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A Toolbox for Monetary Rule Mandates in a New Keynesian Framework

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A Toolbox for Monetary Rule Mandates in a New Keynesian Framework[*]

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Abstract

The toolkit sets out a general mandate for the conduct of monetary policy. It consists of four components: (i) a welfare objective delegated to the central bank in the as a function of either the true household utility or a simple quadratic loss function that penalizes deviations from target macroeconomic variables. (ii) a form of a Taylor-type nominal interest-rate rule that responds to the *same* target variables specified in the loss function; (iii) a zero-lower-bound constraint on the nominal interest rate in the form of a low specified probability of hitting the ZLB and (iv) a long-run (steady-state) inflation target. The mandate is then chosen to maximize the expected inter-temporal utility of the household.

JEL Classification: E52, E58, E61.

Keywords: New Keynesian Model, Mandates, Zero Lower Bound Constraint.

Contents

1   Introduction

This is a toolkit to accompany work in Deak et al. (2020). In that paper we propose a general mandate framework that combines instrument and targeting rules in a consistent fashion. The mandate consists of four components: (i) a welfare objective delegated to the central bank in the form an expected discounted household utility or a simple quadratic loss function that penalizes deviations from target macroeconomic variables. In both case the utility is modified to penalize the variance of the nominal interest rate; (ii) a form of a Taylor-type nominal interest-rate rule that responds to the *same* target variables specified in the loss function; (iii) a zero-lower-bound constraint on the nominal interest rate in the form of a low specified probability of hitting the ZLB and (iv) a long-run (steady-state) inflation target. The mandate is then chosen to maximize the true unmodified inter-temporal utility of the household computed by a second-order perturbation method. With these four features the mandate makes the central bank goal-dependent, but instrument-independent as it remains free to the choose the strength of its response to the targets in the rule. An estimated standard New Keynesian model of Smets and Wouters (2007) is used to compute the household-welfare-optimal mandates with these properties.

In the rest of the paper Section 2 considers the case of the household-welfare mandate for the central bank, but without a ZLB constraint. Section 3.1 is then imposed the ZLB constraint describing the general framework in the form of a delegation game where the central bank designs a household welfare-optimized Taylor-type interest rate rule with targets given by one of the two types of mandate; Section 4 provides an application to the Smets and Wouters (2007) model, where sub-sections 4.1 and 4.2 provide examples for the no ZLB and with ZLB, respectively. Section 5 then presents the same delegation game the only difference being that at stage 3 we use a loss quadratic function as the modified welfare and a compatible simple rule that matches with such a delegated mandate. Section 6 gives full details on how to run this toolbox and the documentation of the new internal matlab functions completes this description. Section 7 concludes by comparing our framework based on optimized simple rules with a linear quadratic approach based on the Ramsey problem as in Debortoli et al. (2019). Appendix A provides full details of this alternative.

2   The Optimized Monetary Rule

We first describe the simple exercise of optimized monetary rule without the ZLB constraint (option specified below in section 6.4).

- **Monetary policy rule:** For the estimation of the model we use the following monetary

policy rule on nominal interest rate to estimate the model:

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\left(\theta_\pi \log\left(\frac{\Pi_t}{\Pi}\right) + \theta_y \log\left(\frac{Y_t^c}{Y^c}\right) + \theta_{dy} \log\left(\frac{Y_t^c}{Y_{t-1}^c}\right)\right) \tag{1}$$

The rule above is preferred under the estimation because it can account for a reliable range of θ under the given prior of normal distribution when ρ_r is closer to 1. However, under the optimization it would be problematic if the optimized rule converges to the price-level targeting rule, i.e. $\rho_r = 1$. Hence, in the optimization mod file we re-parameterize the rule above to the following form:

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + \alpha_\pi \log\left(\frac{\Pi_t}{\Pi}\right) + \alpha_y \log\left(\frac{Y_t^c}{Y^c}\right) + \alpha_{dy} \log\left(\frac{Y_t^c}{Y_{t-1}^c}\right) \tag{2}$$

Notice: We have the first different set of parameters in the rules between the estimation and optimization mod files, namely the θ and α. Since, the estimated θ are irrelevant in our optimized rule exercise, it is valid to replace these θ by the α. However, when we replace these θ by the α in our optimized mod file, we have to replace them at the exact positions of the θ, for instance, if the θ are defined in the estimated mod file at the positions, say 10, 15 and 20, then we must also have the α defined in the optimized mod file at the positions 10, 15 and 20, respectively.

