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The Use and Mis-Use of SVARs for Validating DSGE Models

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The Use and Mis-Use of SVARs for Validating DSGE Models*

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Abstract

This paper studies the potential ability of an SVAR to match impulse response functions of a well-established estimated DSGE model. We study the invertibility (fundamentalness) problem setting out conditions for the RE solution of a linearized Gaussian NK-DSGE model to be invertible taking into account the information sets of agents. We then estimate an SVAR by generating artificial data from the theoretical model. A measure of approximate invertibility, where information can be imperfect, is constructed. Based on the VAR(1) representation of the DSGE model, we compare three forms of SVAR-identification restrictions; zero, sign and bounds on the forecast error variance, for mapping the reduced form residuals of the empirical model to the structural shocks of interest. Separating out two reasons why SVARs may not recover the impulse responses to structural shocks of the DGP, namely non-invertibility and inappropriate identification restrictions, is then the main objective of the paper.

JEL Classification: C11; C18; C32; E32
Keywords: VAR-DSGE impulse response comparisons; NK model; Identification in SVARs; Imperfect information

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1 Introduction

Following a precedent set by Christiano *et al.* (2005), researchers often try to match the impulse responses of an identified SVAR with a DSGE model. But can indeed SVAR methods be employed to recover the structural shocks and impulse response functions if the data generating process (henceforth DGP) is a DSGE model? In principle, this may be possible since the rational expectations (henceforth RE) solution of a linearized DSGE model is a VARMA which may be approximated by a finite SVAR in which the reduced form errors are a linear function of the structural shocks. A necessary and sufficient condition for such a representation is that the VARMA is invertible (or, almost equivalently,[1] satisfies fundamentalness).

The invertibility-fundamentalness problem is often described in the macro-econometrics literature as one of "missing information" when the econometrician does not have all the information had by agents in the DGP. We refer to the econometrician's problem as "E-invertibility". But in our paper, missing information of this form is not at the heart of the problem, but rather imperfect information on the part of both agents and the econometrician takes centre stage; indeed the information sets can be the same for both without removing non-fundamentalness if they are imperfect. We refer to the agents' problem as "A-invertibility" and in its absence the perfect and imperfect information RE solutions of the model diverge. Agents then cannot recover the current and past structural shocks and face a signal extraction problem.

The implication for estimating SVARs is that the resulting time series cannot contain the necessary information to recover the shocks in an SVAR estimation and A-non-invertibility results in E-non-invertibility. As pointed out by Leeper *et al.* (2013) and Blanchard *et al.* (2013), the intuition is very straightforward: if the agents in the DGP are unable to back out structural shocks then, faced with either the same data or a subset of the data (for instance, in a news shocks framework), neither can the econometrician.

The non-E-invertibility of the RE solution of a DSGE model is ubiquitous. The problem for the econometrician occurs when faced with a number of observables that is less than the number of shocks; or with some observable variables of the system observed with a lag; or in models featuring anticipated shocks with a delayed effect on the system such as "news" shocks; and even with square systems when a particular choice of observables is observed with neither delayed effects, nor a lag. In the absence of invertibility, the econometrician estimates an SVAR that, subject to identification, recovers the one-period ahead prediction errors (the "innovation"), *not* the structural shocks. Consequently, impulse response functions based on estimated SVARs can be misleading.

Much of the empirical literature using SVARs is relatively silent on the invertibility (fundamentalness) problem and focuses on identification, i.e., the recovery of structural shocks given the estimated reduced form SVAR. Given an assumed DGP consisting of a DSGE model solved assuming RE, *both* non-invertibility and inappropriate identification can result in impulse response functions that deviate from the true responses to structural shocks in the model. Separating out these two components is the focus of our paper.

