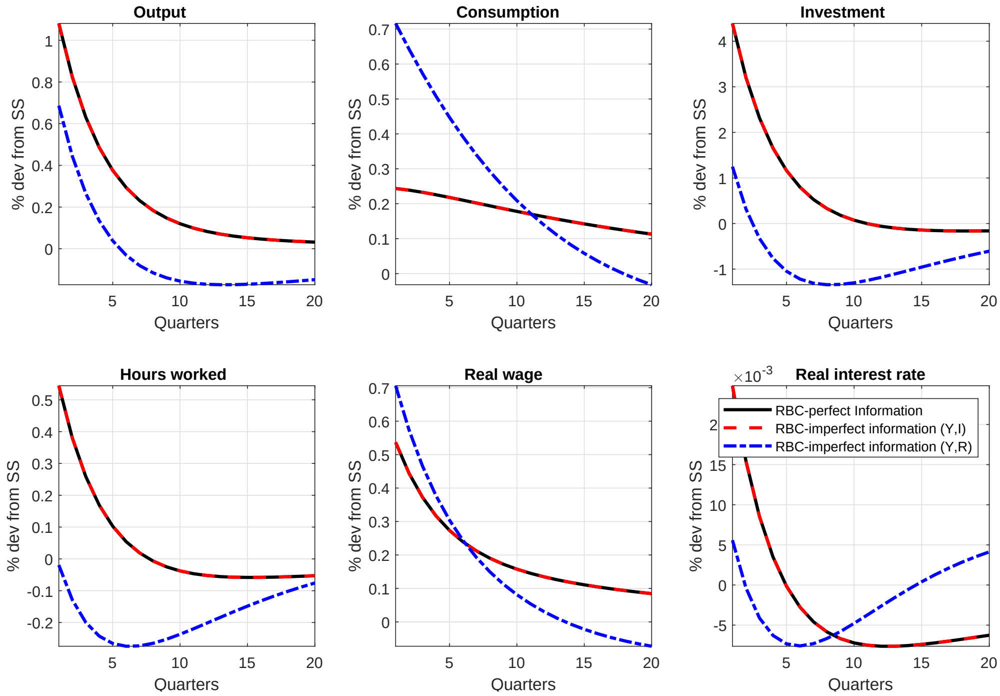
```
--- THE INVERTIBILITY CONDITION IS SATISFIED ---
no. of measurements = no. of shocks,
imperfect information is equivalent to perfect information
```

3.3 A Simple Application

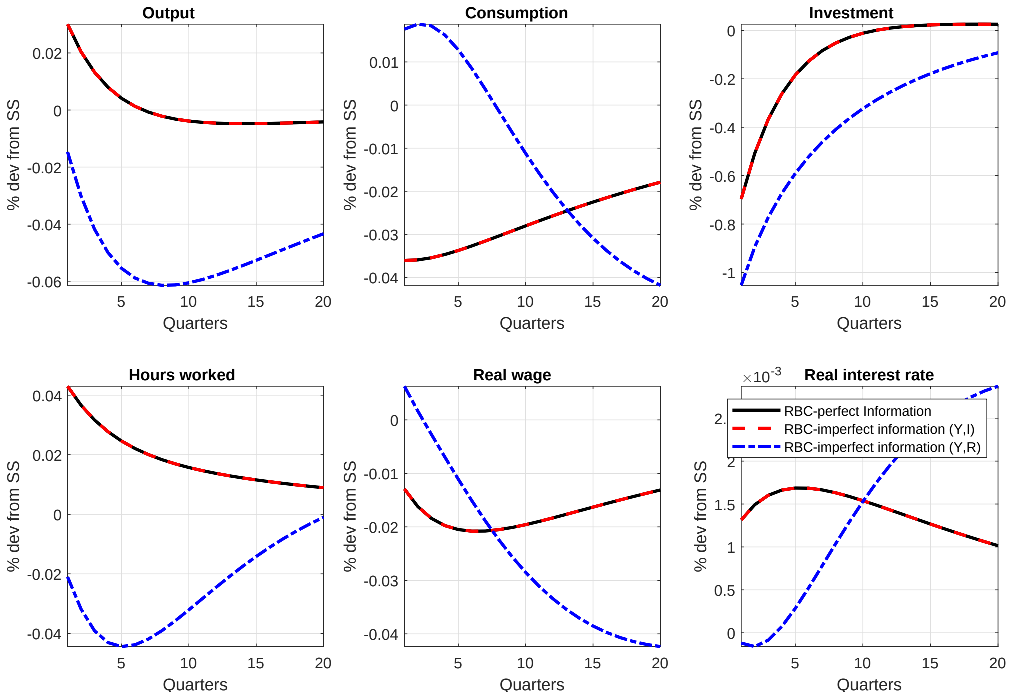
Before proceeding to our seven examples provided in the toolkit we pick out the RBC model of a decentralized economy (`rbc_invertibility.mod`) used in Examples 2 and 3.[15] With two shock processes, A_t and G_t, the following combinations of two observables result in no difference between perfect and imperfect solution procedures (i.e. the invertibility condition holds): (Y_t, C_t), (Y_t, I_t), (I_t, H_t) and (I_t, W_t). On the other hand, the following combinations do generate a difference: (Y_t, R_t), (W_t, R_t) and (C_t, R_t), when the rank condition fails. Figure 1 below plots the simulated deterministic IRFs based on different combinations of the observables. As noted, the Dynare output generates the additional invertibility message in the latter case:

```
SOLUTION UNDER IMPERFECT INFORMATION
OBSERVED VARIABLES Y, R

--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements = no. of shocks,
but imperfect information cannot mimic perfect information

THEORETICAL MOMENTS
VARIABLE          STD. DEV.        VARIANCE
Y              1.1930382312    1.4233402212
C              1.6517040243    2.7281261840
I              5.1967903060   27.0066294846
```

[15]More details and analysis can be found from and Appendix D below, and in Levine *et al.* (2019).



(a) Technology Shock: A_t



(b) Government Spending Shock: G_t

Figure 1: **Impulse Response Functions for RBC: Technology and Fiscal Shocks**

Notes: Each panel plots the mean response corresponding a positive one standard deviation of the shock's innovation. Each response is for a 20 period (5 years) horizon and is level deviation of a variable from its steady-state value in the RBC model. Observations Y_t, R_t (dashed red) lead to a failure of the rank condition so we cannot recover the exact values of the shocks. Observations Y_t, I_t (dashdot blue) show an example of matching IRFs for a case when imperfect information is equal to perfect information.

4 Estimation under Imperfect Information Using Dynare

Internally, the program reflects the fact that the likelihood function for a given draw of parameters depends on the information set. To evaluate the likelihood for a given set of parameters (prior to multiplying by their prior probabilities), the econometrician takes the equations (13)–(16) as representing the dynamics of the system under imperfect information.

It is a standard result for normally distributed observations that apart from constants, we can write the likelihood function as

$$2\ln L = -\text{Tr}\ln(2\pi) - \sum_{t=1}^{T}\left[\ln\det(\text{cov}(\hat{\varepsilon}_t)) + \hat{\varepsilon}_t'(\text{cov}(\hat{\varepsilon}_t))^{-1}\hat{\varepsilon}_t\right] \tag{24}$$

where the innovations process $\hat{\varepsilon}_t \equiv m_t^E - \mathbb{E}_{t-1}m_t^E$, T is the number of time periods and r is the dimension of m_t^E.

In order to obtain $E_{t-1}m_t$, we need to solve the appropriate filtering problem. Defining $\bar{v}_t = \bar{s}_{1t}$, with initial value $\bar{v}_0 = 0$, the Kalman filter updates are given by

$$\bar{v}_{t+1} = A\bar{v}_t + AZ_tE'(EZ_tE')^{-1}\hat{\varepsilon}_t \qquad \hat{\varepsilon}_t \equiv m_t^E - E\bar{v}_t$$

$$Z_{t+1} = AZ_tA' - AZ_tE'(EZ_tE')^{-1}EZ_tA' + P^AJ'(JP^AJ')^{-1}JP^A \tag{25}$$

the latter being a time-dependent Riccati equation. The initial value of Z_t is given by

$$Z_0 = AZ_0A' + P^AJ'(JP^AJ')^{-1}JP^A \tag{26}$$

and $P^A = cov(\bar{z}_0)$, the Riccati matrix defined earlier. Finally, $cov(\hat{\varepsilon}_t) = EZ_tE'$.

An interesting result emerges from examination of (25). We note that the rank k of the positive semi-definite matrix $P^AJ'(JP^AJ')^{-1}JP^A$ is $\leq rank(J)$, where we recall that the number of rows of K is the number of measurements at each period. Thus the updating equations are in effect being driven by a set of k shocks, which yields the following:

Theorem 3. *If $rank(J) <$ the number of observables, then the system under AII is over-identified, or the likelihood function is singular. If this is the case, then we have to exclude a subset of the measurements in order to estimate the system, or to incorporate measurement error into the system.*[16]

5 The .mod File and Syntax

For versions 4.2.x and following, the only change to the `.mod` file that is required is to declare:

```
options_.usePartInfo=1;
```

[16]See, for Proof of Theorem, Levine *et al.* (2019): Theorem 6.

Description: This triggers the partial information estimation software, and must be used in conjunction with the `varobs` command that lists the variables that agents observe. Note that at the moment this is only suitable for estimation under information symmetry as the observable set declared after `varobs VARIABLE NAME...;` is shared by agents and econometrician, where the variables in `varobs` are those that are members of the information set. If, for example, inflation is observed with a lag, then a new variable `piL=pi(-1)` must be defined, and then `piL` is listed in the `varobs` command. If we use observations with a lag and the information set for lag 1 case at time t is $I_t = \{Y_{t-1}, \Pi_{t-1}, R_{n,t}\}$.

`Example.mod`:

```
options_.usePartInfo = 1;
...
piL = pi(-1);
yL  = y(-1);
...

varobs piL yL r;
estimation(datafile=data, OPTIONS, ...);
```

In the future version of Dynare, the partial information estimation should be triggered by the keyword `partial_information` in the `estimation` command and `varobs VARIABLE NAME...;` declares the common set of observed variables.

Same as in moment computations, Kalman filtering and likelihood evaluation for partial information estimation requires a time-dependent solution of the Riccati equation in (25), where P^A is given by (17), to be calculated iteratively using MATLAB's built-in `dare.m` which requires the Control System Toolbox. Unlike the procedure described in Invertibility/Rank section, this part of software does not report the rank condition (that is, the condition defined earlier that shows whether equal numbers of shocks and observables led to an equivalence between perfect and imperfect information[17]).

6 Examples and Results

The partial information Kalman filter based estimation and DSGE-VAR estimation produce new empirical results in the literature. Parameter estimates under AII for DSGE models are often not very different from parameter estimates assuming perfect information on the part of agents. However, because of the endogenous persistence effects of AII,[18]

[17]The same rank condition is used to tell whether the model under imperfect information is over-identified above.

[18]There is more persistence in the model, which is generated endogenously by *learning through Kalman Filter forecasts* of $z_{t,t}$ and $x_{t,t}$ with imperfect information. By construction, the matrices D and F represent

IRFs under AII tend to match those from VAR estimation better than do IRFs under perfect information. One would therefore expect that on balance second moments tend to be better under AII, leading a better model fit (less misspecified measured by a DSGE-VAR benchmark) because all these moments are summarized as described earlier via the likelihood function. For further explanation and more information on the empirical analysis, see, Levine *et al.* (2012) estimating various specifications of a canonical NK-DSGE model assuming AII, and Cantore *et al.* (2015) for an empirical application with slightly modified versions of an industry standard DSGE model (Smets and Wouters (2007)).

The following example working models are run under imperfect information, compared with the conventional perfect information output, and using the latest Dynare version 4.6.2. Examples 1 and 7 in their original form are available for download from the Website (DYNARE: www.dynare.org) and are briefly described in Appendices C and E. Model 2 (for Examples 2 and 3) is set out and reported in Levine *et al.* (2019) and Appendix D presents the functional forms and equilibrium conditions used in this paper. Example 5 is built on Smets and Wouters (2007)'s own code which is available at www.aeaweb.org and again, for convenience, we remind the reader of the linearized version of equilibrium conditions in Appendix G below. To sum up,

- Example 1 in Section 6.1: Fernandez-Villaverde basic non-linear RBC model in the form of the social planner's problem with a labour-augmenting technology shock (file name: `fvrbc_II.mod` and `fvrbc_II_YL.mod`)

- Examples 2 and 3 in Sections 6.2 and 6.3 respectively: Non-linear RBC model of a decentralized economy with two shocks set out and described in Appendix D of this paper and in Levine *et al.* (2019)
(file name: `rbc_invertibility_II.mod`)

- Example 4 in Section 6.4: Estimation and comparison for a small-scale non-linear NK model with sticky prices and flexi wages and with different numbers (3 and 4) of observable variables
(file name: `NK_3_obs.mod` and `NK_4_obs.mod`)

- Example 5 in Section 6.5: The Smets and Wouters (2007) model in linearized form and modified versions of the model adding an inflation target shock and measurement errors to the observations
(file name: `sw07_invert.mod`, `sw07_invert_inf.mod` and `sw07_invert_inf_me.mod`)

- Example 6 in Section 6.6: Estimation and comparison of different versions of the linearized Smets and Wouters (2007) model
(file name: `sw07est_II.mod` and `sw07est_inf_me_II.mod`)

- Example 7 in Section 6.7: Estimation of a standard CIA non-linear model from Schorfheide (2000)
(file name: `fs2000_II.mod` and `fs2000_invert_II.mod`)

pure endogenous persistence mechanisms independent of Kalman Filter learning: Levine *et al.* (2019) provide more explanations and a theoretical example to show this effect.

6.1 Example 1: Invertibility of Social Planner's RBC Model

The following syntax rule first triggers the imperfect information solution and simulation, and produces the second order imperfect information statistics conditional on observing output (Y_t) in the RBC model:

```
alpha   = 0.33;
beta    = 0.99;
delta   = 0.023;
psi     = 1.75;
rho     = 0.95;
sigma   = (0.007/(1-alpha));

shocks;
var e = sigma^2;
end;

varobs Y;
stoch_simul(partial_information, OPTIONS, ...);
```

With one shock in the system, the technology evolution, and one observable under AII (Y_t), the computation results in no difference between perfect and imperfect solution procedures. This first example also shows that, when converting the AII state space to the Blanchard-Kahn form, the non-singularity condition holds for the C_2 matrix so the iterative reduction algorithm described in Appendix A.2 is not required in this case (Appendix L.1 reports the further Dynare output on measures of the invertibility and fundamentalness checks). The output produced below exactly replicates that of the simulation using the original `fvrbc.mod`:

```
STEADY-STATE RESULTS:
Y   1.0301
C   0.793902
K   10.2696
I   0.236201
H   0.331892
z   0

 --- Transformation to Blanchard-Kahn Form ---
Obtain the singular value decomposition of A0
C2 is invertible: go to final stage of conversion
```

```
SOLUTION UNDER PARTIAL INFORMATION

OBSERVED VARIABLES
    Y

--- THE INVERTIBILITY CONDITION IS SATISFIED ---
no. of measurements = no. of shocks,
imperfect information is equivalent to perfect information

THEORETICAL MOMENTS
VARIABLE         STD. DEV.        VARIANCE
Y            0.0387828201    0.0015041071
C            0.0218420076    0.0004770733
K            0.3902171827    0.1522694497
I            0.0214464815    0.0004599516
H            0.0039308552    0.0000154516
z            0.0334596142    0.0011195458
```

If observations are made with a lag this *always* leads to a failure of the rank condition. This is confirmed by Table 1 where JB is not invertible despite J being of full rank (this also applies to EB and E in the case of perfect information). Recall Theorem 2 and Lemma 1. In `rbc_II_YL.mod`, `YL = Y(-1)` defines the lagged output observed by the agents under AII:

```
varobs YL;
stoch_simul(partial_information, OPTIONS, ...);
```

As expected, the invertibility condition is no longer satisfied (the additional output is recorded in Appendix L.2) and the imperfect information results are now *very different* to the perfect information case:

```
SOLUTION UNDER PARTIAL INFORMATION

OBSERVED VARIABLES
    YL

--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
```

```
no. of measurements = no. of shocks,
but imperfect information cannot mimic perfect information

THEORETICAL MOMENTS
VARIABLE          STD. DEV.        VARIANCE
Y              0.0397194642    0.0015776358
C              0.0222620871    0.0004956005
K              0.4012093442    0.1609689378
I              0.0233310511    0.0005443379
H              0.0044545383    0.0000198429
z              0.0334596142    0.0011195458
YL             0.0397194642    0.0015776358
```

Finally, Figure 2 below shows the deterministic IRFs in response to an unanticipated labour technology shock e_t for all the three simulated models: `fvrbc.mod`, `fvrbc_II.mod` and `fvrbc_II_YL.mod`. There are exactly matching IRFs for the former two, the perfect information and AII models, as expected, but the failure of the rank condition indicates the different IRFs as an example for such a case with lagged output observed by the agents (these are the dashdot blue responses).

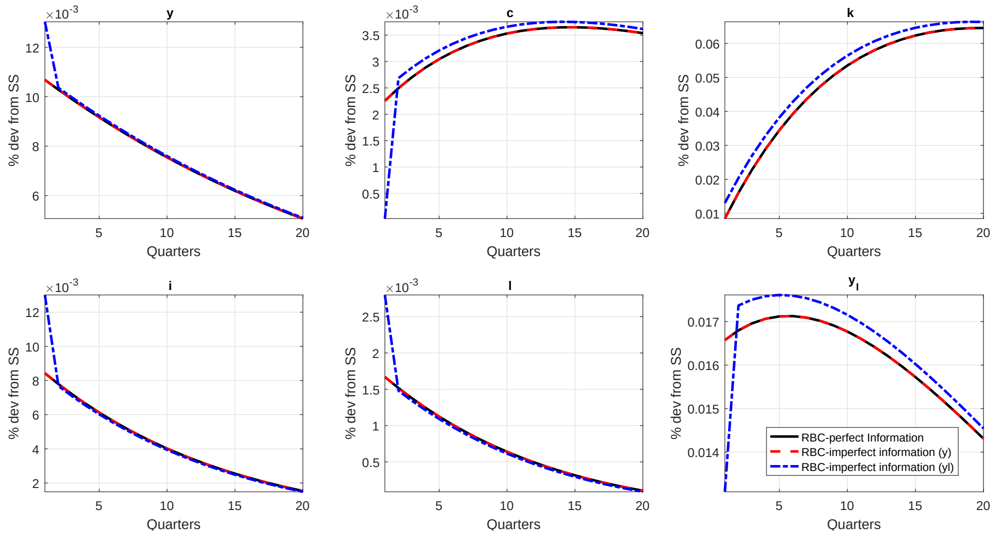


Figure 2: **Impulse Responses to a Labour Technology Shock e_t (Example 1)**

6.2 Example 2: Invertibility of Decentralized Economy RBC Model

Example 2 considers the RBC model in Section 3.2 and in Appendix D for a decentralized economy. With two shock processes, A_t and G_t, the following combinations of two observables are reported in Table 1 that summarises 4 different scenarios. As noted, if observations are made with a lag this obviously always leads to a failure of the rank condition as shown in the last row of Table 1 (i.e. EB is no longer of full rank). The cases in the first row are combinations of two observables (from a set of observables: $(Y_t, C_t, I_t, H_t, W_t, R_t)$) result in no difference between perfect and imperfect information solution procedures and exactly replicate the results that would normally be produced by Dynare without any informational assumptions.[19]

Combinations of observables and $m = k$ (where EB is of full rank)	Theorem 2	Description
$(Y_t, C_t), (Y_t, H_t), (Y_t, I_t)$ $(Y_t, W_t), (C_t, H_t), (C_t, W_t)$ $(I_t, H_t), (I_t, W_t), (H_t, W_t)$	J is of full (row) rank JB is of full (row) rank Q_A is stable	System is invertible; AII is equiv. to API
$(Y_t, R_t), (C_t, R_t), (I_t, R_t)$ $(W_t, R_t), (H_t, R_t)$	J is rank deficient JB is rank deficient	System is not invertible; AII is not equiv. to API
(C_t, I_t)	J, JB of full (row) rank Q_A is **not stable**	System is not invertible; AII is not equiv. to API
Lagged observations (where EB is rank deficient)	J is of full (row) rank JB is rank deficient	System is not invertible; AII is not equiv. to API

Table 1: **Summary of Invertibility (Rank) Condition for RBC (Example 2)**

The most common non-obvious reason for AII not to be equivalent to API is associated with the second row in the table, where J is not of full row rank when EB is invertible.[20] Recall that Theorem 2 establishes an extra condition, given that models under perfect information (with API) are E-invertible, that the square matrix JB is of full rank, *and* $Q_A = F(I - B(JB)^{-1}J)$ is a stable matrix (has all eigenvalues inside the unit circle), for AII to be E-invertible too. In the third row of Table 1, we report the only case with (C_t, I_t) when this eigenvalue condition for AII is not satisfied, despite of J being of full rank. Full details of the invertibility check including the additional eigenvalue condition based on Theorem 2 are reported below in Table 2 and Appendix L.

We now consider a simplified non-linear RBC model without investment adjustment costs and variable hours (i.e. $H_t = \bar{H} = 1$ and $\varrho = 0$), in line with the linearized 'stochastic growth' model of Campbell (1994), with a single observable, the real interest rate $R_{K,t}$. The model is a special case of the full RBC model (set out in Appendix D). In linearized form the structure of the model is described in Section 5 of Levine *et al.* (2019) as a simple analytical example that demonstrates suitable combinations of parameters α and σ for which E-invertibility holds based on the root of the MA component of the model (Figure 1 of Levine *et al.* (2019)). In this exercise, we want to first show there is a complete agreement between the numerical and analytical results with $R_{K,t}$ observable and one shock presented in Section 5 and Table 2 in Levine *et al.* (2019), respectively.

[19] Note also that, as explained, if $JB(EB)$ is not invertible despite $J(E)$ being of full rank, then this implies that the imperfect information set in effect contains a lagged variable (and API is not E-invertible either).

[20] See, for more details, Corollary 2.1 and Corollary 2.2 in Levine *et al.* (2019).