- The steady state inflation is written in a gross level, i.e $\bar{\Pi} = 1 + \bar{\pi}$. Where $\bar{\pi}$ is the net steady state inflation.

- The value of β must be imposed (we calibrated β using the means of nominal interest rate and inflation time series data used for the estimation) before the maximization of stage 3 because in stage 2 the probability of hitting the ZLB is a function of this value β. Hence, this β is a nontrivial value in our exercise, which is in contrast of Coibion et al. (2012) who imposed this value of 0.99 with the same exercise on the probability of the nominal interest rate hitting the ZLB.

3 Optimized Simple Rules with ZLB Considerations

Consider a model with single period utility U_t, a discount factor β, a steady state inflation rate Π and a nominal interest rate rule of the form

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\left(\theta_\pi \log\left(\frac{\Pi_t}{\Pi}\right) + \theta_y \log\left(\frac{Y_t}{Y}\right) + \theta_{dy} \log\left(\frac{\Delta Y_t}{Y}\right)\right) \tag{3}$$

which for optimal policy purposes we re-parameterize as

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + \alpha_\pi \log\left(\frac{\Pi_t}{\Pi}\right) + \alpha_y \log\left(\frac{Y_t}{Y}\right) + \alpha_{dy} \log\left(\frac{\Delta Y_t}{Y}\right) \qquad (4)$$

which allows for the possibility of an integral rule with $\rho_r = 1$. See Section 6 for further details of this parametrization.

Let $\rho \equiv [\rho_r, \alpha_\pi, \alpha_y, \alpha_{dy}]$ be the policy choice of feedback parameters that defines the form of the rule. The equilibrium is solved by backward induction in the following **three-stage delegation game**.

3.1 The Delegation Game

This subsection describes the solution of the equilibrium starting with stage 3.

3.1.1 Stage 3: The CB Mandate

Given a steady state inflation rate target, Π, the Central Bank (CB) receives a mandate to implement the rule (12) and to maximize with respect to ρ a modified welfare criterion

$$\begin{aligned}
\Omega_t^{mod} &\equiv \mathbb{E}_t\left[\sum_{\tau=0}^{\infty} \beta^\tau \left(U_{t+\tau} - w_r \left(R_{n,t+\tau} - R_n\right)^2\right)\right] \\
&= \left(U_t - w_r \left(R_{n,t} - R_n\right)^2\right) + \beta \mathbb{E}_t\left[\Omega_{t+1}^{mod}\right] \qquad (5)
\end{aligned}$$

One can think of this as a mandate with a penalty function $P = w_r \left(R_{n,t} - R_n\right)^2$, penalizing the variance of the nominal interest rate with weight w_r,.

Following Den Haan and Wind (2012), an alternative mandate that only penalizes the zero interest rate in an asymmetric fashion is $P = P(a_t)$ where the OBC is $a_t \equiv R_{n,t} - 1 \geq 0$ with

$$P = P(a_t) = \frac{\exp(-w_r a_t)}{w_r} \qquad (6)$$

and chooses a large w_r. $P(a_t)$ then has the property

$$\begin{aligned}
\lim_{w_r \to \infty} P(a_t) &= \infty \text{ for } a_t < 0 \\
&= 0 \text{ for } a_t > 0
\end{aligned}$$

Thus $P(a_t)$ enforces the ZLB approximately but with more accuracy as w_r becomes large. Stages 3–1 then proceed as before, but we confine ourselves to **large** w_r which will enable Π to be close to unity.

Both mandates result in a probability of hitting the ZLB

$$p = p(\Pi, \rho^*(\Pi, w_r)) \tag{7}$$

where $\rho^*(\Pi, w_r)$ is the optimized form of the rule given the steady state target Π and the weight on the interest rate volatility, w_r.