1.1 Main Results

The paper focuses on the potential ability of a SVAR to match IRFs of a well-established estimated DSGE model (Smets and Wouters, 2007). Failure to do so originates from non-

[1] Invertibility is a more general condition that implies fundamentalness, but in practice they are usually equivalent.

invertibility and a poor choice of identification restrictions. We estimate an SVAR by generating artificial data from the theoretical model. Based on the VAR(1) representation of the NK-DSGE model, we compare three forms of SVAR-identification restrictions; zero, sign and theory-driven bounds on the forecast error variance with bounds (BoundsFEV), for mapping the reduced form residuals of the empirical model to the structural shocks of interest. For the estimated non-invertible (in both E- and A- senses) DSGE models, we assume imperfect information (henceforth II) on the part of both agents and the econometrician. We utilize the II measures of approximate fundamentalness and assess the ability of these measures to predict the non-invertibility of the estimated model.

The results of the paper have strong implications for the researcher using an SVAR to compare impulse responses with those generated by a structural model. First, we can actually report some good news for the estimated Smets and Wouters (2007) model. For the original square case where the number of observations (data sets) equals the number of structural shocks, there is no invertibility problem. In this case, the divergence between the estimated model and SVAR are entirely due to a combination of the finite VAR assumption and the choice of identification strategy. Regarding the latter, we find that of the identification schemes (Cholesky vs Sign vs BoundsFEV) it is very clear that BoundsFEV of Volpicella (2021) delivers the best estimation precision, removing the implausible responses and outperforming the Cholesky- and sign-VARs in replicating the IRFs of the assumed DGP (in terms of the median responses).

Our second finding reports more good news even for the non-square non-invertible case where the number of observations is far greater than the number of structural shocks which include a shock to the inflation target and measurement errors. Namely, the monetary policy and government spending shocks are approximately fundamental as indicated by the IRFs and our approximate fundamentalness measures for the two shocks. This is encouraging as many researchers only focus on these two shocks in the empirical literature. These results are very robust to the alternative identification strategies, but again BoundsFEV delivers the best fit.

However, our third finding is that non-invertibility-fundamentalness does matter in general and a comparison of our approximate fundamentalness measure with the actual impulse function responses of the DGP demonstrates its usefulness. For the non-square case, specific results are that four shocks - investment, preference, price mark-up and inflation target - are not approximately fundamental according to our measure and this is confirmed by the poor matching of the IRFs of the SVARs with those of the DGP even with our preferred identification scheme.

These, to the best of our knowledge, are novel results for both the SVAR and DSGE literature; while there is a substantial body of literature devoted to understanding theory-driven identification strategies, no previous studies have linked this to the informational assumptions in the DGP in the context of constructing data-SVARs that are compatible with the DSGE theory.

1.2 Literature Background and Contributions

Two seminal papers on the invertibility-fundamentalness problem are Lippi and Reichlin (1994) that introduces Blaschke matrices and Fernandez-Villaverde *et al.* (2007) that examines the conditions for a solution of a RE model to have a VAR representation. A popular example of the missing information problem comes from "news shocks" observed by agents but not by the econometrician - see, for example, Leeper *et al.* (2013). "Noisy" news papers by Blanchard *et al.* (2013) and Forni *et al.* (2017) study models closely related to our imperfect information general framework.

Despite a growing literature on the important impact of II on DSGE models, many (indeed most) models of the macro-economy are still solved and/or estimated on the assumption that agents are simply provided with perfect information (henceforth PI), effectively as an *endowment* rather than the consequence of A-invertibility. Levine *et al.* (2022), on which our paper draws, studies how the general nature of the agents' signal extraction problem under II impacts on the econometrician's problem of attempting to infer the nature of structural shocks and associated impulse responses from the data. Levine *et al.* (2020) describes a toolkit that implements the procedures in that paper and provides, as one of a number of examples, the application in this paper.

Based on the earlier studies of informational frictions, II in representative agent (RA) models was initiated by Minford and Peel (1983) and generalized by Pearlman (1986) and Pearlman *et al.* (1986) - henceforth PCL - with major contributions by Woodford (2003) and Collard and Dellas (2010). These papers show that II can act as an endogenous persistence mechanism in the business cycle. More recently, applications with estimation were made by Collard *et al.* (2009), Neri and Ropele (2012) and Levine *et al.* (2012).