Figure 3 below shows the E- and A-invertibility regions for this RBC model with $R_{K,t}$ the only observable and one shock, A_t. For E-invertibility under API, it requires the risk parameter $\sigma_c \ll 1$ and this completely agrees with the numerical results reported in Table 2 in Levine *et al.* (2019). This is also consistent with the analytical results reported on the E-invertibility for the RE solution Campbell (1994)'s RBC model.

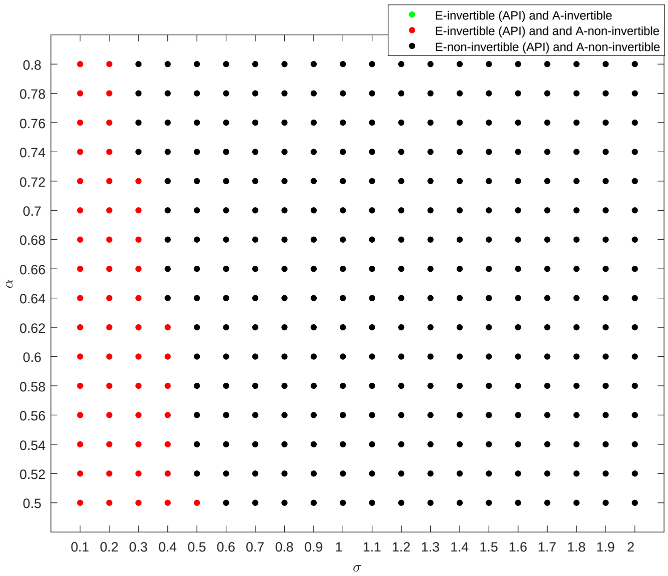


Figure 3: **E- and A-invertibility Regions over Parameters σ_c and α**

Notes: This shows the E- and A-invertibility regions for the linearized model of Campbell (1994) set out as an analytical example in Levine *et al.* (2019), and a simplified non-linear RBC model presented in Table 2 of Levine *et al.* (2019). In line with Figure 1 of Levine *et al.* (2019), we choose $\sigma_c \in [0.1, 2]$ and $\alpha \in [0.5, 0.8]$.

In addition, this second example also shows that using an example of a standard RE model when the invertibility condition fails under AII this requires the iterative reduction algorithm when converting the AII state space to the Blanchard-Kahn form described in Appendix B in Levine *et al.* (2019) and in Appendix A.2 below. This procedure is required to yield a suitable reduced-form system which is to be processed via the Kalman filter to obtain the likelihood function for estimation purposes. At the stage when the calibrated model reports that $C_2 + C_6$ is not invertible, where C_6 is the matrix associated with $s_{t,t}$ (and s_t defines the backward-looking states in the system), Stage 4 in Appendix A.2 is now required to be iterated to reduce the dimension of the forward-looking matrices by a finite number of times and increase the dimension of the backward-looking C_i matrices by the same amount, until $C_2 + C_6$ is non-singular. In particular, the following algorithm is implemented for the RBC example below:

1. Obtain the singular value decomposition for A_0 and partitions of A_0 from (A.3);

2. Transform (A.3) to forward-looking subsystem and and re-define forward-looking system matrices F_i, $i = 1, ..., 5$ according to Stage 2 in Appendix A.2;

3. Transform (A.3) to backward-looking subsystem and and re-define backward-looking

system matrices C_i, $i = 1, ..., 5$ according to Stage 3 in Appendix A.2;

4. The algorithm reports that $C_2 + C_6$ is not invertible, Stage 4 is now required to be iterated once to reduce the dimension of the forward-looking F matrices by 1 and increase the dimension of the backward-looking C matrices by 1 (this is done through the reduction procedure (A.13)–(A.16)). Re-define C_2, C_2+C_6 using (A.16) which is now of full rank;[21]

5. Generate C_2^{-1} and $(C_2 + C_6)^{-1}$, proceed to the following stages, and we have the required Blanchard-Kahn form set out by (A.27) and (A.36).

For example, when the agents in the RBC model observe (Y_t, R_t) with AII, the program now reports a `Singular matrix C2` (where C_2 is the general term for $C_2 + C_6$), as well as the failed rank condition (due to the rank deficiency of JB and J):

```
 --- Transformation to Blanchard-Kahn Form ---
Obtain the singular value decomposition of A0
Singular matrix C2 ...
Start iterative reduction procedure ...
Invertibility and return to conversion

SOLUTION UNDER PARTIAL INFORMATION

OBSERVED VARIABLES
    YY
    RR

--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements = no. of shocks,
but imperfect information cannot mimic perfect information

THEORETICAL MOMENTS
VARIABLE          STD. DEV.        VARIANCE
YY             1.1930382312    1.4233402212
CC             1.6517040243    2.7281261840
II             5.1967903060   27.0066294846
```

6.3 Example 3: Approximate Fundamentalness of Example 2 RBC Model

Before moving to estimation, we want to assess if the models are able to generate the theoretical responses to a fundamental shock. We now consider and implement the multivariate

[21] As a general case, some models may require this stage to be repeated up to a finite number of times until $C_2 + C_6$ is non-singular.

measure of goodness of fit set out in Section 2. We compare numerically the perfect and imperfect information measures of the fit of the innovations to the fundamentals for Model Example 2. The maximum eigenvalue provides a measure of overall non-fundamentalness. In addition, any zero eigenvalues provide information as to which fundamental shocks can be satisfactorily identified (i.e. evidence of partial sufficiency of individual shocks in the system). Table 2 below checks the difference between perfect and imperfect information in terms of identifying the fundamentals from the perspective of VARs via the eigenvalues of $\mathbb{F}^{PI}$ and $\mathbb{F}^{II}$, assuming that the RBC Model is the DGP.

Combinations of observables ($m \leq k$), Lemma 1	Theorem 2, Corollary 2.1 in Levine *et al.* (2019)	Eigenvalues of $\mathbb{F}^{PI}$ Eigenvalues of $\mathbb{F}^{II}$	Diagonal values of $\mathbb{F}^{PI}$ Diagonal values of $\mathbb{F}^{II}$
$(Y_t, C_t), (Y_t, H_t), (Y_t, I_t)$ $(Y_t, W_t), (C_t, H_t), (C_t, W_t)$ $(I_t, H_t), (I_t, W_t), (H_t, W_t)$	rank(E)=rank(EB)=2 rank(J)=rank(JB)=2 $A(I - B(EB)^{-1}E)$ is stable Q_A is stable	$\mathbb{F}^{PI} \equiv \mathbb{F}^{II} = 0$ $eig(\mathbb{F}^{PI}) \equiv eig(\mathbb{F}^{II})$ $= [0, 0]$	
Cases when AII is not equivalent to API			
Rank condition failure for AII			
(Y_t, R_t), E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable	J,JB are rank deficient (=1) Q_A is non-existent	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.1186, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.1190, 0.9996]$
(C_t, R_t), E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable	J,JB are rank deficient (=1) Q_A is non-existent	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.0292, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0345, 0.9946]$
(I_t, R_t), E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable	J,JB are rank deficient (=1) Q_A is non-existent	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.0018, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0166, 0.9852]$
(W_t, R_t), E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable	J,JB are rank deficient (=1) Q_A is non-existent	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.5169, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.5195, 0.9975]$
(H_t, R_t), E, EB are of full rank $A(I - B(EB)^{-1}E)$ is **not stable**	J,JB are rank deficient (=1) Q_A is non-existent	$eig(\mathbb{F}^{PI}) = [0, 0.6593]$ $eig(\mathbb{F}^{II}) = [0.0137, 1]$	$\mathbb{F}_i^{PI} = [0.0057, 0.6597]$ $\mathbb{F}_i^{II} = [0.0149, 0.9988]$
Eigenvalue condition failure for AII			
(C_t, I_t), E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable	J,JB are of full rank (=2) Q_A is **not stable**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0, 0.9167]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0217, 0.8950]$
One observation: rank(E)=rank(J)=1			
(C_t), E, EB are rank deficient (=1) $A(I - B(EB)^{-1}E)$ is non-existent	J,JB are rank deficient (=1) Q_A is non-existent	$eig(\mathbb{F}^{PI}) = [0, 1]$ $eig(\mathbb{F}^{II}) = [0.0901, 1]$	$\mathbb{F}_i^{PI} = [0.0214, 0.9786]$ $\mathbb{F}_i^{II} = [0.0952, 0.9950]$
Lagged observations: rank(E)=rank(J)=2			
(Y_{t-1}, C_{t-1}), EB are rank deficient (=0) $A(I - B(EB)^{-1}E)$ is non-existent	JB are rank deficient (=0) Q_A is non-existent	$eig(\mathbb{F}^{PI}) = [1, 1]$ $eig(\mathbb{F}^{II}) = [0.9967, 0.7620]$	$\mathbb{F}_i^{PI} = [1, 1]$ $\mathbb{F}_i^{II} = [0.9961, 0.7626]$

Table 2: **Summary of Non-fundamentalness Measures for RBC (Example 3)**

Notes: Order of shocks: A_t, G_t. See, for a complete set of results, Levine *et al.* (2019). See also, for the corresponding Dynare output for the cases with $(Y_t, C_t), (Y_t, R_t), (H_t, R_t), (C_t, I_t), (C_t)$ and (Y_{t-1}, C_{t-1}), in Appendices L.3–L.8.

For the case of the system being invertible, and EB is of full rank, the solutions of the Riccati equation (specified in Levine *et al.* (2019) for S and by (17) for P^A) are $S = P^A = BB'$ and, from which it follows that $\mathbb{F}^{PI} = \mathbb{F}^{II} = 0$, and the two processes are perfectly correlated across the perfect information and AII cases. For the case of non-invertibility, the further is $\mathbb{F}^{II}$ from 0, the worse is the fit. Examples $(Y_t, R_t), (C_t, R_t), (I_t, R_t), (W_t, R_t)$ in the table show the cases while the perfect information solution is invertible (or there is complete fundamentalness, i.e., $\mathbb{F}^{PI} = 0$) the imperfect information counterparts are not (i.e. $\mathbb{F}^{II} > 0$ in the positive definite sense). The only way to decide the overall fit of the RBC model approximating the fundamentals by the innovations process is to determine the maximum eigenvalue of $\mathbb{F}^{II}$. In Table 2, the fit of the innovations to the structural shocks under AII is very poor as the eigenvalues are all far from 0, when JB is not of full row rank and the eigenvalue condition fails. The exceptions are for some cases when the symmetric limited information set is contemporaneous, in which case, the first eigenvalue being very close to 0 (e.g. with (I_t, R_t) and (C_t, I_t)) indicates partial fundamentalness or that one of the the two shocks may be satisfactorily identified in this model. When there are large differences in the impulse response functions under imperfect and perfect information, non-fundamentalness may be quantitatively severe, indeed according to Theorem 4 in Levine *et al.* (2019), the simulation appears to indicate that this may be a major issue.

The last column of Table 2 reports the diagonal values of the (non-zero) $\mathbb{F}^{PI}$ and $\mathbb{F}^{II}$ matrices. These tell us explicitly about the goodness of fit of the residuals to the structural shocks (A_t and G_t). Any zero values reported in the diagonal matrices indicate an exact fit for the corresponding individual shocks in the models. Clearly, the goodness of fit deteriorates when switching from API to AII, and as we shall show below, the deterioration is more significant depending on the size of the model and the number of shocks included.[22]

The procedure that computes the '$\mathbb{F}$ Test' for the multivariate measures of correlation requires calling an additional `.m` file in the directory, and reports Table 2, for different combinations of observables. For example for Y_t, R_t:

```
varobs YY RR;
stoch_simul(partial_information,irf=0);
```

This above command produces Appendix L.4, where the program checks the Theorem 2 conditions, the rank of the relevant matrices, the eigenvalue stability conditions and the fundamentalness condition in the form of matrices F^{PI} and F^{II}:

```
Measures of Invertibility and Fundamentalness
Matrix      E  EB   J  JB
Rank        2   2   1   1
The Eigenvalue Condition for PI is satisfied

MATRIX F WITH PI
Shocks       epsA    epsG
epsA      -0.0000  0.0000
epsG       0.0000 -0.0000
Shocks       epsA    epsG
Eigen      0.0000  0.0000

MATRIX F WITH II
```

[22] This is shown by the simulated Smets and Wouters (2007) models and a further illustrative exercise on the RBC model with a news shock can be found in Appendix I of Levine *et al.* (2019).

```
Shocks      epsA    epsG
epsA      0.1190  0.0189
epsG      0.0189  0.9996
Shocks      epsA    epsG
Eigen     0.1186  1.0000
```

If any of the eigenvalue conditions in Theorem 2 fails, the program displays a message: `The Eigenvalue Condition is not satisfied`. The sixth row of Table 2 and Appendix L.5 present an interesting special case with observable set (H_t, R_t), where API is not E-invertible and is not equivalent to AII when $m = k$, since, even though EB is of full rank $A\left(I - B(EB)^{-1}E\right)$ is not a stable matrix.

Now it is important for us to understand, from this example, why where EB is of full rank, but J and JB are not, thus AII becomes E-non-invertible? Note that the AII observable set in this case is (Y_t, R_t), then it is clear that what is causing J to be rank-deficient is the inclusion of R_t observed by the agents. To understand this, we recall that in (15) and (16) J and JB capture the contemporaneous impact of shocks on the observed variables. We have M_1 and M_2 which are observation mapping matrices so the observations under AII can be written in the form

$$m_t^A = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} \varepsilon_{t,t} \\ s_{t-1,t} \\ x_{t-1,t} \\ x_{t,t} \end{bmatrix} \tag{27}$$

Since $J \equiv M_1 - M_2 G_{22}^{-1} G_{21}$ and B captures the direct impact of ε_t in (A.27). From the JB matrix, it is clear that the model's assumption implies that R_t in the RBC model should not respond immediately to either of the two shocks. This is because, in the model used in Appendix D, there is no interest rate rule and the agents are assumed to observe, at time t, only (Y_t, R_t) and (A_t, G_t). This is also the reason, when assuming R_t as part of agents information set, the interest rate is left unchanged by either of the two shocks via the measurement system. The matrix capturing the immediate impact of shocks on the observed variables must always be rank-deficient with $(Y_t, R_t), (C_t, R_t), (I_t, R_t), (W_t, R_t)$ and (H_t, R_t) under AII.[23]

6.4 Example 4: Estimation of a Small-scale NK Model

We extend the RBC model in Appendix D with sticky prices but flexible wages which will eventually lead to the Smets and Wouters (2007)'s setup in stages.[24] This is now an NK model with Calvo price-setting, with capital, costs of investment, consumption habit

[23] Corollary 2.2 in Levine *et al.* (2019) provides some more technical explanations for this result.

[24] The Smets and Wouters (2007)'s model we present in this paper has sticky wages and adds two further features, namely, capacity utilization and a fixed cost of converting the wholesale into a retail good.

and price indexation, and four shocks: shocks to the monetary policy rule, to labour productivity, to government spending and to price mark up. As is standard in NK models, Appendix F sets out the supply side of the economy which consists of the final and intermediate goods producers. Also in Appendix F, we briefly discuss price dispersion that captures the distortion caused by sticky prices. In this exercise, we also include a form of indexation to previous period market price. The linearized full NK model for estimation can be found in Appendix F.

We estimate the NK model and provide results from posterior optimization and provide posterior distribution from posterior simulation. We use the same data set as in Smets and Wouters (2007) in first difference at quarterly frequency. Namely, these observable variables are the log difference of real GDP, the log difference of real consumption, the log difference of the GDP deflator and the federal funds rate. The sample period is 1984:1-2008:2 which starts at observation 143 in the data file. There is a pre-sample period of 4 quarters so the observations actually used for the estimation go from 147:245.

We estimate the linear NK model (Appendix F) with 3 and 4 observables, respectively, and assuming in turn API and AII. The corresponding measurement equations for the 3 observables are

$$\begin{bmatrix} \Delta(\log GDP_t) * 100 \\ \log(GDPDEF_t/GDPDEF_{t-1}) * 100 \\ FEDFUNDS_t/4 * 100 \end{bmatrix} = \begin{bmatrix} Y_t - Y_{t-1} + \text{trend growth} \\ \Pi_t + \text{constant}_{\Pi} \\ R_{n,t} + \text{constant}_{R_n} \end{bmatrix} \tag{28}$$

Note that the quarterly trend growth rate in real GDP; the quarterly steady-state inflation rate and the steady-state nominal interest rate are estimated together with the other parameters.

The following Tables 3–5 report the computed likelihood (log posterior), marginal log data density (Laplace approximation), marginal log data density (Modified harmonic mean (MHM) estimator) and the Bayes factors (BF) compared across the estimated NK models under perfect and imperfect information, respectively. Table 5 summarises the moments analysis based on the estimated models.

	NK_PI (4 obs)	NK_II (4 obs)	LL diff
likelihood (log posterior)	-114.021906	-114.021906	0
log data density (laplace)	-154.701904	-154.701924	-2E-05
computing time	0h00m55s	0h02m22s	

Table 3: **Log Posterior and Data Density Comparison (NK 4 Observables)**

Notes: The two `.mod` file names are `NK_PI_4_obs.mod` and `NK_II_4_obs.mod`. The posterior mode (and log data density [Laplace approximation]) are estimated using `mode_compute=4`. Invertibility and fundamentalness checks satisfied (based on the mode). `periods=1000` for simulating artificial data using the mode (identical data between API and AII).

In Table 3, we report the results from first stage Bayesian log posterior estimation. As expected, for the case when the number of shocks equals the number of observables,

	NK_PI (3 obs)	NK_II (3 obs)	LL diff	BF
log data density (laplace)	-67.765791	-54.936186	12.829605	373101.2091
log data density (MHM)	-69.368172	-56.469285	12.898887	399866.8917
computing time	0h39m07s	2h57m59s		

Table 4: **Log Posterior and Data Density Comparison (NK 3 Observables)**

Notes: The two `.mod` file names are `NK_PI_3_obs_mh.mod` and `NK_II_3_obs_mh.mod`; The mode (and log data density [Laplace approximation]) are estimated using `mode_compute=6`. We first use the prior mean to start the mode-optimizer, then use the computed mode as initial conditions and combinations of `mode_compute=4,5,6` for checking for robustness. We also check the mode solutions using the alternative solvers including `mode_compute=7,8` to ensure no further improvement is possible. Then a sample from the posterior distribution is obtained with the Metropolis-Hastings (MH) algorithm using the inverse Hessian at the estimated posterior mode as the covariance matrix of the jumping distribution. The covariance matrix needs to be adjusted in order to obtain reasonable acceptance rates which are very similar across the chains (28% for both PI and II). Two parallel Markov chains of 250,000 runs each are run from the posterior kernel for the MH, sufficient to ensure convergence according to the indicators recommended by Brooks and Gelman (1998). The first 50,000 draws from each chain have been discarded.

the likelihood values of the parameters should be same, which it is, as is the value of the marginal likelihood. For the case of just three observables, having the additional shock leads to a better fit, under the assumption of imperfect information, as implied by the likelihood values and marginal data densities. To verify this, we also summarize some second order moments and present graphs of the autocorrelations of the observed variables below.

As can be seen, standard deviations are much better under AII than under API for inflation and the interest rate, while AII predicts the wrong sign for the correlations. As regards autocorrelations, AII is better for GDP and for inflation, generating the model persistence and matching the data correlogram well with longer horizons. Figure 4 depicts the estimated ACFs from the data and model. However the likelihood function encompasses all second order moments, so one would need a further analysis of all cross-autocorrelations to establish exactly where AII becomes noticeably superior to API.

Table 6 below reveals an interesting result with the standard NK model. Based on the estimated model, the fiscal shock policy seems to be approximately fundamental using the F^{PI} matrix, but not so using the appropriate F^{II} one. This suggests that the differences between IRFs for API and AII, from the perspective of identifying VARs, should be particularly noticeable. The estimated posterior IRFs are plotted in Appendix J, where we see that AII induces endogenous, hump-shaped persistence, when agents do not observe the shocks directly and have to use their observations and the Kalman filter to form an optimal forecasting rule.

6.5 Example 5: Approximate Fundamentalness of Smets and Wouters (2007)

We run our simulation exercise again using a version of Smets and Wouters (2007) model (henceforth SW). There are seven structural shocks in SW. The model has five AR(1) processes, for the shocks on government spending, technology, preference, investment spe-

Observable	$dlGDP_t$	$dlDEF_t$	INT_t
	Standard Deviation (in %)		
Data	0.5432	0.2392	0.5952
NK_PI (3 obs)	0.6067	0.3291	0.3944
NK_II (3 obs)	0.6411	**0.2940**	**0.4270**
	Cross-correlation with $dlGDP_t$		
Data	1.0000	-0.2013	0.0323
NK_PI (3 obs)	1.0000	0.0120	-0.0968
NK_II (3 obs)	1.0000	0.0453	**-0.0493**
	Autocorrelations (order=1)		
Data	0.1526	0.5364	0.9462
NK_PI (3 obs)	-0.0179	0.6227	0.9022
NK_II (3 obs)	**0.0607**	**0.4741**	**0.9371**

Table 5: **Data and Model Moments for NK Model**

	NK Model (3 obs)	
Theorem 2 Corollary 2.1 in Levine *et al.* (2019)	E, EB J, JB are rank deficient (=3) $A(I - B(EB)^{-1}E)$ is non-existent Q_A is non-existent	
Goodness of Fit	$\mathbb{F}^{PI}_{(4\times4)}$	$\mathbb{F}^{II}_{(4\times4)}$
Eigenvalues	$\begin{bmatrix} 1 \\ 0.0013 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.0410 \\ 0 \\ 0 \end{bmatrix}$
Diagonal values	$\begin{bmatrix} 0.0012 \\ 0.9988 \\ 0 \\ 0.0013 \end{bmatrix}$	$\begin{bmatrix} 0.0393 \\ 0.9998 \\ 0 \\ 0.0019 \end{bmatrix}$

Table 6: **Fundamentalness and Invertibility Measures for Estimated NK (Example 4)**

Notes: Order of shocks: technology, government spending, monetary policy and price markup. The simulation results in this table are based on the posterior estimates of the NK model for the parameters and shocks.

cific, monetary policy, and two ARMA(1,1) processes, for price and wage markup. In this example, we skip the description of the model (the linearized version is summarised in Appendix G) and slightly modify the model by gradually adding more shocks. The SW model is estimated based on seven quarterly macroeconomic time series. When we assume that this exactly coincides with the agents' limited information set so in effect the number of measurements is equal to the number of shocks and EB is non-singular (Case 1: Original SW). In the modified versions of the model, the only changes we make are that (1) we add an inflation target shock so the number of shocks exceeds the number of observables (Case 2: SW with 8 shocks); (2) we further add measurement errors to the observations of real variables and inflation (Case 3: SW with 13 shocks). Table 7 presents the key results from the simulation, based on Levine *et al.* (2019) and the test for non-fundamentalness introduced in Section 6.3.