3.1.2 Stage 2: Choice of the Steady State Inflation Rate Π

Given a target low probability $\bar{p}$ and given w_r, $\Pi = \Pi^*$ is chosen so satisfy

$$p(R_{n,t} \leq 1) \equiv p(\Pi^*, \rho^*(\Pi^*, w_r)) \leq \bar{p} \tag{8}$$

This then achieves the ZLB constraint

$$R_{n,t} \geq 1 \text{ with high probability } 1 - \bar{p} \tag{9}$$

where $R_{n,t}$ is the nominal interest rate.

3.1.3 Stage 1: Design of the Mandate

The policymaker first chooses a per period probability $\bar{p}$ of the nominal interest rate hitting the ZLB (which defines the tightness of the ZLB constraint). Then it maximizes the actual household intertemporal welfare

$$\Omega_t = \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \beta^\tau U_{t+\tau} \right] = (1 - \beta)U_t + \beta\mathbb{E}_t \left[\Omega_{t+1} \right] \tag{10}$$

with respect to w_r.

This three-stage delegation game defines an equilibrium in choice variables w_r^*, ρ^* and Π^* that maximizes the true household welfare subject to the ZLB constraint (9).

4 Application

We consider a Smets Wouter model in a non-linear setup for these exercise. Full details of the model setup and Bayesian estimation are given in **?**.

option **options__.asymetric_penalty** triggers the formation of the modified welfare function

the routine would take. if **options__.asymetric_penalty =0** we have Symmetric penalty function. if **options__.asymetric_penalty =1** we have asymmetric penalty function as in the (6).

4.1   General ZLB Mandate without ZLB Consideration

First, we examine the optimized simple rule of stage 3 (details of the exercise is presented in section **??**)above with a fixed level of steady stage inflation ($\Pi = 1.0$ for the results below) without, at this stage, the ZLB consideration. $options\_.ZLB\_stage\_2 = 0$ and **options__.asymetric_penalty =0**.

wr	Taylor_parameters			probability_ZLB	actual_welfare
2	0.997	3.716	0.037	0.679	0.246	-2599.251
4	0.999	2.588	0.022	0.444	0.236	-2599.259
6	0.998	2.022	0.017	0.328	0.229	-2599.268
8	1	1.666	0.013	0.257	0.223	-2599.277
10	1	1.422	0.011	0.209	0.219	-2599.288

4.2   General ZLB Mandate with ZLB Considerations

We now move to the case where we impose the ZLB (or stage 2) given that the probability of the nominal interest rate hitting the ZLB is allowed at less than or equal to 0.05. $options\_.ZLB\_stage\_2 = 1$ and **options__.asymetric_penalty =0**.

Results_with_ZLB =

 6×5 table

wr	Taylor_parameters			SS_inflation	actual_welfare	Probability
1	0.718	4.869	0.065	0.824	1.008	-2647.785	0.04996
3	0.999	3.162	0.036	0.53	1.007	-2647.73	0.05001
5	1	2.323	0.023	0.367	1.006	-2647.722	0.04993

7	1	1.884	0.017	0.284	1.006	-2647.72	0.04997
9	0.999	1.595	0.014	0.23	1.006	-2647.721	0.05007
11	1	1.379	0.011	0.19	1.006	-2647.724	0.04996

```
Equilibrium =

  1×3 table
```

Optimal_wr	Optimal_Taylor_parameters				Optimal_inflation
7	1	1.884	0.017	0.284	1.006

Results above is the case for symmetric quadratic penalty function on the nominal interest rate. We can also integrate the asymmetric penalty function in form of equation (6). Notice that, we have to impose the grid for w_r without point zero because (6) is unidentified at $w_r = 0$. With $options\_.ZLB\_stage\_2 = 1$ and **options_.asymetric_penalty =1**, we ran an example on this asymmetric function and get the following results:

```
Results_with_ZLB =

  5×5 table
```

wr	Taylor_parameters				SS_inflation	actual_welfare	Probability
1	0.744	4.793	0	0.931	1.008	-2647.779	0.05008
3	0.997	3.9	0	0.773	1.007	-2647.74	0.05
5	0.994	3.246	0	0.623	1.007	-2647.733	0.05014
7	0.999	2.827	0	0.531	1.007	-2647.728	0.05003
9	0.998	2.513	0	0.459	1.006	-2647.726	0.05002

```
Equilibrium =

  1×3 table
```

Optimal_wr	Optimal_Taylor_parameters				Optimal_inflation
----------	-------------------------------				-----------------
9	0.998	2.513	0	0.459	1.006

Apparently, we have to examine a larger grid for w_r to get the most correct equilibrium.