A more recent literature studies imperfect information in a heterogeneous agent framework: e.g.,Pearlman and Sargent (2005), Nimark (2008), Angeletos and La'O (2009), Graham and Wright (2010), Rondina and Walker (2021), Huo and Pedroni (2020), Angeletos and Huo (2020), Angeletos and Huo (2021) and Levine *et al.* (2022). Angeletos and Lian (2016) provide a recent comprehensive survey of what they refer to as the incomplete information literature. Section 2.5 provides a link between the RA framework of our paper and this literature.[2]

If the RE solution of a DSGE model is not E- and A-invertible, all is not lost in the ability of impulse response functions from an SVAR to replicate those in the assumed DGP, the DSGE model. The solution may be *approximately* fundamental, at least for some shocks, in the sense described by Forni and Gambetti (2014), Forni *et al.* (2016), Beaudry *et al.* (2016), Canova and Sahneh (2018), and Levine *et al.* (2022).

Turning to the identification issue, early SVAR studies employed short-run or long-run restrictions on impulse response functions (henceforth IRFs) for the identification of structural shocks. However, recent research has relaxed controversial restrictions and has identified structural shocks with sign restrictions on either the IRFs or the structural parameters. Standard text-books such as Kilian and Lutkepohl (2017) provide an overview of this literature. Volpicella (2021) provides an up-to-date review and a novel identification tool for estimation and inference in SVARs that are set-identified through bound restrictions on the forecast error variance decomposition (FEVD). These restrictions complement the standard sign restrictions approach which are also employed in our paper.

In the light of this review, our paper makes the following three main contributions to the literature.

First contribution: Our paper emphasizes the crucial importance of the information problems of agents and the econometrician when validating a DSGE model by comparing its impulse response functions with those of a SVAR. We distinguish invertibility from the viewpoint of the econometrician and agents, E- and A-invertibility, respectively. An application of the "Poor Man's Invertibility Condition" of Fernandez-Villaverde *et al.* (2007) then states that E-

[2]Here a comment on terminology is called for. Our use of perfect/imperfect Information (PI/II) is widely used in the second strand of literature when describing agents' information of the history of play driven by draws by Nature from the distributions of exogenous shocks. The complete/incomplete framework of the Angeletos-Lian survey (and other work by these authors) incorporates PI/II, but also refers to agent's beliefs regarding each other's payoffs. In our framework this informational friction (leading to "Global Games") is as yet absent.

invertibility only holds if additional conditions for A-invertibility hold, in which case the agents' information problem under II replicates that under PI. We show both generally and in an illustrative example the presence of Blaschke factors in the IRFs in the case where A-invertibility does not hold.

Second contribution: When A- and therefore E-invertibility fails, we construct measures of approximate fundamentalness which generalize results in the literature that implicitly assume PI on the part of agents in the assumed DGP.

Third contribution: In our application to a well-established estimated DSGE model, we use this measure and the identification scheme of Volpicella (2021) to separate out two reasons why SVARs may not recover the impulse responses to structural shocks, namely, non-invertibility and inappropriate identification restrictions.

1.3 Road-Map

The rest of the paper is organized as follows. Section 2 reviews the underlying theory that links the invertibility-fundamentalness issue with the information set assumptions in the model. In Subsection 2.6, an illustrative example motivates the results. Section 3 sets out the log-linearized Smets-Wouters model in Smets and Wouters (2007) (Subsection 3.1) which is then estimated in Subsection 3.2. We consider two forms of the model: a square system as in the original paper which is E- and A-invertible and for which the PI and II RE solutions coincide; and a modified non-square system which is no longer invertible in both E- and A- senses. Subsection 3.3 then provides measures of invertibility-fundamentalness for each shock and confirms the effectiveness of our measure. Section 4 compares the IRFs of the estimated model with those from the SVAR estimated from artificial data comparing the identification schemes described above. To further examine the performance of IRF comparisons, Section 5 computes a metric to measure the cumulative distance for the IRF divergence over the response horizon. Up to this point the paper chooses an SVAR(p) with lag $p = 1$; a final robustness check confirms our main results for lags up to $p = 5$. Section 6 provides concluding remarks.