	Case 1: Original SW Appendix L.9	Case 2: SW with Inflation Obj. Appendix L.10		Case 3: SW with MEs Appendix L.11	
	Measurements = Shocks (=7)	8 Shocks		13 Shocks	
Theorem 2 Corollary 2.1 in Levine *et al.* (2019)	E, EB are full row rank (=7) J, JB are full row rank (=7) $A(I - B(EB)^{-1}E)$ is stable Q_A is stable	E, EB are rank deficient (=7) J, JB are rank deficient (=7) $A(I - B(EB)^{-1}E)$ is non-existent Q_A is non-existent		E, EB are rank deficient (=7) J, JB are rank deficient (=7) $A(I - B(EB)^{-1}E)$ is non-existent Q_A is non-existent	
Goodness of Fit	$\mathbb{F}^{PI} = \mathbb{F}^{II} = 0$	$\mathbb{F}^{PI}_{(8\times8)}$	$\mathbb{F}^{II}_{(8\times8)}$	$\mathbb{F}^{PI}_{(13\times13)}$	$\mathbb{F}^{II}_{(13\times13)}$
Eigenvalues	$eig(\mathbb{F}^{PI}) = eig(\mathbb{F}^{II}) = 0$	$\begin{bmatrix} 1 \\ 0.0013 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.0016 \\ 0.0009 \\ 0.0001 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.0971 \\ 0.0454 \\ 0.0138 \\ 0.0001 \\ 0.0019 \\ 0.0058 \\ 0.0100 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.5404 \\ 0.3627 \\ 0.2975 \\ 0.0302 \\ 0.0011 \\ 0.0044 \\ 0.8182 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
Diagonal values	-	$\begin{bmatrix} 0 \\ 0.0006 \\ 0 \\ 0.0005 \\ 0.0245 \\ 0 \\ 0.0001 \\ 0.9756 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.0006 \\ 0 \\ 0.0004 \\ 0.0256 \\ 0 \\ 0.0001 \\ 0.9761 \end{bmatrix}$	$\begin{bmatrix} 0.2216 \\ 0.0924 \\ 0.5199 \\ 0.1600 \\ 0.1007 \\ 0.2262 \\ 0.2585 \\ 0.9780 \\ 0.4668 \\ 0.7097 \\ 0.9053 \\ 0.8353 \\ 0.6998 \end{bmatrix}$	$\begin{bmatrix} 0.5754 \\ 0.8850 \\ 0.5136 \\ 0.6945 \\ 0.1099 \\ 0.4552 \\ 0.7095 \\ 0.9782 \\ 0.5892 \\ 0.6749 \\ 0.6672 \\ 0.7165 \\ 0.4854 \end{bmatrix}$

Table 7: **Fundamentalness and Invertibility Measures for Calibrated SW Model)**

Notes: Order of shocks: technology, preference, government spending, investment specific, monetary policy, price and wage markup, inflation objective and measurement errors for output growth, consumption growth, investment growth, real wage growth and inflation (Appendices L.9–L.11 record the Dynare output corresponding to the results in this table for the three cases). We assume unit standard deviations so the shocks in this calibrated version are normalized to have unit covariances.

As before, the models are solved and simulated through the conversion procedure set out in Appendix A.2. We find that the original system is completely invertible according to

	Case 1: Original SW Appendix L.12	Case 2: SW with Inflation Obj. Appendix L.13		Case 3: SW with MEs Appendix L.14	
	Measurements = Shocks (=7)	8 Shocks		13 Shocks	
Theorem 2 Corollary 2.1 in Levine *et al.* (2019)	E, EB are full row rank (=7) J, JB are full row rank (=7) $A(I-B(EB)^{-1}E)$ is stable Q_A is stable	E, EB are rank deficient (=7) J, JB are rank deficient (=7) $A(I-B(EB)^{-1}E)$ is non-existent Q_A is non-existent		E, EB are rank deficient (=7) J, JB are rank deficient (=7) $A(I-B(EB)^{-1}E)$ is non-existent Q_A is non-existent	
Goodness of Fit	$\mathbb{F}^{PI}=\mathbb{F}^{II}=0$	$\mathbb{F}^{PI}_{(8\times8)}$	$\mathbb{F}^{II}_{(8\times8)}$	$\mathbb{F}^{PI}_{(13\times13)}$	$\mathbb{F}^{II}_{(13\times13)}$
Eigenvalues	$eig(\mathbb{F}^{PI})=eig(\mathbb{F}^{II})=0$	$\begin{bmatrix}1\\0.0002\\0\\0\\0\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\\0.0064\\0.0058\\0.0018\\0\\0.0009\\0.0006\\0.0005\\1\\1\\1\\1\end{bmatrix}$	$\begin{bmatrix}0.2538\\0.1241\\0.0941\\0.0177\\0.0010\\0\\0\\1\\1\\1\\1\\1\\1\end{bmatrix}$
Diagonal values	-	$\begin{bmatrix}0\\0.0001\\0\\0\\0.0015\\0\\0\\0.9986\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\0\\0.0001\\0\\0\\0.9999\end{bmatrix}$	$\begin{bmatrix}0.0020\\0.0068\\0.0022\\0.0018\\0.0081\\0.0115\\0.0044\\0.9943\\0.9980\\0.9975\\0.9997\\0.9982\\0.9916\end{bmatrix}$	$\begin{bmatrix}0.0023\\0.2401\\0.0007\\0.1007\\0.1222\\0.0076\\0.0189\\0.9998\\1\\0.9999\\1\\0.9999\\0.9985\end{bmatrix}$

Table 8: **Fundamentalness and Invertibility Measures for Estimated SW Model**

Notes: Order of shocks: technology, preference, government spending, investment specific, monetary policy, price and wage markup, inflation objective and measurement errors for output growth, consumption growth, investment growth, real wage growth and inflation (Appendices L.12–L.14 record the Dynare output corresponding to the results in this table for the three cases). The simulation results in this table are based on the estimated posterior mode of the SW models for the parameters and shocks.

the eigenvalue measures and indeed produces exactly the same simulated moments across the perfect and imperfect information assumptions. When we add the additional shock in Case 2, compared to non-invertibility of API the eigenvalues are larger for AII ($\mathbb{F}^{II} > \mathbb{F}^{PI}$), introducing non-fundamentalness into the model. The overall fit for fundamentalness under AII is much improved from the RBC results (in Section 6.3), but with a larger-sized model (e.g. Case 2) the difference between API and AII is less marked. Based on Theorem 3 in Levine *et al.* (2019), this means that the differences between IRFs with API and AII, from the perspective of identifying VARs, are less marked. This result clearly depends on the size of the model and the number of shocks, and via simulation, is consistent with previous literature. When we further add measurement errors to the measurement equations for the 4 real variables and the inflation rate (Case 3), the multivariate fit for fundamentalness or invertibility of SW significantly declines for the both AII and API cases. It is very clear that, even with a medium-sized model like SW, it is the decreasing ratio of observables to shocks that drives a bigger wedge between API and AII, in the sense that the fundamentalness problem worsens for the performance of VARs, and the difference of empirical likelihood between API and AII models increases, with fewer observations by agents.

6.6 Example 6: Estimation of Smets and Wouters (2007)

Example 6 estimates the SW Case 1 and Case 3 used in Section 6.5 and reports the likelihood comparison below. The data sample is 1966Q1-2004Q4 which is the same as in Smets and Wouters (2007). The SW model is estimated based on seven quarterly macroeconomic time series: real output, consumption, investment, and real wage growth, hours, inflation, and interest rates. Appendices record the Dynare output corresponding to the results in this table for the two cases. As discussed, when invertibility fails in Case 3, the estimation under II therefore improves the data density slightly based on the estimated posterior mode and data densities. The corresponding measurement equations for the 7 observables are

$$\begin{bmatrix} output\ growth \\ consumption\ growth \\ investment\ growth \\ real\ wage\ growth \\ hours \\ inflation \\ fed\ rate \end{bmatrix} = \begin{bmatrix} \bar{\gamma} + \Delta y_t \\ \bar{\gamma} + \Delta c_t \\ \bar{\gamma} + \Delta i_t \\ \bar{\gamma} + \Delta w_t \\ \bar{l} + l_t \\ \bar{\pi} + \pi_t \\ \bar{R} + R_t \end{bmatrix} \tag{29}$$

where all variables are measured in percent, $\bar{\pi}$ and $\bar{R}$ measure the steady state level of net inflation and short term nominal interest rates, respectively, $\bar{\gamma}$ captures the deterministic long growth rate of real variables, and $\bar{l}$ captures the mean of hours. Output growth is measured as the percentage growth rate of real GDP, consumption growth as the percentage growth rate of personal consumption expenditure deflated by the GDP deflator and investment growth as the percentage growth rate of the Fixed Private Domestic Investment. Hourly compensation is divided by the GDP price deflator in order to get the real wage variable. The aggregate real variables are expressed per capita by dividing with the population over 16. Inflation is the first difference of the log of the Implicit Price Deflator of GDP and the interest rate is the Federal Funds Rate divided by four.

For the 7-shock case the perfect and imperfect information cases coincide. From Case 3, including the additional shocks under II leads to a relatively small improvement in fitting the data for the Smets and Wouters (2007) model, as implied by the marginal data densities. Again, we report our results from Bayesian maximum-likelihood estimation, log posterior optimization, both of which for this model, are very similar to those from Bayesian MCMC estimation. To show more evidence that supports the likelihood comparison, we also summarize some second order moments and present graphs of the autocorrelations of the observed variables below. The model-implied second moments are much better under AII than under API for hours, inflation and the interest rate, while the

	SW_PI	SW_II	LL diff
likelihood (log posterior)	-821.352518	-821.352518	0
log data density (laplace)	-900.934858	-900.935161	-0.000303
computing time	0h01m20s	0h06m16s	

Table 9: **Log Posterior and Data Density Comparison (SW Case 1)**

Notes: The two `.mod` file names are `sw07est.mod` and `sw07est_II.mod`. The posterior mode (and log data density [Laplace approximation]) are estimated using `mode_compute=4`. Invertibility and fundamentalness checks satisfied (based on the mode). `periods=1000` for simulating artificial data using the mode (identical data between PI and II).

	SW_PI	SW_II	LL diff	BF
log data density (laplace)	-910.631977	-908.170031	2.461946	11.72761128
log data density (MHM)	-905.469531	-904.886754	0.582777	1.791005153
computing time	1h17m32s	8h22m44s		

Table 10: **Log Posterior and Data Density Comparison (SW Case 3)**

Notes: The two `.mod` file names are `sw07est_inf_me_mh.mod` and `sw07est_inf_me_II_mh.mod`; The mode (and log data density [Laplace approximation]) are estimated using `mode_compute=4`. We also check the mode solutions using the alternative solvers including combinations of `mode_compute=4,5,6` for checking for robustness. Then a sample from the posterior distribution is obtained with the Metropolis-Hastings (MH) algorithm using the inverse Hessian at the estimated posterior mode as the covariance matrix of the jumping distribution. The covariance matrix needs to be adjusted in order to obtain reasonable acceptance rates which are very similar across the chains (20-22% for PI and 17% for II). Two parallel Markov chains of 250,000 runs each are run from the posterior kernel for the MH, sufficient to ensure convergence according to the indicators recommended by Brooks and Gelman (1998). The first 50,000 draws from each chain have been discarded.

differences from the other real variables are very small. This explains the data support for the II assumption. When it comes to matching the autocorrelograms in Figure 5, the evidence is now clearer, with the II model fitting better the dynamics seen in the data for most variables. The implied autocorrelograms produced by SW model II match very well the observed autocorrelation of interest rate and hours.

Observable	$dlGDP_t$	$dlCON_t$	$dlINV_t$	dW_t	Lab_t	$dlDEF_t$	INT_t
	Standard Deviation (in percent)						
Data	0.8593	0.6970	2.0627	0.6139	2.8952	0.6073	0.8179
SW_PI	0.9256	0.7709	2.2626	0.6315	3.7724	0.8043	0.9164
SW_II	0.9581	**0.7527**	2.2699	0.6650	**2.5385**	**0.6421**	**0.8089**
	Cross-correlation with $dlGDP_t$						
Data	1.0000	0.6586	0.6755	0.0318	0.1096	-0.3051	-0.2258
SW_PI	1.0000	0.5910	0.6889	0.1962	0.0811	-0.1738	-0.1985
SW_II	1.0000	**0.6093**	0.7110	0.3117	**0.1193**	**-0.2581**	**-0.2416**
	Autocorrelations (order=1)						
Data	0.2486	0.1979	0.5282	0.0926	0.9678	0.8897	0.9341
SW_PI	0.3597	0.4115	0.6159	0.0984	0.9835	0.9327	0.9498
SW_II	0.3712	**0.3670**	**0.6121**	0.2434	**0.9630**	**0.8845**	**0.9322**

Table 11: **Data and Model Moments for Estimated SW Case 3**

The previous simulations either imposed a common 1% standard deviation for all the shocks or were based on the estimated mode. But AII results depend critically on the relative size of these standard deviations which need to be arrived at empirically. Table 12 below addresses this issue by estimating the model by Bayesian methods and posterior simulations. It is clear that, when incorporating measurement errors, compared to non-invertibility of API the eigenvalues are larger for AII, introducing non-fundamentalness into the empirical model, even though the parameter estimates are quite similar between the two models. The finding is consistent with what the calibrated exercise shows.

6.7 Example 7: Estimation of Schorfheide (2000)

Example 7 re-estimates the estimated model in Schorfheide (2000) under imperfect information (the original model `fs2000.mod` is downloadable from DYNARE: www.dynare.org). As noted, at the moment the current setup is only suitable for estimation under information symmetry as the observable set declared after `varobs VARIABLE NAME...;` is shared by agents and econometrician, where the variables in `varobs` are those that are members of the information set:

```
options_.usePartInfo=1;
```

	SW with MEs (13 shocks)	
Theorem 2 Corollary 2.1 in Levine *et al.* (2019)	E, EB J, JB are rank deficient (=7) $A(I-B(EB)^{-1}E)$ is non-existent Q_A is non-existent	
Goodness of Fit	$\mathbb{F}^{PI}_{(13\times13)}$	$\mathbb{F}^{II}_{(13\times13)}$
Eigenvalues	0.1040 0.0417 0.0189 0.0021 0 0.0004 0.0011 1 1 1 1 1 1	0.6043 0.1651 0.0075 0.0055 0.0030 0 0.0002 1 1 1 1 1 1
Diagonal values	0.0447 0.1805 0.3072 0.1283 0.0551 0.8423 0.1894 0.9965 0.9128 0.4779 1 0.1062 0.9993	0.0057 0.9330 0.3383 0.1242 0.0931 0.5505 0.9996 0.9996 0.9898 0.1724 1 0.0005 1

Table 12: **Fundamentalness and Invertibility Measures for Estimated SW (Example 7)**

Notes: Order of shocks: technology, preference, government spending, investment specific, monetary policy, price and wage markup, inflation objective and measurement errors for output growth, consumption growth, investment growth, real wage growth and inflation. The simulation results in this table are based on the posterior estimates of the NK model for the parameters and shocks.

```
...

varobs gp_obs gy_obs;
estimation(datafile=fsdat,nobs=192,loglinear,mh_replic=0);
```

The file fs2000_II.mod reproduces the posterior mode and the model data density estimated using the imperfect information procedures (based on the Laplace approximation):

```
RESULTS FROM POSTERIOR MAXIMIZATION

parameters
  prior mean     mode    s.d. t-stat prior pstdev
alp   0.356   0.4035  0.0207 19.4824 beta 0.0200
bet   0.993   0.9909  0.0020 500.1914 beta 0.0020
gam   0.009   0.0047  0.0009  5.0288 norm 0.0030
mst   1.000   1.0141  0.0015 656.9036 norm 0.0070
rho   0.129   0.8456  0.0344 24.5731 beta 0.2230
psi   0.650   0.6894  0.0481 14.3255 beta 0.0500
del   0.010   0.0017  0.0010  1.6023 beta 0.0050

standard deviation of shocks
  prior mean     mode    s.d. t-stat prior pstdev
e_a   0.035   0.0135  0.0009 15.2019 invg    Inf
e_m   0.009   0.0033  0.0002 18.1618 invg    Inf

Log data density [Laplace approximation] is 1298.520395.
```

The original file fs2000.mod produces the posterior mode and the model data density (based on the Laplace approximation) under the standard perfect information assumption:

```
RESULTS FROM POSTERIOR ESTIMATION

parameters
  prior mean     mode    s.d. prior pstdev
alp   0.356   0.4035  0.0207 beta 0.0200
bet   0.993   0.9909  0.0020 beta 0.0020
gam   0.009   0.0046  0.0009 norm 0.0030
```

```
mst    1.000    1.0143  0.0015 norm 0.0070
rho    0.129    0.8455  0.0341 beta 0.2230
psi    0.650    0.6890  0.0482 beta 0.0500
del    0.010    0.0017  0.0010 beta 0.0050

standard deviation of shocks
  prior mean     mode    s.d. prior pstdev
e_a    0.035    0.0136  0.0009 invg    Inf
e_m    0.009    0.0033  0.0002 invg    Inf

Log data density [Laplace approximation] is 1299.009910.
```

Although the number of measurements are equal to the number of shocks in this model, the aforementioned invertibility condition is not satisfied, the estimation under II therefore does generate a difference, and in fact, improves the data density slightly based on the estimated mode and data density (compared to 1299.009910 under standard perfect information). As noted, the parameter estimates under II are not very different, however, one would expect that on balance second moments tend to be better under II, leading a better model fit overall. The model performance is expected to improve significantly when the number of shocks exceeds the number of observations assuming imperfect information on the part of agents because of the endogenous persistence effects caused by the assumption that agents cannot immediately tell from their measurements of the shocks (more empirical evidence can be found in Collard *et al.* (2009), Levine *et al.* (2012) and Cantore *et al.* (2015)).

Finally, we run a simulated version of this model and report the solution procedure, invertibility condition, simulation output and non-fundamentalness measures for `fs2000_II.mod` as in the RBC and SW examples:

```
 --- Transformation to Blanchard-Kahn Form ---
Obtain the singular value decomposition of A0
Singular matrix C2 ...
Start iterative reduction procedure ...
Invertibility and return to conversion

SOLUTION UNDER PARTIAL INFORMATION

OBSERVED VARIABLES
    gp_obs
    gy_obs
```

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements = no. of shocks,
but imperfect information cannot mimic perfect information
MATRIX      E  EB   J  JB
Rank        2   2   2   2
The Eigenvalue Condition is not satisfied

THEORETICAL MOMENTS
VARIABLE          STD. DEV.        VARIANCE
m              0.0062684346    0.0000392933
P              0.0353367907    0.0012486888
c              0.1171446328    0.0137228650
e              0.0136000000    0.0001849600
W              0.0370836678    0.0013751984
R              0.0041539493    0.0000172553
k              6.9664112313   48.5308854438
d              0.0488127787    0.0023826874
n              0.0122229629    0.0001494008
l              0.0445231024    0.0019823066
gy_obs         0.0098921348    0.0000978543
gp_obs         0.0211463947    0.0004471700
y              0.1181523948    0.0139599884
dA             0.0136627041    0.0001866695
```

As before, Table 13 corresponds to Appendix L.15, showing that imperfect information is not equivalent to perfect information because of the failure of the eigenvalue condition in Theorem 2 for AII only. This is another interesting case: with $m = k$, we find that EB is of full rank and $A\left(I - B(EB)^{-1}E\right)$ is a stable matrix therefore API is E-invertible (See Lemma 1); however, JB is invertible but the eigenvalue stability fails with AII, i.e., $Q_A = F(I - B(JB)^{-1}J)$ has eigenvalues outside the unit circle (the second condition in Theorem 2). As expected. there is complete fundamentalness when $\mathbb{F}^{PI} = 0$ for API but with AII $\mathbb{F}^{II} > 0$ and this confirms the finding based on Theorem 2.

To complete the estimation section, the procedure applies the standard numerical optimization routines to obtain the Hessian matrix which is then used in the Metropolis simulation algorithm to generate a sample from the posterior distribution. The following results are based on minimization by `csminwel`, 20,000 random draws from the posterior density that are obtained via the MCMC-MH algorithm, the reasonable acceptance rates (26%) and are confirmed by the standard convergence indicators (Brooks and Gelman (1998)). The two `.mod` file names are `fs2000_II_Metropolis.mod` and `fs2000_Metropolis.mod` for imperfect and perfect information estimations, respectively:

Observations ($m=k$)	Theorem 2 Corollary 2.1 in Levine *et al.* (2019)	Eigenvalues of $\mathbb{F}^{PI}$ Eigenvalues of $\mathbb{F}^{II}$ Order of shocks: $\epsilon_{A,t}$, $\epsilon_{M,t}$
(gy_{obs}, gp_{obs})	E, EB are of full rank (=2) J, JB are of full rank (=2) $A(I - B(EB)^{-1}E)$ is stable Q_A is not stable	$\mathbb{F}^{PI} = [0, 0]$ $\mathbb{F}^{II} = [0, 1]$

Table 13: **Non-fundamentalness for Estimated Schorfheide (2000) Model (Example 7)**

```
Estimation::mcmc: Current acceptance ratio per chain:
                                                       Chain  1: 26.27%
                                                       Chain  2: 26.4%
Log data density is 1298.679945.