5 The Quadratic Loss Mandate

We next examine the mandate in form of a quadratic loss function. The only difference to the delegation game presented above is that at stage 3 we use a loss quadratic function as the modified welfare and a compatible simple with such delegated mandate.

Stage 3: The CB Mandate

Given a steady state inflation rate target, Π, the Central Bank (CB) receives a mandate to maximize a modified welfare criterion

$$
\begin{aligned}
\Omega_t^{mod} & \equiv \ \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \beta^\tau \left(-\left(\Pi_{n,t+\tau} - \Pi_n\right)^2 - w_y \left(Y_{n,t+\tau} - Y\right)^2 - w_r \left(R_{n,t+\tau} - R_n\right)^2 \right) \right] \\
& = \ \left(-\left(\Pi_{n,t} - \Pi_n\right)^2 - w_y \left(Y_{n,t} - Y\right)^2 - w_r \left(R_{n,t} - R_n\right)^2 \right) + \beta \mathbb{E}_t \left[\Omega_{t+1}^{mod} \right]
\end{aligned}
\tag{11}
$$

One can think of this as a mandate with a penalty function $P = w_r \left(R_{n,t} - R_n\right)^2$, penalizing the variance of the nominal interest rate with weight w_r, w_y is the weight attached on the variance of output from its trend. The following compatible simple rule is used with the delegated mandate above:

$$
\log \left(\frac{R_{n,t}}{R_n} \right) = \rho_r \log \left(\frac{R_{n,t-1}}{R_n} \right) + \alpha_\pi \log \left(\frac{\Pi_t}{\Pi} \right) + \alpha_y \log \left(\frac{Y_t}{Y} \right)
\tag{12}
$$

Results below is for the case where the probability of the nominal interest rate hitting the ZLB is allowed at less than or equal to 0.05, and the values of w_y is equal to 0.1. *options_.quadratic = 1; options_.ZLB_stage_2 = 1*

```
Equilibrium =

  1×4 table
```

Optimal_wr	Optimal_wy	Optimal_Taylor_parameters	Optimal_inflation

3	0.2	0.999	0.386	0.01	0	1.004

6 Instructions on the Use of the Toolbox

The current toolkit has been prepared for Dynare 4.5.7 **and matlab 2019b or later**. This section first describes the structure of the sub-folders that takes the estimation results from a different folder where the model is estimated. Best practice here is that we should have an identical mod file for in both the optimization and estimation stage, except for the estimation block and the monetary policy rule (details are given below). It then describes the different parameterizations of the monetary policy rule for the estimated model and for the policy optimization. It then provides a full documentation of the new internal matlab functions required in the toolbox.

6.1 Organization of Sub-Folders for Dynare and Setting Paths

There are three main folders:

EST_model: folder where the estimation was done. The toolbox takes estimated parameters and shocks to execute the optimization of our problem in a separate folders (Quadratic- or ZLB-mandate). With this routine we do not have to re-estimate the model each time we run the optimization. The estimation mod must identical to the optimization mod file (except the name of the feedback parameters which can be different, but the orders of these parameters must be exactly the same between tow mod files). The path to this folder is specified in the optimization mod file of the optimization folders (note that because we separate the folders, the name of this path must be in according to path on your machine).

Quadratic_mandate: example of the case where the delegated mandate is in form of a quadratic loss function (2 options: With and Without ZLB consideration). The mod file in this folder must contain an identical model in the mod file of the estimation folder (EST_model), except the inclusion of the modified quadratic loss function (any additional parameters in this optimization mod file must be put at the end of the parameter block compared to estimation mod file, so that the toolbox can correctly match the estimated parameters and shocks to the optimization mod file).