2 Invertibility and the Information Sets of Agents

The potential ability of an SVAR to match impulse response functions of a DSGE model depends crucially on the information sets of its agents. We begin by making these explicit in a linearized RA RE model of the following general form

$$A_0 Y_{t+1,t} + A_1 Y_t = A_2 Y_{t-1} + \Psi \epsilon_t \qquad m_t^E = L^E Y_t \qquad m_t^A = L^A Y_t \tag{1}$$

where matrix A_0 may be singular, Y_t is an $n \times 1$ vector of macroeconomic variables; and ϵ_t is a $k \times 1$ vector of Gaussian white noise structural shocks. We assume that the structural shocks are normalized such that their covariance matrix is given by the identity matrix i.e., $\epsilon_t \sim N(0, I)$. The Gaussian assumption is required later to apply the Kalman Filter.

We define $Y_{t,s} \equiv \mathbb{E}\left[Y_t | I_s^A\right]$ where I_t^A is information available at time t to the representative agent, given by $I_t^A = \{m_s^A : s \leq t\}$. We assume that this contains the history of a strict subset of the elements of Y_t, hence information is in general imperfect. In the special case that agents are endowed (somehow) with perfect information, $L^A = I$ (the identity matrix). Note that measurement errors can be accounted for by including them in the vector ϵ_t.

2.1 Conversion to Blanchard-Kahn Form

We first introduce a key result proved in Levine *et al.* (2022) that converts from the very general class of linear RE models (1) into results that are based on the generalized Blanchard-Kahn (BK) form of Pearlman *et al.* (1986).

Theorem 1. *For any information set, (1) can always be converted into the following form, as used by PCL*

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \epsilon_{t+1} \tag{2}$$

$$m_t^A = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \tag{3}$$

where z_t, x_t are vectors of backward and forward-looking variables, respectively. The covariance matrix of shocks is the matrix BB'. Note that, at this stage, we focus solely on the agents' informational problem: m_t^A. We specify the properties of $m \times 1$ vector m_t^E where $m \leq k$, the vector of observables available to the econometrician later.

The expressions involving $z_{t,t}$ and $x_{t,t}$ arise from rewriting the model in PCL form (2). This transformation involves a novel iterative stage which replaces any forward-looking expectations with the appropriate model-consistent updating equations. This reduces the number of equations with forward-looking expectations, while increasing the number of backward-looking equations one-for-one. But, at the same time, it introduces a dependence of the additional backward-looking equations on both state estimates $z_{t,t}$ $\left(\equiv \mathbb{E}[z_t|I_t^A]\right)$ and estimates of forward-looking variables, $x_{t,t}$.

For later convenience, we define matrices G and H conformably with z_t and x_t and define two more structural matrices F and J

$$F \equiv G_{11} - G_{12}G_{22}^{-1}G_{21} \qquad J \equiv M_1 - M_2G_{22}^{-1}G_{21} \tag{4}$$

F and J capture intrinsic dynamics in the system, that are invariant to expectations formation (i.e., by substituting from the second block of equations in (2) we can write $z_t = Fz_{t-1} + \begin{bmatrix} B' & 0 \end{bmatrix}' \epsilon_{t+1}$ plus additional terms involving expectations formed at time t; and $m_t^A = Jz_t +$ additional terms likewise). PCL show that the filtering problem is unaffected by these additional terms.

2.2 From the Perspective of the Agents

First, we consider the solution under ***perfect information***. Here we assume that the representative agent directly observes all elements of Y_t, hence of $\{z_t, x_t\}$, as an endowment. Hence $z_{t,t} = z_t,\ x_{t,t} = x_t$, and using the standard BK solution method there is a saddle path satisfying

$$x_t + Nz_t = 0 \quad \text{where} \quad \begin{bmatrix} N & I \end{bmatrix} (G + H) = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{5}$$

where Λ^U is a matrix with unstable eigenvalues. If the number of unstable eigenvalues of $(G+H)$ is the same as the dimension of x_t, then the system will be determinate.