parameters
      prior mean   post. mean       90% HPD interval    prior        pstdev
alp        0.356       0.4041     0.3684      0.4367    beta         0.0200
bet        0.993       0.9904     0.9871      0.9939    beta         0.0020
gam        0.009       0.0048     0.0032      0.0065    norm         0.0030
mst        1.000       1.0139     1.0116      1.0168    norm         0.0070
rho        0.129       0.8416     0.7859      0.9000    beta         0.2230
psi        0.650       0.6779     0.5981      0.7601    beta         0.0500
del        0.010       0.0025     0.0006      0.0045    beta         0.0050

standard deviation of shocks
      prior mean   post. mean       90% HPD interval    prior        pstdev
e_a        0.035       0.0138     0.0123      0.0153    invg           Inf
e_m        0.009       0.0033     0.0030      0.0037    invg           Inf
Total computing time : 0h07m25s
```

```
Estimation::mcmc: Current acceptance ratio per chain:
                                                       Chain  1: 26.2%
                                                       Chain  2: 26.58%
Log data density is 1299.240091.

parameters
      prior mean   post. mean       90% HPD interval    prior        pstdev
alp        0.356       0.4044     0.3688      0.4397    beta         0.0200
bet        0.993       0.9905     0.9871      0.9939    beta         0.0020
```

```
gam         0.009       0.0046      0.0031      0.0062    norm         0.0030
mst         1.000       1.0142      1.0114      1.0171    norm         0.0070
rho         0.129       0.8461      0.7826      0.9012    beta         0.2230
psi         0.650       0.6834      0.6066      0.7634    beta         0.0500
del         0.010       0.0024      0.0004      0.0042    beta         0.0050

standard deviation of shocks
      prior mean   post. mean        90% HPD interval     prior        pstdev
e_a         0.035       0.0138      0.0123      0.0153    invg            Inf
e_m         0.009       0.0034      0.0030      0.0037    invg            Inf
Total computing time : 0h03m09s
```

6.8 A Note on Posterior Mode Optimization

Finding the posterior mode can often be hard and any of the optimization routines can fail to find a (global) maximum. The key to robust estimation results is to find the highest posterior density point and use sufficiently large number of MH-MCMC replications for getting to the targeted ergodic distribution and sample from it. The aim is to get to the point that has the highest likelihood value which may not be improved by another mode optimizer. The resulting mode (with the inverse Hessian) should provide the most efficient starting conditions for the MCMC. However, it is not mandatory to use the posterior mode (with Hessian at the estimated mode) as the initial conditions to start the chains for running the metropolis. The latter can explore the whole parameter space and asymptotically move to its ergodic distribution.

Newton-type optimizers such as `mode_compute=4` (Chris Sim's csminwel[25]), which provides an estimate of the posterior covariance matrix based on the inverse of the Hessian matrix, tends to find and get stuck at a local maximum, especially if we have more diffuse priors that do not smooth out the likelihood by much. The issue is to find the true mode (global), but the problem is that the inverse of the Hessian matrix computed at the point is not necessarily positive definite if that point is not being at the true mode. A poorly specified model may not be estimable with `mode_compute=4`. `mode_compute=6` uses a Monte-Carlo based optimization routine, involving drawing random numbers, for the mode computation so does not require the inverse of the Hessian matrix when constructing the covariance matrix for the proposal distribution. The call to it avoids the non-invertible Hessian problem when initialising the MCMC but does not necessarily guarantee convergence to the mode and is very time intensive.

In summary, for the estimation, we start from the prior mean and go for a 'brute-force' search for finding the highest posterior density point. Our strategy is to use alternative optimizers and check if we stay at the same point which may not be further improved or whether other optimizers find higher likelihood values. For example, for our Table 10, the mode is computed using `mode_compute=6`, and subsequently using the like-

[25] See, for more details, Chris Sim's homepage: http://www.princeton.edu/ sims/.

lihood values computed as an initial condition and running it through `mode_compute=5` and `mode_compute=4`. As a results, we find the same log-likelihood (and of course the same mode) for both models under AII and API. Similarly, we repeat this procedure using a combination of `mode_compute=7,8` for the log posterior optimization[26] and check it returns the same maximum. We then proceed to starting the chains with a large number of draws consistent with Smets and Wouters (2007), to ensure the MCMC moves to its ergodic distribution. The accedence ratios between the two chains are always very similar and convergence of the chains are checked using the indicators recommended by Brooks and Gelman (1998). We compare the log marginal data density approximated by Laplace around the estimated posterior mode and the log marginal data density approximated by MHM sampling from the posterior. In our applications, the results of both approximations are very close.

7 Summary and Discussion

This paper introduces a MATLAB toolbox in Dynare designed for solving, simulating and estimating RE-DSGE models under the assumption that both econometrician and agents have the same imperfect information set. The implementation of the software provides additional checks on whether the solution of a linearized DSGE model is a VARMA which may be approximated by a finite VAR model. A necessary condition for such a representation is that the VARMA is invertible (or, almost equivalently, satisfies fundamentalness). We then show an extra condition for invertibility and the examples which demonstrate that the imperfect information assumption can make the invertibility problem worse, introducing an important additional source of non-fundamentalness. The examples and results further demonstrate whether and to what extent the solution can be approximated by a finite reduced form VAR with Gaussian shocks which may be identifiable.

The output produced by the toolbox is important to empirical researchers who often try to match the impulse responses of an identified VAR with a DSGE model. The information from the analysis, results and contained in the relevant matrices, e.g., JB and EB, etc., can provide important insights for considering and choosing the appropriate identification strategy in estimating the SVAR model form consistent with the DSGE model assumed to be the true DGP.[27] The examples and results in this paper clearly suggest some potential pitfalls of using VARs to generate the IRFs of the structural shocks and to then validate empirical DSGE models. The problem may be significantly worsened provided that the econometrician is no better informed than the agents. With any forms of invertibility failure, common approaches in empirical work, for example, in Christiano *et al.* (2005)

[26] Based on Dynare's User Guide, `mode_compute=5`: Uses Marco Ratto's routine `newrat`; `mode_compute=7`: Uses the matlab routine `fminsearch` which is a simplex-based optimization routine; `mode_compute=8`: Uses the Nelder-Mead simplex-based optimization routine. `mode_compute=5` is often quite good but can be slow. `mode_compute=7,8` can be useful for computing some initial estimates.

[27] If you use this toolbox or substantial parts of the code please cite : Levine, P., Pearlman, J. and Yang, B. (2020). DSGE Models under Imperfect Information: A Dynare-based Toolkit. University of Surrey Discussion Papers.

and Kehoe (2006), that are often used for comparisons of IRFs could produce seriously misleading results since the reduced form residuals in the data VAR cannot be a linear transformation of the structural shocks, regardless of the choice of identification schemes.

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Appendix

A Transformation of Model to Blanchard-Kahn Form

A.1 The Problem Stated

The only general results on imperfect information solutions to rational expectations models date back to Pearlman *et al.* (1986), who utilize the Blanchard-Kahn setup, which is given by

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \varepsilon_{t+1} \qquad \text{(A.1)}$$

with agents' measurements given by

$$m_t^A = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \qquad \text{(A.2)}$$

and these can be solved together to yield a reduced-form system. The latter can then be processed via the Kalman filter to obtain the likelihood function for estimation purposes. Note that measurement errors on observations can be incorporated into ϵ_t.

Dynare does not accept models in the form of (A.1). In linearized form, the typical Dynare modfile setup will lead to a system of the form

$$A_0 y_{t+1,t} + A_1 y_t = A_2 y_{t-1} + \Psi \varepsilon_t \qquad \text{(A.3)}$$

with measurements

$$m_t^E = L y_t \qquad \text{(A.4)}$$

The next section describes a completely novel algorithm for converting the state space (A.3), (A.4) under partial information to the form (A.1), (A.2). We assume that the system is 'proper', by which we mean the matrix A_1 is invertible; this precludes the possibility of a system that includes equations of the form $h^T y_{t+1} = 0$, but it is fairly easy to take account of these as well.

A.2 Conversion to Pearlman *et al.* (1986) Setup

Although complicated, the basic stages for the conversion are fairly simple:

1. We first (Stages 1 to 3) find the singular value decomposition for the $n \times n$ matrix A_0 (which is typically of reduced rank $m < n$) which allows us to define a vector of m forward-looking variables that are linear combinations of the original y_t.

2. We then introduce a novel iterative stage (Stage 4) which replaces any forward-looking expectations that use model-consistent updating equations. This reduces the number of equations with forward-looking expectations, while increasing the number of backward-looking equations one-for-one. But at the same time it introduces a dependence of the additional backward-looking equations on both state estimates $z_{t,t}$ $\left(\equiv E_t z_t | I_t^A\right)$ and estimates of forward-looking variables, $x_{t,t}$. This in turn implies that both (A.26) and (A.2) in general contain such terms.

3. A simple example may help to provide intuition for this iterative stage: Suppose two of the equations in the system are of the form: $z_t = \rho z_t + \varepsilon_t$, $y_t = z_{t+1,t}$ (where both y_t and z_t are scalars) i.e., we have one backward-looking (BL) equation and one forward-looking (FL) equation. However using the first equation we can write $z_{t+1,t} = E_t z_{t+1} = \rho z_{t,t}$, hence substituting into the second equation, $y_t = \rho z_{t,t}$: i.e., we can use a model-consistent updating equation. Note, however, a crucial feature: since under II we cannot assume that z_t is directly observable, this updating equation is expressed in terms of the filtered state estimate $z_{t,t}$ rather than directly in terms of x_t We thus now have two BL equations, but one of these is expressed in term of a state estimate.

4. The iterative Stage 4 may need to be repeated a finite number of times. In the case of perfect information this is all that is needed, apart from defining what are the $t+1$ variables.

5. For imperfect information, we retain the same backward and forward looking variables as in the perfect information case, but the solution process is a little more intricate.

The detailed procedure for conversion of (A.3) and (A.4) to the form in (A.26) and (A.2) is as follows:

Stage 1: SVD and partitions of A_0

Obtain the singular value decomposition for matrix A_0: $A_0 = U_0 S_0 V_0^T$, where U_0, V_0 are unitary matrices. Assuming that only the first m values of the diagonal matrix S_0 are non-zero ($m = FL\_RANK =$ the rank of S_0), we can rewrite this as $A_0 = U_1 S_1 V_1^T$, where U_1 are the first m columns of U_0, S_1 is the first $m \times m$ block of S_0 and V_1^T are the first m rows of V_0^T. In addition, U_2 are the remaining $n - m$ columns of U_0, and V_2^T are the remaining $n - m$ rows of V_0^T (A_0 is $n \times n$).

Stage 2: Transform (A.3) to FL subsystem using S_1 and U_1
Multiply (A.3) by $S_1^{-1}U_1^T$, which yields:

$$V_1^T y_{t+1,t} + S_1^{-1}U_1^T A_1 y_t = S_1^{-1}U_1^T A_2 y_{t-1} + S_1^{-1}U_1^T \Psi \varepsilon_t \tag{A.5}$$

Now define forward-looking $x_t = V_1^T y_t$, backward-looking $s_t = V_2^T y_t$, and use the fact that $I = VV^T = V_1V_1^T + V_2V_2^T$ to rewrite (A.5) as (note that $y_t = V_1 x_t + V_2 s_t$):

$$x_{t+1,t} + S_1^{-1}U_1^T A_1(V_1 x_t + V_2 s_t) = S_1^{-1}U_1^T A_2(V_1 x_{t-1} + V_2 s_{t-1}) + S_1^{-1}U_1^T \Psi \varepsilon_t \tag{A.6}$$

or simply:

$$x_{t+1,t} + F_1 x_t + F_2 s_t = F_3 x_{t-1} + F_4 s_{t-1} + F_5 \varepsilon_t \tag{A.7}$$

where $F_1 = S_1^{-1}U_1^T A_1 V_1$, $F_2 = S_1^{-1}U_1^T A_1 V_2$, $F_3 = S_1^{-1}U_1^T A_2 V_1$, $F_4 = S_1^{-1}U_1^T A_2 V_2$ and $F_5 = S_1^{-1}U_1^T \Psi$

Stage 3: Transform (A.3) to BL subsystem using U_2
Multiply (A.3) by U_2^T which yields:

$$U_2^T A_1 y_t = U_2^T A_2 y_{t-1} + U_2^T \Psi \varepsilon_t \tag{A.8}$$

which can be rewritten as

$$U_2^T A_1(V_1 x_t + V_2 s_t) = U_2^T A_2(V_1 x_{t-1} + V_2 s_{t-1}) + U_2^T \Psi \varepsilon_t \tag{A.9}$$

or more simply:

$$C_1 x_t + C_2 s_t = C_3 x_{t-1} + C_4 s_{t-1} + C_5 \varepsilon_t \tag{A.10}$$

where $C_1 = U_2^T A_1 V_1$, $C_2 = U_2^T A_1 V_2$, $C_3 = U_2^T A_2 V_1$, $C_4 = U_2^T A_2 V_2$ and $C_5 = U_2^T \Psi$.

If C_2 is invertible then multiply (A.10) by C_2^{-1}, and go straight to Stage 6. If C_2 is not invertible, then write (A.7) and (A.10) in the more general form:

$$x_{t+1,t} + F_1 x_t + F_2 s_t = F_3 x_{t-1} + F_4 s_{t-1} + F_5 \varepsilon_t \tag{A.11}$$
$$C_1 x_t + C_2 s_t + C_7 x_{t,t} + C_6 s_{t,t} = C_3 x_{t-1} + C_4 s_{t-1} + C_5 \varepsilon_t \tag{A.12}$$

where by comparison of (A.12) with (A.10) we have introduced two new matrices, C_6 and C_7 that must be zero in the first stage of iteration. However, at the end of the first iteration of Stage 4 below we shall increase the dimension of s_t, and reduce the dimension of x_t one-for-one, which will require us to re-define all the matrices in (A.11) and (A.12), such that, from the second iteration onwards, C_6 and C_7 will be non-zero. The whole of Stage 4 may then need to be iterated a finite number of times.

Stage 4: $C_2 + C_6$ singular
Find a matrix J_2 such that $J_2^T(C_2 + C_6)$=0 (by using the SVD of $C_2 + C_6$). Then take forward expectations of (A.12) and pre-multiply by J_2^T to yield:

$$J_2^T(C_1 + C_7)x_{t+1,t} = J_2^T C_3 x_{t,t} + J_2^T C_4 s_{t,t} \tag{A.13}$$

Then reduce the number of forward-looking variables by substituting for $x_{t+1,t}$ from (A.11). In addition find a matrix Q that has the same number of columns as $J_2^T(C_1 + C_7)$ and is made up of rows that are orthogonal to it. Then we define

$$\begin{bmatrix} \bar{x}_t \\ \hat{x}_t \end{bmatrix} = \begin{bmatrix} Q \\ J_2^T(C_1 + C_7) \end{bmatrix} x_t \qquad x_t = M_1\bar{x}_t + Q_2\hat{x}_t \tag{A.14}$$

where $[Q_1 \;\; Q_2] = \begin{bmatrix} Q \\ J_2^T(C_1 + C_7) \end{bmatrix}^{-1}$ From the substitution of $x_{t+1,t}$ into (A.13), we can rewrite the system in terms of forward-looking variables $\bar{x}_t$ and backward-looking variables $s_t,\ \hat{x}_t$:

$$\begin{aligned} &\bar{x}_{t+1,t} + QF_1Q_1\bar{x}_t + [QF_2 \;\; QF_1Q_2] \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \\ &= QF_3Q_1\bar{x}_{t-1} + [QF_4 \;\; QF_3Q_2] \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + QF_5\varepsilon_t \end{aligned} \tag{A.15}$$

$$\begin{aligned} &\begin{bmatrix} C_1Q_1 \\ J_2^T(C_1 + C_7)F_1Q_1 \end{bmatrix} \bar{x}_t + \begin{bmatrix} C_2 & C_1Q_2 \\ J_2^T(C_1 + C_7)F_2 & J_2^T(C_1 + C_7)F_1Q_2 \end{bmatrix} \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \\ &+ \begin{bmatrix} C_7Q_1 \\ J_2^TC_3Q_1 \end{bmatrix} \bar{x}_{t,t} + \begin{bmatrix} C_6 & C_7Q_2 \\ J_2^TC_4 & J_2^TC_3Q_2 \end{bmatrix} \begin{bmatrix} s_{t,t} \\ \hat{x}_{t,t} \end{bmatrix} \\ &= \begin{bmatrix} C_3Q_1 \\ J_2^T(C_1 + C_7)F_3Q_1 \end{bmatrix} \bar{x}_{t-1} + \begin{bmatrix} C_4 & C_3Q_2 \\ J_2^T(C_1 + C_7)F_4 & J_2^T(C_1 + C_7)F_3Q_2 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} \\ &+ \begin{bmatrix} C_5 \\ J_2^T(C_1 + C_7)F_5 \end{bmatrix} \varepsilon_t \end{aligned} \tag{A.16}$$

The number of forward-looking states has decreased because $\bar{x}_t = Q_1x_t$, and the number of backward-looking states $\bar{s}_t = [s_t, \hat{x}_t]'$ has increased by the same amount. In addition the relationship $y_t = V_1x_t + V_2s_t$ has changed to

$$y_t = V_1Q_1\bar{x}_t + \begin{bmatrix} V_2 & V_1Q_2 \end{bmatrix} \bar{s}_t \tag{A.17}$$

The system is now again the form of (A.11) and (A.12). Repeat this stage until $C_2 + C_6$ is of full rank. In the perfect information case, the form (A.11), (A.12) with $s_t = s_{t,t}, x_t = x_{t,t}$ is generated after a finite number of iterations of Stage 3 – the number of iterations cannot exceed the number of variables. The forward looking variables are now x_t and the backward looking variables are s_t and x_{t-1}, and the system can be set up in Blanchard-Kahn form by defining $z_{t+1} = [s_t, x_t]'$. The only additional calculation is to invert $C_2 + C_6$ to obtain the equation for s_t, and to substitute into (A.11). From this point, we eschew the details of matrix manipulations, as these are much more straightforward to understand conceptually compared with those above.

Stage 5: C_2 non-singular
Firstly form expectations of (A.12) and invert C_2+C_6 to obtain $s_{t,t}$ in terms of $x_{t,t}, x_{t-1,t}, s_{t-1,t}$ and $\epsilon_{t,t}$. Then substitute this back into (A.12), and invert C_2 to yield an expression for s_t in terms of the above expected values and also $x_t, x_{t-1}, s_{t-1}, \epsilon_t$. This can be further substituted into (A.11) to yield an expression for $x_{t+1,t}$ in terms of these variables and their expectations. Similarly the measurement equations $m_t = Ly_t$ can now be expressed in terms of all these variables. It follows that if we define $z_{t+1} = [\varepsilon_{t+1}, s_t, x_t]'$ then the system can now be described by (A.1).

Stage 6: C_2 singular
We again start from (A.11) and (A.12), and regard x_t as the forward looking variable and s_t, x_{t-1} as the backward looking variables. Now advance these equations by changing t to $t+k: \ k = 1, 2, 3, ...$ and take expectations using information at time t, implying that $E_t s_{t+k} = E_t s_{t+k,t+k}$. Because $C_2 + C_6$ is invertible, we can rewrite these equations with just $x_{t+k+1,t}$ and $s_{t+k,t}$ on the LHS, which implies the relationship

$$\begin{bmatrix} x_{t+k,t} \\ s_{t+k,t} \\ x_{t+k+1,t} \end{bmatrix} = AA \begin{bmatrix} x_{t+k-1,t} \\ s_{t+k-1,t} \\ x_{t+k,t} \end{bmatrix} \tag{A.18}$$

where,

$$AA = \begin{bmatrix} 0 & 0 & I \\ (C_2+C_6)^{-1}C_3 & (C_2+C_6)^{-1}C_4 & -(C_2+C_6)^{-1}(C_1+C_7) \\ F_3 - F_2(C_2+C_6)^{-1}C_3 & F_4 - F_2(C_2+C_6)^{-1}C_4 & -F_1 + F_2(C_2+C_6)^{-1}(C_1+C_7) \end{bmatrix} \tag{A.19}$$

Then the usual Blanchard-Kahn conditions for stable and unstable roots imply a saddlepath relationship of the form

$$x_{t+k+1,t} + N_1 s_{t+k,t} + N_2 x_{t+k,t} = 0 \tag{A.20}$$

where $[I \ \ N_1 \ \ N_2]$ represents the eigenvectors of the unstable eigenvalues. In particular, this holds for $k = 0$, so if we substitute for $x_{t+1,t} = -N_1 s_{t,t} - N_2 x_{t,t}$ into (A.11), then together with (A.12) we obtain solutions for x_t, s_t in terms of $x_{t,t}, s_{t,t}, x_{t-1}, s_{t-1}, \epsilon_t$. This is possible, because we have assumed the system is proper i.e. A_1 is invertible[28], and any manipulations of A_1 in the previous stages have been simple linear transformations of it to yield the matrices F_1, F_2, C_1, C_2. From (A.20), (A.11) and (A.12) become

$$F_1 x_t + F_2 s_t = N_1 s_{t,t} + N_2 x_{t,t} + F_3 x_{t-1} + F_4 s_{t-1} + F_5 \varepsilon_t \tag{A.21}$$
$$C_1 x_t + C_2 s_t = -C_6 s_{t,t} - C_7 x_{t,t} + C_3 x_{t-1} + C_4 s_{t-1} + C_5 \varepsilon_t \tag{A.22}$$

Taking expectations at t of (A.21) and (A.22) and solving jointly for $[x_{t,t}, s_{t,t}]'$, in terms

[28] The algorithm can be reworked without too much much difficulty if for example some of the forward looking equations in (A.3) are of the form $S_0 E_t Y_{t+1} = 0$.

of $[x_{t-1,t}, s_{t-1,t}, \varepsilon_{t,t}]'$ yield:

$$\begin{bmatrix} x_{t,t} \\ s_{t,t} \end{bmatrix} = \begin{bmatrix} F_1 - N_2 & F_2 - N_1 \\ C_1 + C_7 & C_2 + C_6 \end{bmatrix}^{-1} \begin{bmatrix} F_3 & F_4 & F_5 \\ C_3 & C_4 & C_5 \end{bmatrix} \begin{bmatrix} x_{t-1,t} \\ s_{t-1,t} \\ \varepsilon_{t,t} \end{bmatrix} \tag{A.23}$$

Substituting (A.23) into (A.21) and (A.22) and now solving jointly for $[x_t, s_t]'$, in terms of $[x_{t-1,t}, s_{t-1,t}, \varepsilon_{t,t}, x_{t-1}, s_{t-1}, \varepsilon_t]'$ yield (A.24) below with $FF_1 = 0$ and $GG_{13} = 0$:

$$s_t = G_{13}x_t + G_{12}x_{t-1} + G_{11}s_{t-1} + P_1\varepsilon_t + FF_1x_{t,t} + FF_2x_{t-1,t} + FF_3s_{t-1,t} + FF_4\varepsilon_{t,t} \tag{A.24}$$

Further substituting this expression into (A.11) to yield an expression for $x_{t+1,t}$

$$x_{t+1,t} = G_{33}x_t + G_{32}x_{t-1} + G_{31}s_{t-1} + P_3\varepsilon_t + FF_5x_{t,t} + FF_6x_{t-1,t} + FF_7s_{t-1,t} + FF_8\varepsilon_{t,t} \tag{A.25}$$

The system is now again the form described by (A.1). Finally, to summarise the required Blanchard-Kahn setup

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{A.26}$$

and we define $z_{t+1} = [\varepsilon_{t+1}, s_t, x_t]'$, the converted form (A.26) becomes (when invertibility of A_0 holds)

$$\begin{aligned} \begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ P_1 & G_{11} & G_{12} & G_{13} \\ 0 & 0 & 0 & I \\ P_3 & G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ FF_4 & FF_3 & FF_2 & FF_1 \\ 0 & 0 & 0 & 0 \\ FF_8 & FF_7 & FF_6 & FF_5 \end{bmatrix} \begin{bmatrix} \varepsilon_{t,t} \\ s_{t-1,t} \\ x_{t-1,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{t+1} \end{aligned} \tag{A.27}$$

where $G_{13} = -C_2^{-1}C_1$, $G_{12} = C_2^{-1}C_3$, $G_{11} = C_2^{-1}C_4$, $P_1 = C_2^{-1}C_5$, $G_{33} = -F_2G_{13} - F_1$, $G_{32} = -F_2G_{12} + F_3$, $G_{31} = -F_2G_{11} + F_4$, $P_3 = -F_2P_1 + F_5$, $FF_1 = -C_2^{-1}C_7 + C_2^{-1}C_6(C_2 + C_6)^{-1}(C_1 + C_7)$, $FF_2 = -C_2^{-1}C_6(C_2 + C_6)^{-1}C_3$, $FF_3 = -C_2^{-1}C_6(C_2 + C_6)^{-1}C_4$, $FF_4 = -C_2^{-1}C_6(C_2 + C_6)^{-1}C_5$, $FF_5 = -F_2FF_1$, $FF_6 = -F_2FF_2$, $FF_7 = -F_2FF_3$ and $FF_8 = -F_2FF_4$ (**for the case when C_2 is non-singular**).

Now define

$$\begin{bmatrix} F_1 & F_2 \\ C_1 & C_2 \end{bmatrix}^{-1} = \begin{bmatrix} \overline{F}_1 & \overline{F}_2 \\ \overline{C}_1 & \overline{C}_2 \end{bmatrix} \tag{A.28}$$

we have $G_{13} = 0$, $G_{12} = \overline{C_1}F_3 + \overline{C_2}C3$, $G_{11} = \overline{C_1}F_4 + \overline{C_2}C4$, $P_1 = \overline{C_1}F_5 + \overline{C_2}C5$, $G_{33} = -F_2G_{13} - F_1 = -F_1$, $G_{32} = -F_2G_{12} + F_3$, $G_{31} = -F_2G_{11} + F_4$, $P_3 = -F_2P_1 + F_5$, $FF_1 = 0$, $FF_2 = \overline{C_1}BB_1 + \overline{C_2}DD_1$, $FF_3 = \overline{C_1}BB_2 + \overline{C_2}DD_2$, $FF_4 = \overline{C_1}BB_3 + \overline{C_2}DD_3$, and $FF_5 = -F_2FF_1 = 0$, $FF_6 = -F_2FF_2$, $FF_7 = -F_2FF_3$ and $FF_8 = -F_2FF_4$ (**for the case when C_2 is singular**), where, if we define

$$\left[\begin{array}{cc} F_1 - N_2 & F_2 - N_1 \\ C_1 + C_7 & C_2 + C_6 \end{array}\right]^{-1} = \left[\begin{array}{cc} \overline{F_1 - N_2} & \overline{F_2 - N_1} \\ \overline{C_1 + C_7} & \overline{C_2 + C_6} \end{array}\right] \tag{A.29}$$

the BB and DD matrices take the form of

$$BB_1 = N_1\overline{C_1 + C_7}F_3 + N_1\overline{C_2 + C_6}C_3 + N_2\overline{F_1 - N_2}F_3 + N_2\overline{F_2 - N_1}C_3 \tag{A.30}$$
$$BB_2 = N_1\overline{C_1 + C_7}F_4 + N_1\overline{C_2 + C_6}C_4 + N_2\overline{F_1 - N_2}F_4 + N_2\overline{F_2 - N_1}C_4 \tag{A.31}$$
$$BB_3 = N_1\overline{C_1 + C_7}F_5 + N_1\overline{C_2 + C_6}C_5 + N_2\overline{F_1 - N_2}F_5 + N_2\overline{F_2 - N_1}C_5 \tag{A.32}$$
$$DD_1 = -C_6\overline{C_1 + C_7}F_3 - C_6\overline{C_2 + C_6}C_3 - C_7\overline{F_1 - N_2}F_3 - C_7\overline{F_2 - N_1}C_3 \tag{A.33}$$
$$DD_1 = -C_6\overline{C_1 + C_7}F_4 - C_6\overline{C_2 + C_6}C_4 - C_7\overline{F_1 - N_2}F_4 - C_7\overline{F_2 - N_1}C_4 \tag{A.34}$$
$$DD_1 = -C_6\overline{C_1 + C_7}F_5 - C_6\overline{C_2 + C_6}C_5 - C_7\overline{F_1 - N_2}F_5 - C_7\overline{F_2 - N_1}C_5 \tag{A.35}$$

The C and F matrices are the reduction system matrices in (A.15) and (A.16) in the form of (A.11) and (A.12) (i.e. the iterative procedure that ensures invertibility to be achieved).

The measurements $m_t = Ly_t$ can be written in terms of the states as $m_t = L(V_1x_t + V_2s_t)$, where V_1, V_2 have been updated by (A.17) through the same reduction procedure as above. Using (A.27), we show that m_t can be rewritten as

$$\begin{aligned} m_t &= \left[\begin{array}{cccc} LV_2P_1 & LV_2G_{11} & LV_2G_{12} & LV_1 + LV_2G_{13} \end{array}\right] \left[\begin{array}{c} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{array}\right] \\ &+ \left[\begin{array}{cccc} LV_2FF_4 & LV_2FF_3 & LV_2FF_2 & LV_2FF_1 \end{array}\right] \left[\begin{array}{c} \varepsilon_{t,t} \\ s_{t-1,t} \\ x_{t-1,t} \\ x_{t,t} \end{array}\right] \end{aligned} \tag{A.36}$$

So the observations (A.36) can now be cast into the form in (A.2)

$$m_t = \left[\begin{array}{cc} M_1 & M_2 \end{array}\right] \left[\begin{array}{c} z_t \\ x_t \end{array}\right] + \left[\begin{array}{cc} M_3 & M_4 \end{array}\right] \left[\begin{array}{c} z_{t,t} \\ x_{t,t} \end{array}\right] \tag{A.37}$$

where $M_1 = [LV_2P_1 \; LV_2G_{11} \; LV_2G_{12}]$ and $M_2 = LV_1 + LV_2G_{13}$. Similarly, $M_3 = [LV_2FF_4 \; LV_2FF_3 \; LV_2FF_2]$ and $M_4 = LV_2FF_1$. Thus the setup is as required, with the vector of predetermined variables given by $[\varepsilon_t' \; s_{t-1}' \; x_{t-1}']'$, and the vector of jump variables given by x_t.

A.3 Example of Stage 6 Being Needed for Imperfect Information

Suppose that at the end of Stage 4, the system appears as

$$x_{t+1,t} + \alpha x_t + s_t = \beta s_{t-1} + \varepsilon_t \qquad x_t - x_{t,t} + s_{t,t} = \gamma s_{t-1} \tag{A.38}$$

It is clear from examining these equations that they cannot be manipulated into B-K form directly. However, if we now advance these equations by k periods and take expectations subject to I_t, one obtains two equations relating $x_{t+k+1,t}, s_{t+k,t}$ to $x_{t+k,t}, s_{t+k-1,t}$. Since this is true for all $k \geq 1$, and provided there is exactly one unstable eigenvalue corresponding to these dynamic relationships, it follows that there must be an expectational saddlepath relationship $x_{t+1,t} = -ns_{t,t}$. Substituting this into the first of the above equations allows us to solve in particular for s_t in terms of $x_t, s_{t,t}, s_{t-1}, \varepsilon_t$; from the second equation we can solve for $s_{t,t}$ in terms of $s_{t-1,t}$, so that we can replace the second equation by an equation for s_t in terms of $x_t, s_{t-1,t}, s_{t-1}, \varepsilon_t$. Redefining $z_{t+1} = s_t$, it is now straightforward to obtain the B-K form for the first equation and the new second equation.

So for example, we set out (A.38) in Dynare, but to avoid confusion, as the definitions of x_t, s_t are constantly changing through the stages, we rewrite (A.38) as $u_{t+1,t}+\alpha u_t+v_t = \beta v_{t-1} + \varepsilon_t$, $u_t - u_{t,t} + v_{t,t} = \gamma v_{t-1}$:

```
alpha = 1.50; beta = 0.90; gamma = 0.50;
model;
v1 = v(-1); u1 = u(-1);
u(+1) + alpha * u + v = beta * v(-1) + e;
u - u1(+1) + v1(+1) = gamma * v(-1);
end;
```

From Stage 3, we obtain equations for the 2-dimensional vectors x_t, s_t, where $x_{2t} = -u_t$, $s_{2t} = v_t$, with $C_2 = \begin{bmatrix} -0.7071 & 0 \\ 0.7071 & 0 \end{bmatrix}$ being rank deficient so that Stage 4 is required. (A.11) and (A.12) in the more general form become

$$x_{t+1,t} + \begin{bmatrix} 0 & -0.7071 \\ 0 & 1.5 \end{bmatrix} x_t + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} s_t \qquad \text{(A.39)}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x_{t-1} + \begin{bmatrix} 0 & 0.3536 \\ 0 & -0.9 \end{bmatrix} s_{t-1} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \varepsilon_t$$

$$\begin{bmatrix} -0.7071 & 0 \\ -0.7071 & 0 \end{bmatrix} x_t + \begin{bmatrix} -0.7071 & 0 \\ 0.7071 & 0 \end{bmatrix} s_t + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x_{t,t} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s_{t,t} \qquad \text{(A.40)}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x_{t-1} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} s_{t-1} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \varepsilon_t$$

Now from Stage 4, $x_t = \bar{x}_t$ becomes a 1-dimensional vector, and s_t becomes a 3-dimensional vector $[s_t' \;\; \hat{x}_t]'$, with (A.15) and (A.16) given by

$$\bar{x}_{t+1,t} + [1.5]\,\bar{x}_t + \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \tag{A.41}$$

$$= [0]\,\bar{x}_{t-1} + \begin{bmatrix} 0 & -0.9 & 0 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + [-1]\varepsilon_t$$

$$\begin{bmatrix} 0 \\ 0 \\ 0.7071 \end{bmatrix} \bar{x}_t + \begin{bmatrix} -0.7071 & 0 & 0.7071 \\ 0.7071 & 0 & 0.7071 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \tag{A.42}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ -0.7071 \end{bmatrix} \bar{x}_{t,t} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -0.7071 & 0 \end{bmatrix} \begin{bmatrix} s_{t,t} \\ \hat{x}_{t,t} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \bar{x}_{t-1} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -0.3536 & 0 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_t$$

At this stage, because C_2 is singular but C_2+C_6 is invertible, we move to **Stage 6** recalling (A.18) we compute AA as follows

$$\begin{bmatrix} x_{t+k,t} \\ s_{t+k,t} \\ x_{t+k+1,t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ -0.7071 & 0 & 0.7071 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ -0.7071 & 0 & -0.7071 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & -1.5 \end{bmatrix} \begin{bmatrix} x_{t+k-1,t} \\ s_{t+k-1,t} \\ x_{t+k,t} \end{bmatrix} \tag{A.43}$$

The saddlepath relationship of the form (A.20) solves for

$$N_2 = 0 \qquad N_1 = \begin{bmatrix} 0 & 0.2 & 0 \end{bmatrix} \tag{A.44}$$

This is consistent with the saddlepath relationship obtained from the original setup in terms of $u_t,\ v_t$:

$$\begin{bmatrix} v_{t+k,t} \\ v_{t+k+1,t} \\ u_{t+k,t} \\ u_{t+k+1,t} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0.9 & -1 & 0 & -1.5 \end{bmatrix} \begin{bmatrix} v_{t+k-1,t} \\ v_{t+k,t} \\ u_{t+k-1,t} \\ u_{t+k,t} \end{bmatrix} \tag{A.45}$$

The unstable eigenvalue of the matrix above is -1.5, with eigenvector $[-0.6 \;\; 0.8 \;\; 0 \;\; 1]$, implying that $u_{t+1,t} = 0.6v_{t,t} - 0.8v_{t+1,t}$; it is straightforward to check from the second dynamic equation for (u, v) that $v_{t+1,t} = 0.5v_{t,t}$, which implies that $u_{t+1,t} = 0.2v_{t,t}$; this corresponds to (A.44). We can now substitute this into the first of the (u, v) equations, so that they become $0.2v_{t,t} + 1.5u_t + v_t = 0.9v_{t-1} + \varepsilon_t,\;\; u_t - u_{t,t} + v_{t,t} = 0.5v_{t-1}$. A little bit of manipulation of these equations results in:

$$v_t = 0.15v_{t-1} + 0.35v_{t-1,t} + \epsilon_t - \epsilon_{t,t} \tag{A.46}$$

We can then eliminate $x_{t+1,t}$ from $x_{t+1,t} = -N_1 s_{t,t} - N_2 x_{t,t}$ and (A.11) to obtain equation (A.21). Take expectations at t of (A.21) and (A.22) and solve jointly for $[x_{t,t}, s_{t,t}]'$, in terms of $[x_{t-1,t}, s_{t-1,t}, \varepsilon_{t,t}]'$; Then substitute (A.23) into (A.21) and (A.22) and now solving jointly for $[x_t, s_t]'$, in terms of $[x_{t-1,t}, s_{t-1,t}, \varepsilon_{t,t}, x_{t-1}, s_{t-1}, \varepsilon_t]'$ yield (A.24) below with $FF_1 = 0$ and $GG_{13} = 0$. Retain just the solution for s_t which is (A.24), substituting this expression into (A.11) to yield (A.25). We now have all the matrices required for this simple testing model set out in (A.38):

$$FF_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.35 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ FF_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},\ P_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ G_{11} = \begin{bmatrix} 0 & 0.7071 & 0 \\ 0 & 0.15 & 0 \\ 0 & -0.7071 & 0 \end{bmatrix}$$

It is now obvious that the middle element of the vector s_t in (A.24) corresponds precisely to v_t in (A.46).

B Generating Artificial Data

This Appendix describes how artificial data is obtained from tochastic simulations of the model.

First, we rewrite the system (**??**) with a one-period lead

$$\begin{bmatrix} z_{t+1,t} \\ \tilde{z}_{t+1} \end{bmatrix} = \begin{bmatrix} A & A\left[P^A J'(JP^A J')^{-1} J - I\right] \\ 0 & F[I - P^A J'(JP^A J')^{-1} J] \end{bmatrix} \begin{bmatrix} z_{t,t-1} \\ \tilde{z}_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{B.47}$$

To obtain the impulse response for the underlying variables y_t we use the relationship

$$y_t = V_1 x_t + V_2 s_t \tag{B.48}$$

Recalling that $z_{t+1} = [\varepsilon_{t+1}, s_t, x_t]'$, it follows that $s_t = [0 \;\; I \;\; 0] z_{t+1}$, and we may write

$$y_t = V_1 x_t + \begin{bmatrix} 0 & V_2 & 0 \end{bmatrix} \left(A z_t + A\left[P^A J'(JP^A J')^{-1} J - I\right] \tilde{z}_t \right) \tag{B.49}$$

or more simply

$$y_t = \begin{bmatrix} 0 & V_2 & V_1 \end{bmatrix} z_{t+1} = \begin{bmatrix} 0 & V_2 & V_1 \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \end{bmatrix} \tag{B.50}$$

To calculate the IRFs of observable states s_t, we know that, at time t, the first period response, using (B.47), is

$$I_{s,1} = \begin{bmatrix} A & A\left[P^A J'(JP^A J')^{-1} J - I\right] \\ 0 & F[I - P^A J'(JP^A J')^{-1} J] \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} \sigma_\varepsilon \tag{B.51}$$

where σ_ε is the standard error of ε_t. So the first period IRF of y_t can be obtained using (B.50) after a one-time shock.

To obtain a simulation with shocks happening every single period, we use the same strategy as above for simulating data. The only thing that is different is that we compute the sum of the IRFs from all of the past shocks when at each point in time a new random shock hits the above system. In other words, for the length of the simulation (`periods=1000`), at each t, $[B, 0]'\sigma_\varepsilon$ is produced by multiplying a sequence of normally distributed random numbers by the standard error of ε_t.