ZLB_mandate: General ZLB mandate case (2 options: with and without ZLB consideration), including other 2 options of the formation of the penalty function (Symmetric and Asymmetric formations). Preparation of this mod file is the same as in the Quadratic_mandate folder, except the modified welfare function.

6.2 Details of the Internal Matlab Functions

- *varstats__esti.m*. This function calculates the variables statistics given value of estimated parameters, it resolves the model instead of going through the stoch_simul when we put it in a loop. More details are given the .m file.

- *ZLB__mandate__stage3.m*. Giving the optimization problem in stage 3, and calculating the probability of the nominal interest rate hititng the ZLB. More details are given in the .m file description.

- *ZLB__mandate.m* (The quadratic loss function mandate is the file *QUAD__mandate.m* instead). This is where the stage 2 of finding the optimal level of inflation is done.

These three internal MATLAB function must be put inside the optimization folders (i.e. **Quadratic_mandate**, **ZLB_mandate**).

6.3 To execute the routine

1. First, we have to prepare two identical-model .mod files (one for estimation and one for optimization).

2. Second, the estimation is done in a separate folder (i.e. **EST__model** folder above), where we utilize the standard linear estimation routine in DYNARE, the estimation results (estimated parameters and shocks in a standard-DYNARE-structures) produced by DYNARE will then be transferred and matched with the parameters and shocks in the .mod file of the optimization exercise by the toolkit once you have correctly specified the path to this estimation folder (path name as shown up on your machine) in the optimization .mod file.

3. Once the Estimation exercise has been done by DYNARE, we put the second identical-model .mod file into the optimization folders (i.e. **Quadratic_mandate**, **ZLB_mandate** mentioned above), where the correct options must be specified in the optimization .mod file (details are in section (6.4) below). Note that before these options, we must first run the stochastic simulation with calibrated model as in the .mod file in order to get the structures of the model as the starting point for the toolkit, then after triggering the commands *ZLB__mandate.m* or *QUAD__mandate.m* the toolkit will take the estimated values from the estimation folder.

4. Once the options have all been specified in the .mod file in the optimization folders, you just need to run that .mod file as usual in DYNARE.

6.4 Mod file options in the optimization folders

The following options need to be specified in the mod file:

- We firs specify inputs for the main function **ZLB_mandate** which two options: $(= 0)$ we fix the steady state inflation specified in the (optimization) mod file above (PIE_bar equals to parameter, or no ZLB); $(= 1)$ doing optimal simple with moving steady stage inflation and ZLB constraint; i.e. with the ZLB option we set

  ```
  options_.ZLB_stage_2  = 1;
  ```

- We next sepcify the options for the functional form of the penalty function: $(= 0)$ Symmetric penalty function, i.e. $P(R_{n,t}) = -w_r(R_{n,t} - Rn)^2$; $(= 1)$ Asymmetric penalty function $P(R_{n,t}) = \frac{\exp(-w_r(R_{n,t}-1))}{w_r}$. i.e. we have the option for asymmetric penalty function as below in the mod file

  ```
  options_.asymetric_penalty    = 1;
  ```

- Path to the estimated mod file, the programme assumes that the estimation is already done in a different folder, which loads the estimated results for the exercise.

  ```
  modpath_estim  = {'...\EST_MODEL'};
  ```

  And the name of the estimated mod file (without extension) in the estimated folder.

  ```
  modname_estim  = {'NK_SW'};
  ```

- We then specify the endgenous variables needed for our optimization exercise:

  (1) The actual welfare function of the social planner in the stage 1, named V as in the mod file in our example. In particular, in the mod file the actual welfare is defined as follows:$V = U + \beta(1 + g)^{(1-\sigma_c)}V(+1)$;

  ```
  options_.ZLB_act_obj_func = {'V'};
  ```

  (2) The name of the modified welfare function delegated to the central bank in the stage 3, additional component of this objective function, w_r, is also defined below. In the mod file this variable is specified as follows: $V^{MOD} = U + P(R_{n,t}) + \beta(1 + g)^{(1-\sigma_c)}V^{MOD}(+1)$;

  ```
  options_.ZLB_mod_obj_func  = {'V_MOD'}; %
  ```

(3) The weight attached on the variation of the nominal interest rate of the delegated mandate. Notice that, the toolbox only examines through a grid of this parameters to find an optimal value, it does not solve for an exact optimal weight.

```
options_.ZLB_wr      = {'wr'}; %
```

And the grid for the weight attached on the nominal interest rate variations.

```
options_.ZLB_mand_wr_value     = [5 15 25];
```

Notice that the wr cannot be zero in this case when we also integrate the asymmetric formation of the modified welfare function which is unidentified when wr = 0. Importantly, alternative forms of the delegated mandate will also be specified with the exact approach.

(4) The name of the long-run target inflation level or the steady state inflation level which is specified in the mod file:

```
options_.ZLB_ss_pi   = {'PIE_bar'};
```

(5) Simple rule's feedback parameters, the optimized value of these parameters is found from the optimization in stage 3.

```
options_.ZLB_params  = {'rho_r'; 'theta_pie'; 'theta_y' ; 'theta_dy'};
```

(6) The lower bound on the feedback parameters for the optimization. notice that for some model the Taylor' principle must be satisfied, so the lower bound on inflation feedback parameter must be larger than 1

```
options_.lb   = [0 0 0 0];
```

And the upper bound on the feedback parameters

```
options_.ub     = [1 20 1 1];
```

(7) The nominal interest rate, name is compatible with that in the mod file.

```
options_.instrument_rate   = {'Rn'};
```

(8) The theoretical means and standard deviations of the specified variables here will be plotted

```
options_.vars_stats  = {'Rn';'PIE';'V'};
```

(9) the allowed probability of the nominal interest rate hitting the

```
options_.pbar  = [0.05];
```

- **Changes for the mod file in case of a quadratic loss function mandate:**

  (1) name of the weight attached on the output variances (w_y)

```
options_.ZLB_wy = {'wy'}; %
```

And the grid for the output variance weight:

```
options_.ZLB_mand_wy_value      = [0.1 0.2 0.3];  %
```

(2) the initial interval guest for ss inflation, if the optimal steady state is above 4% anually the programme returns the value for the upper bound of 4%. In case of without the ZLB (stage 2) we impose the steady state inflation value as in the mod file in the parameters block.

```
options_.PIEss0                 = [1 1.01]; %
```

(3) The modified welfare function is now specified as above.

```
V_MOD    = (-(PIE - STEADY_STATE(PIE))^2
   - wy*(Y - STEADY_STATE(Y))^2 - wr*(Rn - STEADY_STATE(Rn))^2)
             + betta*(1+g)^(1-sigma_c)*V_MOD(+1);
```

6.5   Results

The main results of the toolbox will be save in the sub-struct **oo__.ZLB_mandate_results**. In this sub-struct we first have risky steady state and standard variation of the list of endogenous variables listed in **options__.vars_stats** above. And the summary of the results for the full grid of the weight w_r and the equilibrium in the variables **oo__.ZLB_mandate_results.summary_ZLB** and **oo__.ZLB_mandate_results.equilibrium**, respectively.

The completed programme can be download directly from the following link.

7 Concluding Remarks

Our paper relates to an optimal mandate's literature which is reviewed in Deak et al. (2020). A close paper to ours is Debortoli et al. (2019) (henceforth DKLN) who develop a mandate framework in an alternative linear quadratic framework but still leaving the central bank as instrument-independent and goal-dependent. Our paper differs from this paper in a number of ways. First, we formalize the ZLB constraint on nominal interest rate as described below. This small probability is interpreted as the tightness level of the ZLB. This approach enables us to employ a second-order perturbation solution combined with a penalty function to calculate the exact social welfare value used to design optimal mandates thus avoiding a quadratic approximation in DKLN Second, we require the central bank to conduct its monetary policy in the form of an optimized interest rate rule with targets corresponding to those in the welfare goal mandate and with an imposed optimal shift in the steady state inflation rate. Finally, we formalize our framework as a delegation game.

The LQ approach of DKLN closely follows Benigno and Woodford (2004) and Levine, Pearlman and Pierse (2008). The LQ approach has the advantage that it facilitates, even for large models, the computation of a discretionary equilibrium at Stage 3 of the delegation game. Details of this approach with an analytical example are provided in Appendix A.