```
varobs a_obs b_obs;
stoch_simul(partial_information, periods=1000, OPTIONS, ...);
```

C Fernandez-Villaverde Basic RBC Model

The modelling example starts from the basic RBC prototype which is an infinite horizon model with logarithmic utility, inelastic labour supply, Cobb-Douglas technology, and with a zero growth steady state. We analyse the canonical *social planner's problem*

$$\max \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t + \psi \log(1 - H_t) \right\} \tag{C.52}$$

subject to a resource constraint

$$\begin{aligned} C_t + K_t &= K_{t-1}^{\alpha} (e^{z_t} H_t)^{1-\alpha} + (1-\delta) K_{t-1}\,; \;\; \forall t > 0 && \text{(C.53)} \\ z_t &= \rho z_{t-1} + \varepsilon_t\,; \;\; \varepsilon_t \sim N(0, \sigma^2) && \text{(C.54)} \end{aligned}$$

where C_t is consumption, $z_t \equiv \log(A_t)$ where A_t is labour productivity, H_t is the proportion of hours worked out of time available and K_t is defined as *end-of-period* t capital stock. The following equilibrium conditions are derived from the household problem, the firms' problem and aggregate conditions

$$\begin{aligned} \frac{1}{C_t} &= \beta E_t \left\{ \frac{1}{C_{t+1}} \left(1 + \alpha K_t^{\alpha-1} (e^{z_{t+1}} H_{t+1})^{1-\alpha} - \delta \right) \right\} && \text{(C.55)} \\ \psi \frac{C_t}{1 - L_t} &= (1-\alpha) K_{t-1}^{\alpha} (e^{z_t})^{1-\alpha} H_t^{-1} && \text{(C.56)} \\ Y_t &= K_t^{\alpha} (e^{z_t} H_t)^{1-\alpha} && \text{(C.57)} \\ I_t &= K_t - (1-\delta) K_{t-1} && \text{(C.58)} \\ Y_t &= C_t + I_t && \text{(C.59)} \end{aligned}$$

where Y_t is output and I_t is investment.

D The RBC Model in Levine *et al.* (2019)

We now consider the standard RBC model of a *decentralized economy.* There are now two shock AR(1) exogenous processes, A_t and G_t where G_t is government spending.

Euler consumption and the household behaviour is summarised by

$$\text{Utility}: U_t = U(C_t, H_t) \tag{D.60}$$

$$\text{Euler Consumption}: U_{C,t} = \beta R_t \mathbb{E}_t \left[U_{C,t+1}\right] \tag{D.61}$$

$$\text{Labour Supply}: \frac{U_{H,t}}{U_{C,t}} = -W_t \tag{D.62}$$

where $U_{C,t} \equiv \frac{\partial U_t}{\partial C_t}$ is the marginal utility of consumption and $\mathbb{E}_t[\cdot]$ denotes rational expectations based on the agents' information set, describes the optimal consumption-savings decisions of the household. It equates the marginal utility from consuming one unit of income in period t with the discounted marginal utility from consuming the gross income acquired, R_t, by saving the income. For later use define $\Lambda_{t,t+1} \equiv \beta \frac{U_{C,t+1}}{U_{C,t}}$ is the *real stochastic discount factor* over the interval $[t, t+1]$. (D.62) equates the real wage with the marginal rate of substitution between consumption and leisure.

Output and the firm behaviour is summarised by

$$\text{Output}: Y_t = F(A_t, H_t, K_{t-1}) \tag{D.63}$$

$$\text{Labour Demand}: F_{H,t} = W_t \tag{D.64}$$

$$\text{Capital Demand}: 0 = \mathbb{E}_t \left[\Lambda_{t+1}(F_{K,t+1} - R_t + 1 - \delta)\right] \tag{D.65}$$

$$\text{Stochastic Discount Factor}: \Lambda_{t,t+1} = \beta \frac{U_{C,t+1}}{U_{C,t}} \tag{D.66}$$

(D.63) is a production function where again K_t is *end-of-period* t capital stock. Equation (D.64), where $F_{H,t} \equiv \frac{\partial F_t}{\partial H_t}$, equates the marginal product of labour with the real wage. (D.65), where $F_{K,t} \equiv \frac{\partial F_t}{\partial K_{t-1}}$, equates the marginal product of capital with the cost of capital. The model is completed with an output equilibrium, law of motion for capital and a balanced budget constraint with fixed lump-sum taxes.

The equilibrium of the model can be summarised by a vector with endogenous variables $[U_t, U_{C,t}, C_t, U_{H,t}, H_t, R_t, \Lambda_t, W_t, R_{K,t}, Y_t, K_t, I_t]'$ and a vector of two shock processes $[A_t, G_t]'$. New variables are utility U_t which is of Cobb-Douglas form, marginal utilities of consumption and hours, $U_{C,t}$ and C_t, $U_{H,t}$ respectively, the gross real interest rate R_t, the real wage W_t and the gross return on capital $R_{K,t}$. These 12 endogenous variables satisfy the following 12 equations

$$
\begin{aligned}
U_t &= \frac{(C_t^{(1-\varrho)}(1-H_t)^{\varrho})^{1-\sigma}-1}{1-\sigma} && \text{(D.67)}\\
U_{C,t} &= (1-\varrho)C_t^{(1-\varrho)(1-\sigma)-1}(1-H_t)^{\varrho(1-\sigma)} && \text{(D.68)}\\
U_{H,t} &= -\varrho C_t^{(1-\varrho)(1-\sigma)}(1-H_t)^{\varrho(1-\sigma)-1} && \text{(D.69)}\\
1 &= R_t\mathbb{E}_t\left[\Lambda_{t,t+1}\right] && \text{(D.70)}\\
\frac{U_{H,t}}{U_{C,t}} &= -W_t && \text{(D.71)}\\
\Lambda_t &= \beta\frac{U_{C,t}}{U_{C,t-1}} && \text{(D.72)}\\
Y_t &= (A_tH_t)^{\alpha}K_{t-1}^{1-\alpha} && \text{(D.73)}\\
R_{K,t} &= \frac{(1-\alpha)Y_t}{K_{t-1}}+1-\delta && \text{(D.74)}\\
W_t &= \frac{\alpha Y_t}{H_t} && \text{(D.75)}\\
1 &= \mathbb{E}_t[\Lambda_{t,t+1}R_{K,t+1}] && \text{(D.76)}\\
Y_t &= C_t+G_t+I_t && \text{(D.77)}\\
I_t &= K_t-(1-\delta)K_{t-1} && \text{(D.78)}
\end{aligned}
$$

In the `.mod` file we define scaled variables $YY_t \equiv \frac{Y_t}{Y}$, $CC_t \equiv \frac{C_t}{C}$ etc where Y, C denotes the zero-growth steady state.

E The Monetary CIA Model in Schorfheide (2000)

In this standard cash-in-advance model[29] where decisions of the agents are made after the current period surprise change in money growth $\epsilon_{M,t} \sim N(0,\sigma_M^2)$ and technology $\epsilon_{A,t} \sim N(0,\sigma_A^2)$, there are three agents who solve for their optimality conditions. The household chooses consumption C_t, hours worked H_t, and deposits D_t to maximise the sum of discounted expected future utility. Firms and the financial intermediary are owned by households. The firm chooses desired capital, K_{t+1}, labour demand, N_t, dividends F_t and loans L_t to maximise a discounted unit of date t nominal dividends in terms of the consumption it enables during $t+1$. Similarly the financial intermediary values the dividends from the financial intermediaries B_t and chooses B_t, L_t and D_t to maximise a discounted unit of date t nominal dividends. The equilibrium conditions are summarised

[29]See also Nason and Cogley (1994).

as follows

$$E_t\left\{\frac{P_t}{C_{t+1}P_{t+1}}\right\} = \beta E_t\left\{\frac{P_{t+1}\alpha K_t^{\alpha-1}(A_{t+1}N_{t+1})^{1-\alpha}+1-\delta}{C_{t+2}P_{t+2}}\right\} \tag{E.79}$$

$$\frac{1}{C_tP_t} = \beta E_t\left\{\frac{R_t}{C_{t+1}P_{t+1}}\right\} \tag{E.80}$$

$$W_t = \frac{\psi}{1-\psi}\frac{C_tP_t}{1-N_t} \tag{E.81}$$

$$R_t = \frac{P_t(1-\alpha)K_{t-1}^{\alpha}A_t^{1-\alpha}N_t^{-\alpha}}{W_t} \tag{E.82}$$

$$W_t = \frac{L_t}{N_t} \tag{E.83}$$

$$L_t = M_t - M_{t-1} + D_t \tag{E.84}$$

$$M_t = P_tC_t \tag{E.85}$$

$$Y_t = K_{t-1}^{\alpha}(A_tN_t)^{1-\alpha} \tag{E.86}$$

$$I_t = K_t - (1-\delta)K_{t-1} \tag{E.87}$$

$$Y_t = C_t + I_t \tag{E.88}$$

$$g_{A,t} = \frac{A_t}{A_{t-1}} \tag{E.89}$$

$$g_{M,t} = \frac{M_t}{M_{t-1}} \tag{E.90}$$

$$\log g_{A,t} = \gamma + \epsilon_{A,t} \tag{E.91}$$

$$\log g_{M,t} = (1-\rho)\log g_{M^*} + \rho\log g_{M,t-1} + \epsilon_{M,t} \tag{E.92}$$

where A_t is a labour-augmenting technology; P_t the price index and the central bank lets the money stock M_t grow at $g_{M,t}$. The innovations $\epsilon_{M,t}$ capture unexpected changes of the money growth rate due to 'normal' policy making g_{M^*} and changes in M^* correspond to regime shifts.

F A Small-scale New Keynesian Model

F.1 Supply Side

The retail sector uses a homogeneous wholesale good to produce a basket of differentiated goods for consumption

$$C_t = \left(\int_0^1 C_t(m)^{(\zeta-1)/\zeta}dm\right)^{\zeta/(\zeta-1)} \tag{F.93}$$

where ζ is the elasticity of substitution. For each m, the consumer chooses $C_t(m)$ at a price $P_t(m)$ to maximize (F.93) given total expenditure $\int_0^1 P_t(m)C_t(m)dm$. This results in a set of consumption demand equations for each differentiated good m with price $P_t(m)$ of the form

$$C_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta}C_t \tag{F.94}$$

where $P_t = \left[\int_0^1 P_t(m)^{1-\zeta} dm\right]^{\frac{1}{1-\zeta}}$. P_t is the aggregate price index. Note that C_t and P_t are Dixit-Stiglitz aggregators. So in aggregate

$$Y_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} Y_t \tag{F.95}$$

where $Y_t(m)$ is the quantities of output needed in the wholesale sector to produce good m in the retail sector. Integrating over m we then have

$$\int_0^1 Y_t(m)dm = Y_t^W = \left(\int_0^1 \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} dm\right) Y_t = \Delta_t Y_t \tag{F.96}$$

where $\Delta_t \equiv \int_0^1 \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} dm$ is *price dispersion.*

Following Calvo (1983), we now assume that there is a probability of $1-\xi$ at each period that the price of each retail good $m$is set optimally to $P_t^O(m)$. If the price is not re-optimized, then it is held fixed.[30] For each retail producer m, given its real marginal cost $MC_t = \frac{P_t^W}{P_t}$, the objective is at time t to choose $\{P_t^O(m)\}$ to maximize discounted real profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \frac{\Lambda_{t,t+k}}{P_{t+k}} Y_{t+k}(m) \left[P_t^O(m) - P_{t+k} MC_{t+k}\right] \tag{F.97}$$

subject to

$$Y_{t+k}(m) = \left(\frac{P_t^O(m)}{P_{t+k}}\right)^{-\zeta} Y_{t+k} \tag{F.98}$$

where $\Lambda_{t,t+k} \equiv \beta^k \frac{U_{C,t+k}}{U_{C,t}}$ is the stochastic discount factor over the interval $[t, t+k]$. The solution to this is

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \frac{\Lambda_{t,t+k}}{P_{t+k}} Y_{t+k}(m) \left[P_t^O(m) - \frac{1}{(1-1/\zeta)} P_{t+k} MC_{t+k}\right] = 0 \tag{F.99}$$

Using (F.98) and rearranging this leads to

$$P_t^O = \frac{1}{(1-1/\zeta)} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \frac{\Lambda_{t,t+k}}{P_{t+k}} (P_{t+k})^{\zeta} Y_{t+k} MC_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \frac{\Lambda_{t,t+k}}{P_{t+k}} (P_{t+k})^{\zeta} Y_{t+k}} \tag{F.100}$$

where the m index is dropped as all firms face the same marginal cost so the right-hand side of the equation is independent of firm size or price history.

By the law of large numbers the evolution of the price index is given by

$$P_t^{1-\zeta} = \xi P_{t-1}^{1-\zeta} + (1-\xi)(P_t^O)^{1-\zeta} \tag{F.101}$$

Prices now are indexed to last period's aggregate inflation, with a price indexation parameter γ_p. Then the price trajectory with no re-optimization is given by $P_t^O(j)$, $P_t^O(j)\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_p}$, $P_t^O(j)\left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_p}$, $\cdots$. where $Y_{t+k}(m)$ is given by (F.95) with indexing so that

$$Y_{t+k}(m) = \left(\frac{P_t^O(m)}{P_{t+k}}\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_p}\right)^{-\zeta} Y_{t+k} \tag{F.102}$$

[30] Thus we can interpret $\frac{1}{1-\xi}$ as the average duration for which prices are left unchanged.

F.2 Linearization

The log linearized equilibrium conditions of the full NK model are summarised as follows

$$a_t = \rho_A a_{t-1} + \varepsilon_{A,t} \tag{F.103}$$

$$g_t = \rho_G g_{t-1} + \varepsilon_{G,t} \tag{F.104}$$

$$ms_t = \rho_{ms} ms_{t-1} + \varepsilon_{MS,t} \tag{F.105}$$

$$k_t = (1-\delta)k_{t-1} + \delta i_t \tag{F.106}$$

$$\mathbb{E}_t[u_{C,t+1}] = u_{C,t} - r_t \tag{F.107}$$

$$u_{C,t} = -(1 + (\sigma_c - 1)(1-\varrho))c_t + (\sigma_c - 1)\varrho \frac{H}{1-H} h_t \tag{F.108}$$

$$u_{L,t} = u_{C,t} + c_t + \frac{H}{1-H} h_t \tag{F.109}$$

$$w_t = u_{L,t} - u_{C,t} \tag{F.110}$$

$$y_t = \alpha(a_t + h_t) + (1-\alpha)k_{t-1} \tag{F.111}$$

$$y_t = c_y c_t + i_y i_t + g_y\, g_t \tag{F.112}$$

$$g_t = t_t \tag{F.113}$$

$$r_t = E_t[x_{t+1}] - q_t \tag{F.114}$$

$$Rx_t \equiv (R - 1 + \delta)(y_t - k_{t-1}) + (1-\delta)q_t \tag{F.115}$$

$$\left(1 + \frac{1}{R}\right) i_t = \frac{1}{R} E_t i_{t+1} + i_{t-1} + \frac{1}{S''(1)} q_t \tag{F.116}$$

$$w_t = y_t - h_t \tag{F.117}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\beta\xi)(1-\xi)}{\xi}(mc_t + ms_t) \tag{F.118}$$

$$mc_t = p_t^w - p_t = w_t + h_t - y_t \tag{F.119}$$

$$r_{n,t} = \rho_r r_{n,t-1} + (1-\rho_r)(\theta_\pi \pi_t + \theta_y y_t) + \epsilon_{M,t} \tag{F.120}$$

where the NK model has four structural shocks: three AR(1) which are for technology a_t, government g_t and price markup ms_t, and one IID monetary policy $\epsilon_{M,t}$.

G The SW Model Linearized Equilibrium Conditions

The log linearized equilibrium conditions of Smets and Wouters (2007) are summarised as follows

$$y_t = C/Y c_t + I/Y i_t + R^k K/Y z_t + e^g_t \tag{G.121}$$
$$c_t = c_1 c_{t-1} + (1-c_1)Ec_{t+1} + c_2(h_t - Eh_{t+1}) - c_3(r_t - E\pi_{t+1} + e^b_t) \tag{G.122}$$
$$i_t = i_1 i_{t-1} + (1-i_1)E_t i_{t+1} + i_2 q_t + \epsilon^i_t \tag{G.123}$$
$$q_t = q_1 Eq_{t+1} + (1-q_1)Er^k_{t+1} - (r_t - E\pi_{t+1} + e^b_t) \tag{G.124}$$
$$y_t = \alpha\phi_p k_t + (1-\alpha)\phi_p h_t + \phi_p \epsilon^a_t \tag{G.125}$$
$$k^s_t = k_{t-1} + z_t \tag{G.126}$$
$$z_t = \psi/(1-\psi) r^k_t \tag{G.127}$$
$$k_t = k_1 k_{t-1} + (1-k_1) i_t + k_2 \epsilon^i_t \tag{G.128}$$
$$mp_t = \alpha(k^s_t - h_t) + e^a_t - w_t \tag{G.129}$$
$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E\pi_{t+1} - \pi_3 mp_t + e^p_t \tag{G.130}$$
$$r^k_t = -(k_t - h_t) + w_t \tag{G.131}$$
$$mw_t = w_t - \left(\sigma_n h_t + \frac{1}{1+\lambda/\gamma}(c_t - \lambda/\gamma c_{t-1})\right) \tag{G.132}$$
$$w_t = w_1 w_{t-1} + (1-w_1)E(\pi_{t+1} + w_{t+1}) - w_2\pi_t + w_3\pi_{t-1} + mw_t + e^w_t \tag{G.133}$$
$$r_t = \rho_r r_{t-1} + (1-\rho_r)(\rho_\pi \pi_t + \rho_y(y_t - y^f_t) + \rho_{\Delta y}\Delta(y_t - y^f_t)) + e^r_t$$
$$+ \ \mathit{flexible\ economy\ equations} \tag{G.134}$$

where variables with time subscript are variables from the original non-linear model expressed in log deviation from the steady state. Variables without time subscript are the corresponding balanced growth steady state with growth rate γ (these are Y, C, I, R^k, K, W, H and e.g. $y_t = \log\left(\frac{Y_t}{Y}\right)$,where Y_t is output from the non-linear equilibrium conditions). The notation is consistent with the Smets and Wouters (2007) paper and with the Dynare code in Section 6.5. Flexible output is defined as the level of output that would prevail under flexible prices and wages in the absence of the two mark-up shocks. There are even structural shocks. The model has five AR(1), government, technology, preference, investment specific, monetary policy, and two ARMA(1,1) processes, price and wage markup.

The nominal interest rate rule in the SW model (G.134) differs from that used in the small-scale NK model (F.120) in that the latter does not require knowledge of the output gap $y_t - y^f_t$ and is referred to as 'implementable' by Schmitt-Grohe and Uribe (2007). This is a more natural choice of rule in our imperfect information set-up. Indeed in the version of the SW model with measurement errors neither output nor inflation is directly observed so we introduce an implementable form of (G.134):

$$r_t = \rho_r r_{t-1} + (1-\rho_r)(\rho_\pi \pi_{t,t} + \rho_y y_{t,t} + \rho_{\Delta y}\Delta y_{t,t}) + e^r_t \tag{G.135}$$

H Data and Model Autocorrelations (NK with 3 Observables)

See Figure 4.

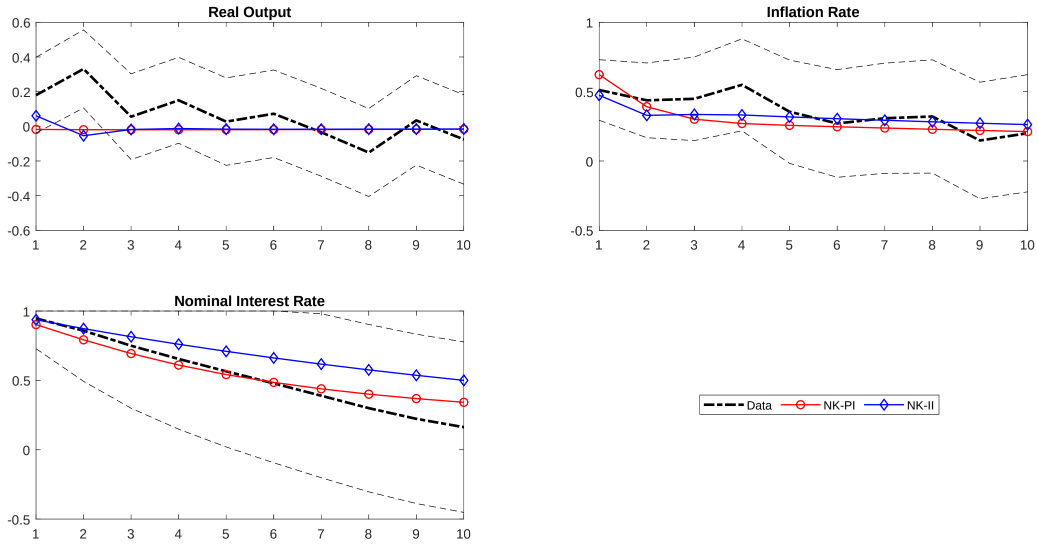


Figure 4: **Autocorrelations of Observables in the Actual Data and in the Estimated NK Models**

Note: The approximate 95% confidence bands are constructed using the large-lag standard errors (see Anderson (1976)).

I Data and Model Autocorrelations (SW Case 3)

See Figure 5.

J Impulse Response Functions (NK with 3 Observables)

See Figure 6.

K Impulse Response Functions (SW Case 3)

See Figure 7.

L Additional Dynare Output on Theorem 2 and Fundamentalness

Here we refer the reader to the additional output produced in Dynare for checking the invertibility and fundamentalness conditions for all our example models under API and

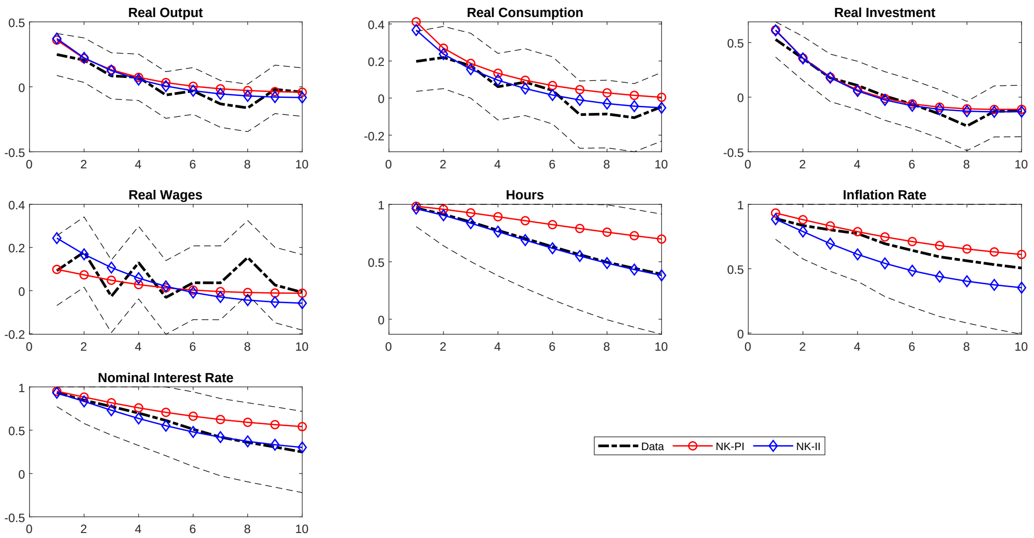


Figure 5: **Autocorrelations of Observables in the Actual Data and in the Estimated SW Models**

Note: The approximate 95% confidence bands are constructed using the large-lag standard errors (see Anderson (1976)).

AII. The results in the tables above have Dynare output counterparts reported in this Appendix below.

L.1 Example 1: Simulation with One Observable (y_t): `rbc_II.mod`

```
--- THE INVERTIBILITY CONDITION IS SATISFIED ---
no. of measurements = no. of shocks,
imperfect information is equivalent to perfect information

Measures of Invertibility and Fundamentalness
Matrix      E  EB   J  JB
Rank        1   1   1   1
The Eigenvalue Condition for PI is satisfied
The Eigenvalue Condition for II is satisfied

MATRIX F WITH PI
Shocks          e
e          0.0000
Shocks          e
Eigen      0.0000
```

```
MATRIX F WITH II
Shocks         e
e         0.0000
Shocks         e
Eigen     0.0000

MATRIX B FOR SQUARE SYSTEMS AND PI ONLY
Shocks         e
e         0.0000
Shocks         e
Eigen     0.0000
```

Notes: There is one shock and one observable; imperfect information is equivalent to perfect information and this is verified by the rank and eigenvalue conditions: EB is of full rank; $A\left(I - B(EB)^{-1}E\right)$ has stable eigenvalues; JB is of full rank and $F(I - B(JB)^{-1}J)$ has stable eigenvalues. As expected, there is complete fundamentalness when $\mathbb{F}^{PI} = 0$ and $\mathbb{F}^{II} = 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $eig(\mathbb{F}) = 0$. Finally, the program also reports $\mathbb{B}^{PI} = EP^{E}E' - EBB'E' = 0$ only when $m = k$.

L.2 Example 1: Simulation with One Lagged Observable (y_{t-1}): `rbc_yl_II.mod`

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements = no. of shocks,
but imperfect information cannot mimic perfect information

Measures of Invertibility and Fundamentalness
Matrix     E  EB   J  JB
Rank       1   0   1   0

MATRIX F WITH PI
Shocks         e
e         1.0000
Shocks         e
Eigen     1.0000

MATRIX F WITH II
Shocks         e
e         0.8630
```

```
Shocks          e
Eigen      0.8630

MATRIX B FOR SQUARE SYSTEMS
Shocks          e
e          1.0000
Shocks          e
Eigen      1.0000
```

Notes: There is one shock and one lagged observable; imperfect information is not equivalent to perfect information and this is verified by the rank conditions: EB is not of full rank; JB is not of full rank despite E, J being of full rank; $A\left(I - B(EB)^{-1}E\right)$ and $F(I - B(JB)^{-1}J)$ are non-existent. There is no complete fundamentalness for both cases when $\mathbb{F}^{PI} > 0$ and $\mathbb{F}^{II} > 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. Finally, the program also reports $\mathbb{B}^{PI} = EP^E E' - EBB'E'$ only when $m = k$.