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APPENDIX

A   The Linear Quadratic Approximation Approach

From Levine, Pearlman and Pierse (2008) consider the following general deterministic optimization problem

$$\max \sum_{t=0}^{\infty} \beta^t U(X_t, W_t) \ \ s.t. \ \ X_t = f(X_{t-1}, W_t) \tag{A.1}$$

where X_t is vector of state variables and W_t a vector of instruments.[1] There are given initial and the usual tranversality conditions. For our purposes, we consider this as including models with forward-looking expectations, so that the optimal solution to the latter setup is the pre-commitment solution. Suppose the solution converges to a steady state X, W as $t \to \infty$ for the states X_t and the policies W_t. Define $x_t = X_t - X$ and $w_t = W_t - W$ as representing the first-order approximation to absolute deviations of states and policies from their steady states.[2]

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[1]An alternative representation of the problem is $U(X_t, W_t)$ and $\mathbb{E}_t[X_{t+1}] = f(X_t, W_t)$ where X_t includes forward-looking non-predetermined variables and $\mathbb{E}_t[X_{t+1}] = X_{t+1}$ for the deterministic problem where perfect foresight applies. Whichever one uses, it is easy to switch from one to the other by a simple re-definition. As we demonstrate in Levine, McAdam and Pearlman (2008), although the inclusion of forward-looking variables significantly alters the nature of the optimization problem, these changes only affect the boundary conditions and the second-order conditions, but not the steady state of the optimum which is all we require for LQ approximation.

[2]Alternatively $x_t = (X_t - X)/X$ and $w_t = (W_t - W)/W$, depending on the nature of the economic variable. Then the Theorem follows in a similar way with an appropriate adjustment to the Jacobian Matrix.

The Lagrangian for the general problem is defined as,

$$\sum_{t=0}^{\infty} \beta^t [U(X_t, W_t) - \lambda_t'(X_t - f(X_{t-1}, W_t))] \tag{A.2}$$

where λ_t is the Lagrange multiplier so that a necessary condition for the solution to (A.1) is that the Lagrangian is stationary at all $\{X_s\}$, $\{W_s\}$ i.e.

$$U_W + \lambda_t' f_W = 0 \qquad U_X - \lambda_t' + \beta \lambda_t' f_X = 0 \tag{A.3}$$

Assume a steady state λ for the Lagrange multipliers exists as well. Now define the Hamiltonian $H_t = U(X_t, W_t) - \lambda_t' f(X_{t-1}, W_t)$. The following is the discrete time version of Magill (1977):

Theorem: If a steady state solution (X, W, M) to the optimization problem (A.1) exists, then any perturbation (x_t, w_t) about this steady state can be expressed as the solution to

$$\max \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} x_t' & w_t' \end{bmatrix} \begin{bmatrix} H_{XX} & H_{XW} \\ H_{WX} & H_{WW} \end{bmatrix} \begin{bmatrix} x_t \\ w_t \end{bmatrix} \quad \text{s.t. } x_t = f_X x_t + f_W w_t \tag{A.4}$$

where H_{XX}, etc denote second-order derivatives evaluated at (X, W). This can be directly extended to the case incorporating disturbances.

Thus our general procedure is as follows:

1. Set out the deterministic non-linear problem for the Ramsey Problem, to maximize the representative agents' utility subject to non-linear dynamic constraints.

2. Write down the Lagrangian for the problem.

3. Calculate the first order conditions. We do not require the initial conditions for an optimum since we ultimately only need the steady-state of the Ramsey problem.

4. Calculate the steady state of the first-order conditions. The terminal condition implied by this procedure is such that the system converges to this steady state.

5. Calculate a second-order Taylor series approximation, about the steady state, of the Hamiltonian associated with the Lagrangian in 2 (note that this involves only the steady state M of the multipliers).

6. Calculate a first-order Taylor series approximation, about the steady state, of the first-order conditions and the original constraints.