L.3 Table 2: RBC Model with Observables (Y_t, C_t): `rbc_invertibility.mod`

```
--- THE INVERTIBILITY CONDITION IS SATISFIED ---
no. of measurements = no. of shocks,
imperfect information is equivalent to perfect information

Measures of Invertibility and Fundamentalness
Matrix      E  EB   J  JB
Rank        2   2   2   2
The Eigenvalue Condition for PI is satisfied
The Eigenvalue Condition for II is satisfied

MATRIX F WITH PI
Shocks        epsA    epsG
epsA       0.0000  0.0000
epsG       0.0000  0.0000
Shocks        epsA    epsG
Eigen      0.0000  0.0000

MATRIX F WITH II
Shocks        epsA    epsG
epsA       0.0000 -0.0000
```

```
epsG       0.0000  0.0000
Shocks       epsA    epsG
Eigen      0.0000  0.0000

MATRIX B FOR SQUARE SYSTEMS
Shocks       epsA    epsG
epsA       0.0000 -0.0000
epsG      -0.0000  0.0000
Shocks       epsA    epsG
Eigen      0.0000  0.0000
```

Notes: There are two shocks and two observables; imperfect information is equivalent to perfect information and this is verified by both the rank and eigenvalue conditions: EB is of full rank; $A\left(I - B(EB)^{-1}E\right)$ has stable eigenvalues; JB is of full rank and $F(I - B(JB)^{-1}J)$ has stable eigenvalues. There is complete fundamentalness when $\mathbb{F}^{PI} = 0$ and $\mathbb{F}^{II} = 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. Finally, the program also reports $\mathbb{B}^{PI} = EP^{E}E' - EBB'E' = 0$ only when $m = k$.

L.4 Table 2: RBC Model with Observables (Y_t, R_t): `rbc_invertibility.mod`

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements = no. of shocks,
but imperfect information cannot mimic perfect information

Measures of Invertibility and Fundamentalness
Matrix      E  EB   J  JB
Rank        2   2   1   1
The Eigenvalue Condition for PI is satisfied

MATRIX F WITH PI
Shocks       epsA    epsG
epsA      -0.0000  0.0000
epsG       0.0000 -0.0000
Shocks       epsA    epsG
Eigen      0.0000  0.0000

MATRIX F WITH II
Shocks       epsA    epsG
```

```
epsA       0.1190  0.0189
epsG       0.0189  0.9996
Shocks       epsA    epsG
Eigen      0.1186  1.0000

MATRIX B FOR SQUARE SYSTEMS
Shocks       epsA    epsG
epsA      -0.0000  0.0000
epsG       0.0000  0.0000
Shocks       epsA    epsG
Eigen      0.0000  0.0000
```

Notes: There are two shocks and two observables; imperfect information is not equivalent to perfect information and this is verified by both the rank and eigenvalue conditions: EB is of full rank; $A\left(I - B(EB)^{-1}E\right)$ has stable eigenvalues; JB is not of full rank and $F(I - B(JB)^{-1}J)$ is non-existent. There is complete fundamentalness for API, as expected, when $\mathbb{F}^{PI} = 0$ but not for AII when $\mathbb{F}^{II} > 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. Finally, the program also reports $\mathbb{B}^{PI} = EP^E E' - EBB'E' = 0$ only when $m = k$.

L.5 Table 2: RBC Model with Observables (H_t, R_t): `rbc_invertibility.mod`

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements = no. of shocks,
but imperfect information cannot mimic perfect information

Measures of Invertibility and Fundamentalness
Matrix      E  EB   J  JB
Rank        2   2   1   1
The Eigenvalue Condition for PI is not satisfied

MATRIX F WITH PI
Shocks       epsA    epsG
epsA       0.0060 -0.0627
epsG      -0.0627  0.6533
Shocks       epsA    epsG
Eigen      0.0000  0.6593

MATRIX F WITH II
```

```
Shocks      epsA    epsG
epsA      0.0149  0.0345
epsG      0.0345  0.9988
Shocks      epsA    epsG
Eigen     0.0137  1.0000

MATRIX B FOR SQUARE SYSTEMS
Shocks      epsA    epsG
epsA      0.0006  0.0191
epsG      0.0191  0.6588
Shocks      epsA    epsG
Eigen     0.0000  0.6593
```

Notes: There are two shocks and two observables; imperfect information is not equivalent to perfect information and this is verified by both the rank and eigenvalue conditions: although EB is of full rank **API is not invertible because** $A\left(I - B(EB)^{-1}E\right)$ **is not stable** – `The Eigenvalue Condition for PI is not satisfied`; JB is not of full rank; There is no complete fundamentalness when both $\mathbb{F}^{PI} > 0$ and $\mathbb{F}^{II} > 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. Finally, the program also reports $\mathbb{B}^{PI} = EP^E E' - EBB'E' > 0$ only when $m = k$.

L.6 Table 2: RBC Model with Observables (C_t, I_t): `rbc_invertibility.mod`

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements = no. of shocks,
but imperfect information cannot mimic perfect information

Measures of Invertibility and Fundamentalness
Matrix      E  EB   J  JB
Rank        2   2   2   2
The Eigenvalue Condition for PI is satisfied
The Eigenvalue Condition for II is not satisfied

MATRIX F WITH PI
Shocks      epsA    epsG
epsA      0.0000  0.0000
epsG      0.0000  0.0000
Shocks      epsA    epsG
Eigen     0.0000  0.0000
```

```
MATRIX F WITH II
Shocks       epsA    epsG
epsA       0.0217  0.1394
epsG       0.1394  0.8950
Shocks       epsA    epsG
Eigen      0.0000  0.9167

MATRIX B FOR SQUARE SYSTEMS
Shocks       epsA    epsG
epsA       0.0000 -0.0000
epsG      -0.0000  0.0000
Shocks       epsA    epsG
Eigen      0.0000  0.0000
```

Notes: There are two shocks and two observables; imperfect information is not equivalent to perfect information and this is verified by both the eigenvalue conditions only for AII: although JB is of full rank **AII is not invertible because** $F(I - B(JB)^{-1}J)$ **has unstable roots** – `The Eigenvalue Condition for PI is not satisfied`; There is no complete fundamentalness when both $\mathbb{F}^{PI} = 0$ but not surprisingly $\mathbb{F}^{II} > 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. Finally, the program also reports $\mathbb{B}^{PI} = EP^EE' - EBB'E' > 0$ only when $m = k$.

L.7 Table 2: RBC Model with One Observable (C_t): `rbc_invertibility.mod`

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements < no. of shocks,
imperfect information cannot mimic perfect information

Measures of Invertibility and Fundamentalness
Matrix      E  EB   J  JB
Rank        1   1   1   1

MATRIX F WITH PI
Shocks       epsA    epsG
epsA       0.0126  0.1100
epsG       0.1100  0.9877
Shocks       epsA    epsG
Eigen      0.0003  1.0000
```

```
MATRIX F WITH II
Shocks       epsA    epsG
epsA       0.0172  0.0109
epsG       0.0109  0.9999
Shocks       epsA    epsG
Eigen      0.0171  1.0000
```

Notes: Number of measurements < number of shocks; imperfect information is not equivalent to perfect information and this is verified by the rank conditions: EB and JB are not of full rank therefore API and AII are not invertible. There is no complete fundamentalness when both $\mathbb{F}^{PI} > 0$ and $\mathbb{F}^{II} > 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. $\mathbb{B}^{PI}$ is no longer applicable for a non-square case ($m < k$).

L.8 Table 2: RBC Model with Observables (Y_{t-1}, C_{t-1}): `rbc_invertibility.mod`

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements = no. of shocks,
but imperfect information cannot mimic perfect information

Measures of Invertibility and Fundamentalness
Matrix      E  EB   J  JB
Rank        2   0   2   0

MATRIX F WITH PI
Shocks       epsA    epsG
epsA       1.0000  0.0000
epsG       0.0000  1.0000
Shocks       epsA    epsG
Eigen      1.0000  1.0000

MATRIX F WITH II
Shocks       epsA    epsG
epsA       0.9776 -0.0007
epsG      -0.0007  0.8908
Shocks       epsA    epsG
Eigen      0.9776  0.8908
```

```
MATRIX B FOR SQUARE SYSTEMS
Shocks        epsA     epsG
epsA        1.0000   0.0000
epsG        0.0000   1.0000
Shocks        epsA     epsG
Eigen       1.0000   1.0000
```

Notes: There are two shocks and two observables; imperfect information is not equivalent to perfect information and this is verified by the rank conditions: JB is not of full rank despite J being of full rank and this must imply there are lagged observations in the AII information set. There is no complete fundamentalness when both $\mathbb{F}^{PI} > 0$ and $\mathbb{F}^{II} > 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. Finally, the program also reports $\mathbb{B}^{PI} = EP^{E}E' - EBB'E' > 0$ only when $m = k$.

L.9 Table 7: SW Model Case 1: `sw07_invertibility.mod`

```
--- THE INVERTIBILITY CONDITION IS SATISFIED ---
no. of measurements = no. of shocks,
imperfect information is equivalent to perfect information

Measures of Invertibility and Fundamentalness
Matrix      E   EB    J   JB
Rank        7    7    7    7
The Eigenvalue Condition for PI is satisfied
The Eigenvalue Condition for II is satisfied

MATRIX F WITH PI
Shocks       eta_a    eta_b    eta_g    eta_i    eta_r    eta_p    eta_w
eta_a       0.0000   0.0000   0.0000  -0.0000   0.0000   0.0000   0.0000
eta_b       0.0000   0.0000  -0.0000  -0.0000  -0.0000   0.0000  -0.0000
eta_g       0.0000  -0.0000   0.0000  -0.0000   0.0000   0.0000   0.0000
eta_i      -0.0000  -0.0000  -0.0000   0.0000  -0.0000  -0.0000  -0.0000
eta_r       0.0000  -0.0000   0.0000  -0.0000   0.0000   0.0000  -0.0000
eta_p       0.0000   0.0000   0.0000  -0.0000   0.0000   0.0000  -0.0000
eta_w       0.0000  -0.0000   0.0000  -0.0000  -0.0000  -0.0000   0.0000
Shocks       eta_a    eta_b    eta_g    eta_i    eta_r    eta_p    eta_w
Eigen       0.0000   0.0000   0.0000   0.0000   0.0000   0.0000   0.0000
```

```
MATRIX F WITH II
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w
eta_a      0.0000  0.0000  0.0000 -0.0000  0.0000  0.0000  0.0000
eta_b      0.0000  0.0000 -0.0000 -0.0000 -0.0000  0.0000 -0.0000
eta_g      0.0000 -0.0000  0.0000 -0.0000  0.0000  0.0000  0.0000
eta_i     -0.0000 -0.0000 -0.0000  0.0000 -0.0000 -0.0000 -0.0000
eta_r      0.0000 -0.0000  0.0000 -0.0000  0.0000  0.0000 -0.0000
eta_p      0.0000  0.0000  0.0000 -0.0000  0.0000  0.0000 -0.0000
eta_w      0.0000 -0.0000  0.0000 -0.0000 -0.0000 -0.0000  0.0000
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w
Eigen      0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000

MATRIX B FOR SQUARE SYSTEMS
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w
eta_a      0.0000  0.0000 -0.0000  0.0000 -0.0000 -0.0000  0.0000
eta_b      0.0000  0.0000 -0.0000 -0.0000 -0.0000 -0.0000  0.0000
eta_g     -0.0000 -0.0000  0.0000 -0.0000  0.0000  0.0000 -0.0000
eta_i      0.0000 -0.0000 -0.0000  0.0000  0.0000 -0.0000 -0.0000
eta_r     -0.0000 -0.0000  0.0000  0.0000  0.0000 -0.0000 -0.0000
eta_p     -0.0000 -0.0000  0.0000 -0.0000 -0.0000  0.0000  0.0000
eta_w      0.0000  0.0000 -0.0000 -0.0000 -0.0000  0.0000  0.0000
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w
Eigen      0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
```

Notes: There are seven shocks and seven observables (as in the original Smets and Wouters (2007)); imperfect information is equivalent to perfect information and this is verified by both the rank and eigenvalue conditions: EB is of full rank; $A\left(I - B(EB)^{-1}E\right)$ has stable eigenvalues; JB is of full rank and $F(I - B(JB)^{-1}J)$ has stable eigenvalues. There is complete fundamentalness when $\mathbb{F}^{PI} = 0$ and $\mathbb{F}^{II} = 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. Finally, the program also reports $\mathbb{B}^{PI} = EP^{E}E' - EBB'E' = 0$ only when $m = k$.

L.10 Table 7: SW Model Case 2: `sw07_invertibility_inf.mod`

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements < no. of shocks,
imperfect information cannot mimic perfect information

Measures of Invertibility and Fundamentalness
```

```
Matrix      E  EB   J  JB
Rank        7   7   7   7

MATRIX F WITH PI
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t
eta_a      0.0000 -0.0000 -0.0000  0.0000  0.0000  0.0000 -0.0000 -0.0000
eta_b     -0.0000  0.0006  0.0000  0.0005 -0.0038  0.0001  0.0002 -0.0232
eta_g     -0.0000  0.0000  0.0000 -0.0000 -0.0000 -0.0000  0.0000  0.0000
eta_i      0.0000  0.0005 -0.0000  0.0005 -0.0034  0.0001  0.0002 -0.0223
eta_r      0.0000 -0.0038 -0.0000 -0.0034  0.0245 -0.0006 -0.0015  0.1504
eta_p      0.0000  0.0001 -0.0000  0.0001 -0.0006  0.0000  0.0000 -0.0041
eta_w     -0.0000  0.0002  0.0000  0.0002 -0.0015  0.0000  0.0001 -0.0096
eta_t     -0.0000 -0.0232  0.0000 -0.0223  0.1504 -0.0041 -0.0096  0.9756
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t
Eigen      1.0000  0.0013  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000

MATRIX F WITH II
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t
eta_a      0.0000  0.0000  0.0000 -0.0001  0.0000 -0.0000 -0.0000 -0.0000
eta_b      0.0000  0.0006 -0.0000 -0.0004  0.0000  0.0000 -0.0002 -0.0000
eta_g      0.0000 -0.0000  0.0000 -0.0000  0.0000 -0.0000  0.0000 -0.0000
eta_i     -0.0001 -0.0004 -0.0000  0.0004 -0.0000 -0.0000  0.0001  0.0000
eta_r      0.0000  0.0000  0.0000 -0.0000  0.0256 -0.0000  0.0000  0.1526
eta_p     -0.0000  0.0000 -0.0000 -0.0000 -0.0000  0.0000 -0.0000  0.0000
eta_w     -0.0000 -0.0002  0.0000  0.0001  0.0000 -0.0000  0.0001 -0.0000
eta_t     -0.0000 -0.0000 -0.0000  0.0000  0.1526  0.0000 -0.0000  0.9761
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t
Eigen      1.0000  0.0016  0.0009  0.0001  0.0000  0.0000  0.0000  0.0000
```

Notes: Number of measurements < number of shocks; imperfect information is not equivalent to perfect information and this is verified by the rank conditions: EB and JB are not of full rank therefore both API and AII are not invertible. There is no complete fundamentalness when both $\mathbb{F}^{PI} > 0$ and $\mathbb{F}^{II} > 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. $\mathbb{B}^{PI}$ is not applicable for a non-square case $(m < k)$.

L.11 Table 7: SW Model Case 3: `sw07_invertibility_inf_me.mod`

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements < no. of shocks,
```

```
imperfect information cannot mimic perfect information

Measures of Invertibility and Fundamentalness
Matrix     E  EB   J  JB
Rank       7   7   7   7

MATRIX F WITH PI
Shocks     eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t    me_y    me_c    me_i    me_w   me_pi
eta_a     0.2216  0.0505  0.2269 -0.0324  0.0098  0.0237 -0.0341 -0.0013 -0.2978 -0.1207 -0.0524  0.0806 -0.0100
eta_b     0.0505  0.0924 -0.0722 -0.0712 -0.0195 -0.0157 -0.0250 -0.0206 -0.0677 -0.2191  0.0043  0.0000  0.0376
eta_g     0.2269 -0.0722  0.5199  0.0098  0.0600  0.0169 -0.0027 -0.0107 -0.3237  0.2712  0.0931  0.0379 -0.0064
eta_i    -0.0324 -0.0712  0.0098  0.1600  0.0083  0.0206  0.0108 -0.0219  0.0493  0.1631 -0.2551  0.0222 -0.0322
eta_r     0.0098 -0.0195  0.0600  0.0083  0.1007 -0.0958 -0.0904  0.1372 -0.0057  0.0806 -0.0051  0.0210  0.1940
eta_p     0.0237 -0.0157  0.0169  0.0206 -0.0958  0.2262  0.1182  0.0128 -0.0009 -0.0078  0.0039  0.1149 -0.3026
eta_w    -0.0341 -0.0250 -0.0027  0.0108 -0.0904  0.1182  0.2585  0.0068 -0.0299  0.0314  0.0112 -0.3117 -0.2136
eta_t    -0.0013 -0.0206 -0.0107 -0.0219  0.1372  0.0128  0.0068  0.9780  0.0001 -0.0137 -0.0048 -0.0053 -0.0332
me_y     -0.2978 -0.0677 -0.3237  0.0493 -0.0057 -0.0009 -0.0299  0.0001  0.4668  0.1745  0.0764  0.0967 -0.0040
me_c     -0.1207 -0.2191  0.2712  0.1631  0.0806 -0.0078  0.0314 -0.0137  0.1745  0.7097  0.0013 -0.0207  0.0065
me_i     -0.0524  0.0043  0.0931 -0.2551 -0.0051  0.0039  0.0112 -0.0048  0.0764  0.0013  0.9053  0.0024 -0.0069
me_w      0.0806  0.0000  0.0379  0.0222  0.0210  0.1149 -0.3117 -0.0053  0.0967 -0.0207  0.0024  0.8353 -0.0753
me_pi    -0.0100  0.0376 -0.0064 -0.0322  0.1940 -0.3026 -0.2136 -0.0332 -0.0040  0.0065 -0.0069 -0.0753  0.6998
Shocks     eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t    me_y    me_c    me_i    me_w   me_pi
Eigen     0.0971  0.0454  0.0138  0.0001  0.0019  0.0058  0.0100  1.0000  1.0000  1.0000  1.0000  1.0000  1.0000

MATRIX F WITH II
Shocks     eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t    me_y    me_c    me_i    me_w   me_pi
eta_a     0.5754  0.0324  0.1915 -0.0129  0.0799  0.0718 -0.0552 -0.0125 -0.0918 -0.0911 -0.0486 -0.0201  0.0069
eta_b     0.0324  0.8850 -0.0436 -0.0261  0.0465  0.0116 -0.0049 -0.0073 -0.0183 -0.0781 -0.0180 -0.0060 -0.0065
eta_g     0.1915 -0.0436  0.5136  0.0280  0.0565  0.0100 -0.0068 -0.0088 -0.3519  0.2408  0.1185  0.0415 -0.0126
eta_i    -0.0129 -0.0261  0.0280  0.6945  0.0222  0.0057 -0.0067 -0.0035  0.0207  0.1334 -0.3107 -0.0108 -0.0061
eta_r     0.0799  0.0465  0.0565  0.0222  0.1099 -0.1121 -0.0521  0.1394  0.0478  0.1168  0.0117 -0.0173  0.1291
eta_p     0.0718  0.0116  0.0100  0.0057 -0.1121  0.4552  0.1214  0.0176  0.0479  0.0148  0.0038  0.1195 -0.4220
eta_w    -0.0552 -0.0049 -0.0068 -0.0067 -0.0521  0.1214  0.7095  0.0082 -0.0473  0.0150 -0.0054 -0.2786 -0.0721
eta_t    -0.0125 -0.0073 -0.0088 -0.0035  0.1394  0.0176  0.0082  0.9782 -0.0075 -0.0183 -0.0018  0.0027 -0.0202
me_y     -0.0918 -0.0183 -0.3519  0.0207  0.0478  0.0479 -0.0473 -0.0075  0.5892  0.1870  0.0862  0.0272 -0.0013
me_c     -0.0911 -0.0781  0.2408  0.1334  0.1168  0.0148  0.0150 -0.0183  0.1870  0.6749  0.0886 -0.0191 -0.0109
me_i     -0.0486 -0.0180  0.1185 -0.3107  0.0117  0.0038 -0.0054 -0.0018  0.0862  0.0886  0.6672 -0.0186 -0.0038
me_w     -0.0201 -0.0060  0.0415 -0.0108 -0.0173  0.1195 -0.2786  0.0027  0.0272 -0.0191 -0.0186  0.7165 -0.0764
me_pi     0.0069 -0.0065 -0.0126 -0.0061  0.1291 -0.4220 -0.0721 -0.0202 -0.0013 -0.0109 -0.0038 -0.0764  0.4854
Shocks     eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t    me_y    me_c    me_i    me_w   me_pi
Eigen     0.5404  0.3627  0.2975  0.0302  0.0011  0.0044  0.8182  1.0000  1.0000  1.0000  1.0000  1.0000  1.0000
```

Notes: Number of measurements < number of shocks; imperfect information is not equivalent to perfect information and this is verified by the rank conditions: EB and JB are not of full rank therefore both API and AII are not invertible. There is no complete fundamentalness when both $\mathbb{F}^{PI} > 0$ and $\mathbb{F}^{II} > 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$ which is now much worse than Case 2. $\mathbb{B}^{PI}$ is not applicable for a non-square case ($m < k$).

L.12 Table 8: SW Model Case 1: `sw07_estimation_invertib.mod`

```
--- THE INVERTIBILITY CONDITION IS SATISFIED ---
no. of measurements = no. of shocks,
imperfect information is equivalent to perfect information

Measures of Invertibility and Fundamentalness
Matrix     E  EB   J  JB
Rank       7   7   7   7
The Eigenvalue Condition for PI is satisfied
```

```
The Eigenvalue Condition for II is satisfied

MATRIX F WITH PI
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w
eta_a      0.0000  0.0000  0.0000 -0.0000  0.0000  0.0000  0.0000
eta_b      0.0000  0.0000 -0.0000 -0.0000 -0.0000 -0.0000 -0.0000
eta_g      0.0000 -0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
eta_i     -0.0000 -0.0000  0.0000  0.0000 -0.0000  0.0000 -0.0000
eta_r      0.0000 -0.0000  0.0000 -0.0000  0.0000  0.0000 -0.0000
eta_p      0.0000 -0.0000  0.0000  0.0000  0.0000  0.0000 -0.0000
eta_w      0.0000 -0.0000  0.0000 -0.0000 -0.0000 -0.0000  0.0000
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w
Eigen      0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000

MATRIX F WITH II
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w
eta_a     -0.0000 -0.0000 -0.0000  0.0000 -0.0000 -0.0000 -0.0000
eta_b     -0.0000  0.0000 -0.0000 -0.0000 -0.0000 -0.0000 -0.0000
eta_g     -0.0000 -0.0000  0.0000  0.0000  0.0000  0.0000 -0.0000
eta_i      0.0000 -0.0000  0.0000  0.0000  0.0000 -0.0000  0.0000
eta_r     -0.0000 -0.0000  0.0000  0.0000  0.0000  0.0000 -0.0000
eta_p     -0.0000 -0.0000  0.0000 -0.0000  0.0000 -0.0000  0.0000
eta_w     -0.0000 -0.0000 -0.0000  0.0000 -0.0000  0.0000 -0.0000
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w
Eigen      0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000

MATRIX B FOR SQUARE SYSTEMS
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w
eta_a      0.0000  0.0000 -0.0000  0.0000 -0.0000 -0.0000  0.0000
eta_b      0.0000  0.0000 -0.0000 -0.0000 -0.0000 -0.0000  0.0000
eta_g     -0.0000 -0.0000  0.0000 -0.0000  0.0000  0.0000 -0.0000
eta_i      0.0000 -0.0000 -0.0000  0.0000  0.0000 -0.0000 -0.0000
eta_r     -0.0000 -0.0000  0.0000  0.0000  0.0000 -0.0000 -0.0000
eta_p     -0.0000 -0.0000  0.0000 -0.0000 -0.0000  0.0000  0.0000
eta_w      0.0000  0.0000 -0.0000 -0.0000 -0.0000  0.0000  0.0000
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w
Eigen      0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
```

Notes: There are seven shocks and seven observables (as in the original Smets and Wouters (2007)); imperfect information is equivalent to perfect information and this is verified by both the rank and eigenvalue conditions: EB is of full rank; $A\left(I - B(EB)^{-1}E\right)$ has stable eigenvalues; JB is of full rank and $F(I - B(JB)^{-1}J)$ has stable eigenvalues. There is complete fundamentalness when $\mathbb{F}^{PI} = 0$ and $\mathbb{F}^{II} = 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. Finally, the program also reports $\mathbb{B}^{PI} = EP^E E' - EBB'E' = 0$ only when $m = k$.

L.13 Table 8: SW Model Case 2: `sw07_estimation_inf_invertib.mod`

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements < no. of shocks,
imperfect information cannot mimic perfect information

Measures of Invertibility and Fundamentalness
Matrix     E  EB   J  JB
Rank       7   7   7   7

MATRIX F WITH PI
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t
eta_a      0.0000 -0.0000 -0.0000  0.0000  0.0000  0.0000 -0.0000 -0.0000
eta_b     -0.0000  0.0001  0.0000  0.0000 -0.0003  0.0000  0.0000 -0.0082
eta_g     -0.0000  0.0000  0.0000 -0.0000 -0.0000 -0.0000  0.0000  0.0000
eta_i      0.0000  0.0000 -0.0000  0.0000 -0.0001  0.0000  0.0000 -0.0038
eta_r      0.0000 -0.0003 -0.0000 -0.0001  0.0015 -0.0001 -0.0001  0.0360
eta_p      0.0000  0.0000 -0.0000  0.0000 -0.0001  0.0000  0.0000 -0.0028
eta_w     -0.0000  0.0000  0.0000  0.0000 -0.0001  0.0000  0.0000 -0.0026
eta_t     -0.0000 -0.0082  0.0000 -0.0038  0.0360 -0.0028 -0.0026  0.9986
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t
Eigen      1.0000  0.0002  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000

MATRIX F WITH II
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t
eta_a      0.0000 -0.0000  0.0000  0.0000  0.0000  0.0000 -0.0000 -0.0000
eta_b     -0.0000  0.0000 -0.0000 -0.0000 -0.0000 -0.0000  0.0000  0.0000
eta_g      0.0000 -0.0000  0.0000  0.0000  0.0000  0.0000 -0.0000 -0.0000
eta_i      0.0000 -0.0000  0.0000  0.0000  0.0000  0.0000  0.0000 -0.0000
eta_r      0.0000 -0.0000  0.0000  0.0000  0.0001  0.0000  0.0000  0.0071
eta_p      0.0000 -0.0000  0.0000  0.0000  0.0000 -0.0000  0.0000 -0.0000
eta_w     -0.0000  0.0000 -0.0000  0.0000  0.0000  0.0000 -0.0000 -0.0000
eta_t     -0.0000  0.0000 -0.0000 -0.0000  0.0071 -0.0000 -0.0000  0.9999
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t
Eigen      1.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
```

Notes: Number of measurements < number of shocks; imperfect information is not equivalent to perfect information and this is verified by the rank conditions: EB and JB are not of full rank therefore both API and AII are not invertible. There is no complete fundamentalness when both $\mathbb{F}^{PI} > 0$ and $\mathbb{F}^{II} > 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. $\mathbb{B}^{PI}$ is not applicable for a non-square case ($m < k$).

L.14 Table 8: SW Model Case 3: `sw07_estimation_inf_me_invertib.mod`

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements < no. of shocks,
imperfect information cannot mimic perfect information

Measures of Invertibility and Fundamentalness
Matrix     E  EB   J  JB
Rank       7   7   7   7

MATRIX F WITH PI
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t    me_y    me_c    me_i    me_w   me_pi
eta_a      0.0020  0.0004  0.0014 -0.0003  0.0006  0.0007 -0.0002 -0.0000 -0.0315 -0.0000 -0.0000  0.0049  0.0000
eta_b      0.0004  0.0068 -0.0017 -0.0019 -0.0023  0.0001  0.0002 -0.0164 -0.0004 -0.0358  0.0008 -0.0032 -0.0025
eta_g      0.0014 -0.0017  0.0022  0.0002  0.0008  0.0003  0.0001 -0.0001 -0.0304  0.0286  0.0088  0.0020 -0.0000
eta_i     -0.0003 -0.0019  0.0002  0.0018 -0.0007  0.0011  0.0004 -0.0076  0.0021  0.0132 -0.0142  0.0020 -0.0075
eta_r      0.0006 -0.0023  0.0008 -0.0007  0.0081 -0.0028 -0.0021  0.0727 -0.0046  0.0133  0.0003  0.0036  0.0362
eta_p      0.0007  0.0001  0.0003  0.0011 -0.0028  0.0115  0.0040 -0.0054 -0.0057 -0.0002  0.0009  0.0190 -0.0731
eta_w     -0.0002  0.0002  0.0001  0.0004 -0.0021  0.0040  0.0044 -0.0050  0.0007  0.0027  0.0013 -0.0372 -0.0400
eta_t     -0.0000 -0.0164 -0.0001 -0.0076  0.0727 -0.0054 -0.0050  0.9943  0.0003 -0.0015 -0.0001 -0.0004 -0.0034
me_y      -0.0315 -0.0004 -0.0304  0.0021 -0.0046 -0.0057  0.0007  0.0003  0.9980  0.0009  0.0003  0.0004 -0.0002
me_c      -0.0000 -0.0358  0.0286  0.0132  0.0133 -0.0002  0.0027 -0.0015  0.0009  0.9975 -0.0000 -0.0001 -0.0004
me_i      -0.0000  0.0008  0.0088 -0.0142  0.0003  0.0009  0.0013 -0.0001  0.0003 -0.0000  0.9997  0.0000  0.0000
me_w       0.0049 -0.0032  0.0020  0.0020  0.0036  0.0190 -0.0372 -0.0004  0.0004 -0.0001  0.0000  0.9982 -0.0002
me_pi      0.0000 -0.0025 -0.0000 -0.0075  0.0362 -0.0731 -0.0400 -0.0034 -0.0002 -0.0004  0.0000 -0.0002  0.9916
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t    me_y    me_c    me_i    me_w   me_pi
Eigen      1.0000  1.0000  0.0064  0.0058  0.0018  0.0000  0.0009  0.0006  0.0005  1.0000  1.0000  1.0000  1.0000

MATRIX F WITH II
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t    me_y    me_c    me_i    me_w   me_pi
eta_a      0.0023  0.0046 -0.0002 -0.0106 -0.0107  0.0023 -0.0005  0.0002 -0.0017  0.0001 -0.0000  0.0003  0.0001
eta_b      0.0046  0.2401 -0.0035 -0.0333  0.0264  0.0029 -0.0035 -0.0004  0.0000 -0.0074 -0.0001 -0.0000  0.0001
eta_g     -0.0002 -0.0035  0.0007  0.0068 -0.0058  0.0011 -0.0005  0.0001 -0.0014  0.0013  0.0004  0.0001  0.0000
eta_i     -0.0106 -0.0333  0.0068  0.1007 -0.0103  0.0012 -0.0030  0.0001 -0.0000  0.0018 -0.0029 -0.0000  0.0000
eta_r     -0.0107  0.0264 -0.0058 -0.0103  0.1222 -0.0236  0.0101  0.0127 -0.0000  0.0003 -0.0000  0.0001 -0.0008
eta_p      0.0023  0.0029  0.0011  0.0012 -0.0236  0.0076 -0.0039  0.0003 -0.0004  0.0000  0.0000  0.0006 -0.0380
eta_w     -0.0005 -0.0035 -0.0005 -0.0030  0.0101 -0.0039  0.0189 -0.0001 -0.0000  0.0002 -0.0000 -0.0075 -0.0064
eta_t      0.0002 -0.0004  0.0001  0.0001  0.0127  0.0003 -0.0001  0.9998  0.0000 -0.0000  0.0000 -0.0000  0.0000
me_y      -0.0017  0.0000 -0.0014 -0.0000 -0.0000 -0.0004 -0.0000  0.0000  1.0000  0.0000  0.0000  0.0000 -0.0000
me_c       0.0001 -0.0074  0.0013  0.0018  0.0003  0.0000  0.0002 -0.0000  0.0000  0.9999  0.0000  0.0000  0.0000
me_i      -0.0000 -0.0001  0.0004 -0.0029 -0.0000  0.0000 -0.0000  0.0000  0.0000  0.0000  1.0000 -0.0000  0.0000
me_w       0.0003 -0.0000  0.0001 -0.0000  0.0001  0.0006 -0.0075 -0.0000  0.0000  0.0000 -0.0000  0.9999 -0.0000
me_pi      0.0001  0.0001  0.0000  0.0000 -0.0008 -0.0380 -0.0064  0.0000 -0.0000  0.0000  0.0000 -0.0000  0.9985
Shocks      eta_a   eta_b   eta_g   eta_i   eta_r   eta_p   eta_w   eta_t    me_y    me_c    me_i    me_w   me_pi
Eigen      0.2538  0.1241  0.0941  0.0177  0.0010  0.0000  0.0000  1.0000  1.0000  1.0000  1.0000  1.0000  1.0000
```

Notes: Number of measurements < number of shocks; imperfect information is not equivalent to perfect information and this is verified by the rank conditions: EB and JB are not of full rank therefore both API and AII are not invertible. There is no complete fundamentalness when both $\mathbb{F}^{PI} > 0$ and $\mathbb{F}^{II} > 0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$ which is now much worse than Case 2. $\mathbb{B}^{PI}$ is not applicable for a non-square case ($m < k$).

L.15 Table 13: Simulation of the Estimated Model: `fs2000_invertibility_II.mod`

```
--- THE INVERTIBILITY CONDITION IS NOT SATISFIED ---
no. of measurements = no. of shocks,
but imperfect information cannot mimic perfect information

Measures of Invertibility and Fundamentalness
Matrix      E  EB   J  JB
Rank        2   2   2   2
The Eigenvalue Condition for PI is satisfied
The Eigenvalue Condition for II is not satisfied

MATRIX F WITH PI
Shocks         e_a     e_m
e_a      -0.0000 -0.0000
e_m      -0.0000  0.0000
Shocks         e_a     e_m
Eigen      0.0000  0.0000

MATRIX F WITH II
Shocks         e_a     e_m
e_a        0.0667  0.2495
e_m        0.2495  0.9333
Shocks         e_a     e_m
Eigen      0.0000  1.0000

MATRIX B FOR SQUARE SYSTEMS
Shocks         e_a     e_m
e_a      -0.0000  0.0000
e_m        0.0000 -0.0000
Shocks         e_a     e_m
Eigen      0.0000  0.0000
```

Notes: There are two shocks and two observables; imperfect information is not equivalent to perfect information and this is verified by just the failure of the eigenvalue condition for AII: EB is of full rank and $A\left(I - B(EB)^{-1}E\right)$ is a stable matrix therefore API is E-invertible; JB is of full rank **but the eigenvalue condition fails with AII:** $F(I - B(JB)^{-1}J)$ **has eigenvalues outside the unit circle** – `The Eigenvalue Condition for II is not satisfied`. There is complete fundamentalness when $\mathbb{F}^{PI} = 0$ for API but with AII $\mathbb{F}^{II} > 0$ and this is consistent with the finding based on Theorem 2. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$. Finally, the program also reports $\mathbb{B}^{PI} = EP^E E' - EBB'E' = 0 = \mathbb{F}^{PI}$ only when $m = k$.

M The PartInfoDyn Toolbox: Instructions

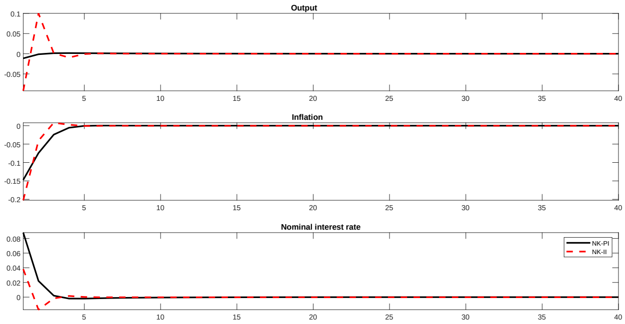
1. Click here for viewing and downloading `PartInfoDyn.zip`.

2. Download and install `dynare-4.6.2` in the usual way, download the zip-file `PartInfoDyn.zip` and extract its content which contains:

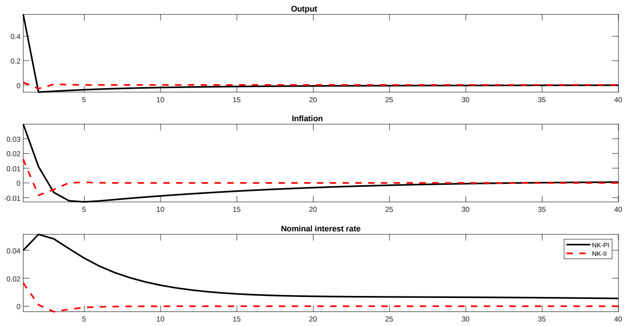
   - `partial_information`
   - `examples`
   - `partinfo_doc`

3. Replace subfolder `...\dynare-4.6.2\matlab\partial_information` with the above folder `partial_information` (i.e. copy the codes from `partial_information`, paste them into the `dynare-4.6.2` subfolder, overwriting the content).

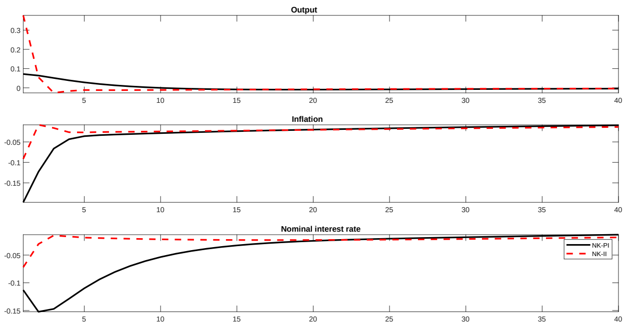
4. Run standard `.mod` files as usual; Run `.mod` files under imperfect information by following the syntax rules introduced above for simulation and estimation respectively.



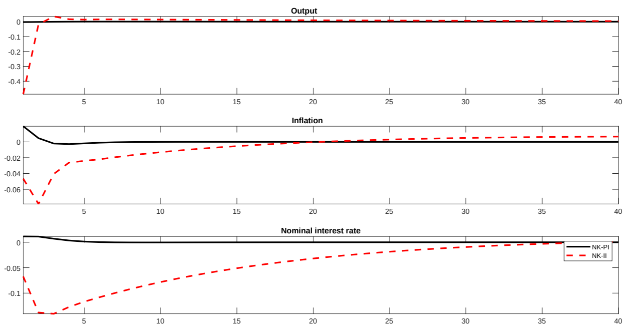
(a) Monetary Policy Shock



(b) Government Spending Shock



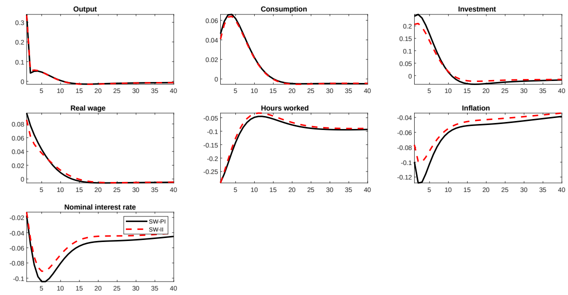
(c) Technology Shock



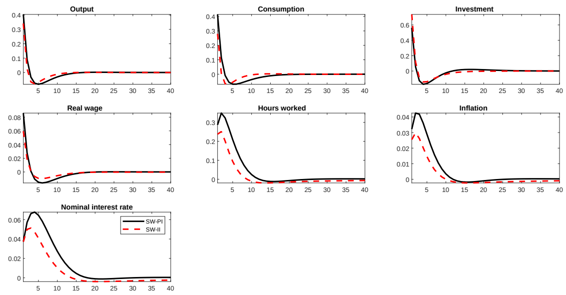
(d) Mark-up Shock

Figure 6: **Impulse Response Functions (NK Model with 3 Observables)**

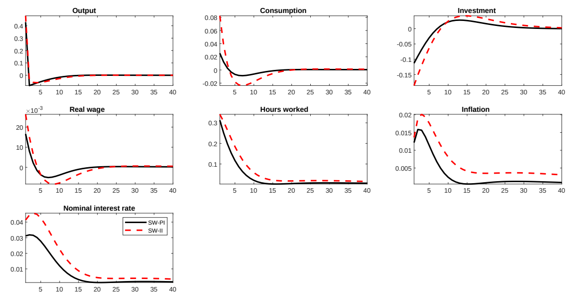
Notes: Solid black line PI responses. Dashed red line II responses. Each panel plots the mean response corresponding a positive one standard deviation of the shock's innovation. Each response is for a 40 period (10 years) horizon and is level deviation of a variable from its steady-state value.



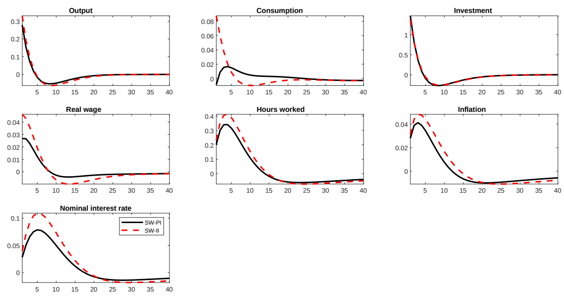
(a) Technology



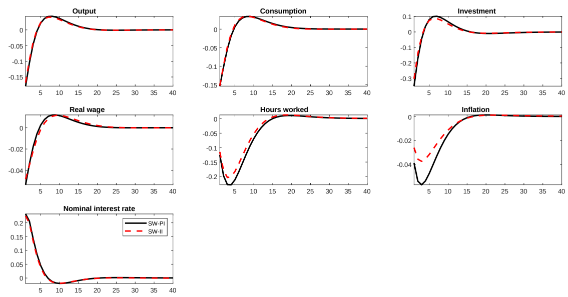
(b) Preference



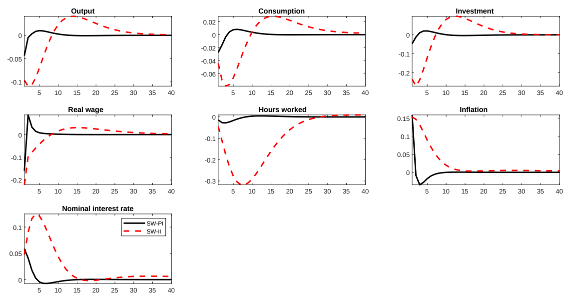
(c) Government Spending



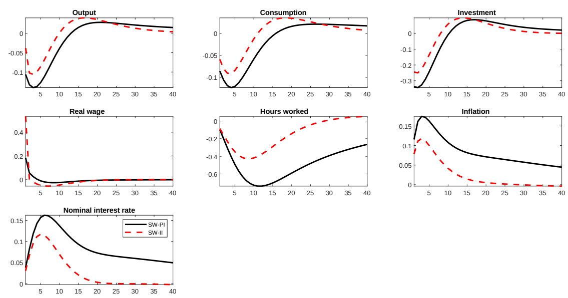
(d) Investment Specific



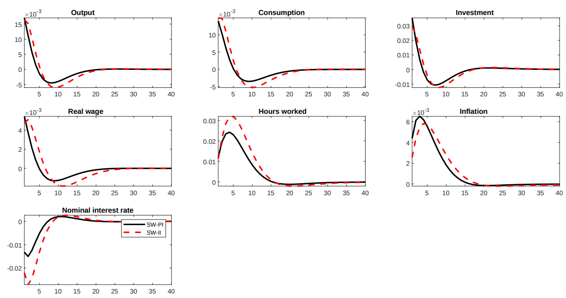
(e) Monetary Policy



(f) Price Markup



(g) Wage Markup



(h) Inflation Objective

Figure 7: **Impulse Response Functions (SW Model Case 3)**

Notes: Solid black line PI responses. Dashed red line II responses. Each panel plots the mean response corresponding a positive one standard deviation of the shock's innovation. Each response is for a 40 period (10 years) horizon and is level deviation of a variable from its steady-state value.