7. Use 4. to eliminate the steady-state Lagrangian multipliers in 5. By appropriate elimination both the Hamiltonian and the constraints can be expressed in minimal form. This then

gives us the accurate LQ approximation of the original non-linear optimization problem in the form of a minimal linear state-space representation of the constraints and a quadratic form of the utility expressed in terms of the states.

The Lagrangian for the NK model is

$$\sum_{t=0}^{\infty} \beta^t \Big[U(C_t, H_t) \quad + \quad \lambda_{1,t} \left(Y_t - \frac{(A_t H_t)^\alpha}{\Delta_t} \right) + \lambda_{2,t} \left(\frac{U_{H,t}}{U_{C,t}} + W_t \right)$$

$$+ \quad \lambda_{3,t} \left(W_t - \alpha MC_t A_t^\alpha H_t^{\alpha-1} \right) + \lambda_{4,t} \left(\Lambda_{t,t+1} - \beta \frac{U_{C,t+1}}{U_{C,t}} \right)$$

$$+ \quad \lambda_{5,t} \left(JJ_t - \xi \mathbb{E}_t[\Lambda_{t,t+1} \Pi_{t+1}^{\zeta-1} JJ_{t+1}] - Y_t \right)$$

$$+ \quad \lambda_{6,t} \left(J_t - \xi \mathbb{E}_t[\Lambda_{t,t+1} \Pi_{t+1}^{\zeta} J_{t+1}] - \left(\frac{1}{1 - \frac{1}{\zeta}} \right) Y_t MC_t MS_t \right)$$

$$+ \quad \lambda_{7,t} \left(1 - \xi \Pi_t^{\zeta-1} + (1-\xi) \left(\frac{J_t}{JJ_t} \right)^{1-\zeta} \right)$$

$$+ \quad \lambda_{8,t} \left(MC_t - \frac{W_t}{\alpha A_t^\alpha H_t^{\alpha-1}} \right)$$

$$+ \quad \lambda_{9,t} \left(\Delta_t - \xi \Pi_t^{\zeta} \Delta_{t-1} - (1-\xi) \left(\frac{J_t}{JJ_t} \right)^{-\zeta} \right) + \lambda_{10,t}(Y_t - C_t - C_t) \Big]$$

From Levine, Pearlman and Pierse (2008) with no habit ($h_C = 0$) and $\Phi \equiv 1 - \frac{1}{\zeta}$ the inefficiency wedge in the model here with wage flexibility (defined as α in Levine, Pearlman and Pierse (2008)) we finally arrive at the *correct quadratic approximation to the nonlinear Ramsey problem* as the maximization of $E_0 \left[\sum_{t=0}^{\infty} \beta^t U_t \right]$ with respect to $\{\pi_t\}$, subject to

$$\pi_t = \beta \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi}((\phi + \sigma)y_t - (1+\phi)a_t) \tag{A.5}$$

where

$$U_t = -\frac{1}{2\Phi} \frac{Y^{1+\phi}}{A^{1+\phi}} \Big[\sigma y_t^2 + \phi(\Phi + \lambda_6(1+\phi))y_t^2 - 2(1+\phi)(\Phi + \lambda_6(1+\phi))y_t a_t + 2\lambda_6 \sigma y_t^2$$

$$- \quad \lambda_6 \sigma(\sigma+1)y_t^2 + \frac{\xi \zeta}{(1-\xi)(1-\beta\xi)}(\Phi + (1+\phi)\lambda_6)\pi_t^2 \Big] \tag{A.6}$$

where $\lambda_6 = \frac{1-\Phi}{\sigma+\phi}$. Then in the efficient case we have $\Phi = 1$ and $\lambda_6 = 0$ in which case the term in big brackets in (A.6) reduces to

$$-\frac{1}{2} \left((\sigma + \phi) \left(y_t - \frac{1+\phi}{\sigma+\phi} a_t \right)^2 + \frac{\xi \zeta}{(1-\beta\xi)(1-\xi)} \pi_t^2 \right) \tag{A.7}$$

plus terms in $\hat{a}_t$ which are independent of policy. Since $\frac{1+\phi}{\sigma+\phi}a$ is the flexi-price level of output this is a quadratic loss function in terms of inflation and the output gap. In fact it is the welfare-based loss function of the canonical NK model as derived by Woodford (2003).