To find N, consider the matrix of eigenvectors W satisfying

$$W(G+H) = \Lambda^U W \tag{6}$$

Then, as for G and H, partitioning W conformably with z_t and x_t, from PCL, we have

$$N = W_{22}^{-1} W_{21} \tag{7}$$

From the saddle-path relationship (7), the saddle-path stable RE solution under PI is

$$z_t = Az_{t-1} + B\epsilon_t \qquad x_t = -Nz_t \tag{8}$$

where

$$A \equiv G_{11} + H_{11} - (G_{12} + H_{12})N \tag{9}$$

is a non-structural matrix dependent on the saddle-path solution through the matrix N.

Turning to ***imperfect information***, for the general case of imperfect information, the transformation of (1) into the form (2) and (3) in Theorem 1 allows us to apply the solution techniques originally derived in PCL. We briefly outline this solution method. Following Pearlman *et al.* (1986), we apply the Kalman filter updating given by

$$\begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} = \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} + K \left[m_t^A - \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} - \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \right] \tag{10}$$

The Kalman filter was developed in the context of backward-looking models, but extends here to forward-looking models. The representative agent's best estimate of $\{z_t, x_t\}$ based on current information is a weighted average of their best estimate using last period's information and the new information m_t^A. Thus the best estimator of $\{z_t, x_t\}$ at time $t-1$ is updated by the "Kalman gain" K of the error in the predicted value of the measurement. K is solved endogenously as

$$K = \begin{bmatrix} P^A J' \\ -NP^A J' \end{bmatrix} [(M_1 - M_2 N)P^A J']^{-1} \tag{11}$$

where P^A is defined below, but is not directly incorporated into the solution for $\{z_t, x_t\}$.

Using the Kalman filter, the solution under II as derived by Pearlman *et al.* (1986) is given by the pre-determined and non-predetermined variables z_t and x_t, described by processes for the predictions $z_{t,t-1}$ and for the prediction errors $\tilde{z}_t \equiv z_t - z_{t,t-1}$

$$\textbf{Predictions}: \quad z_{t+1,t} = A\left(z_{t,t-1} + \mathcal{K}J\tilde{z}_t\right) \tag{12}$$

$$\textbf{Prediction Errors}: \quad \tilde{z}_t = Q^A \tilde{z}_{t-1} + B\epsilon_t \tag{13}$$

$$\textbf{Non-predetermined}: \quad x_t = -N\left(z_{t,t-1} + \mathcal{K}J\tilde{z}_t\right) - G_{22}^{-1}G_{21}\left(I - \mathcal{K}J\right)\tilde{z}_t \tag{14}$$

$$\textbf{Measurement Equation}: \quad m_t^A = E\left(z_{t,t-1} + \mathcal{K}\tilde{z}_t\right) \tag{15}$$

where

$$\mathcal{K} = P^A J \left(JP^A J'\right)^{-1}; \qquad Q^A = F\left[I - \mathcal{K}J\right] \tag{16}$$

F and J are as defined above in (4), $\mathcal{K}$ is an alternative Kalman gain matrix after stripping out the predictable variation in the state variables z_{t+1} arising from dependence on x_t. The matrix A, defined in (9) is the autoregressive matrix of the states z_t in the solution under PI; and we have introduced another non-structural matrix E defined by

$$E \equiv M_1 + M_3 - (M_2 + M_4)N \tag{17}$$

which captures the impact of predictions and prediction errors for z_t on observable variables. B captures the direct (but unobservable) impact of the structural shocks ϵ_t and $P^A = \mathbb{E}[\tilde{z}_t\tilde{z}_t']$ is the solution of a Riccati equation given by

$$P^A = Q^A P^A Q^{A'} + BB' \tag{18}$$

To ensure stability of the solution P^A, we also need to satisfy the convergence condition, that Q^A has all eigenvalues in the unit circle. Since the matrix Q^A is also the autoregressive matrix of the prediction errors $\tilde{z}_t$ in (13), this is equivalent to requiring that prediction errors are stable. Since there is a unique solution of the Riccati equation under mild conditions that satisfies this condition, it follows that the solution (12)–(15) is also unique thereby extending this property of the PI BK solution to the II case.

We can thus see that the solution procedure above is a generalization of the Blanchard-Kahn solution for PI and that the determinacy of the system is independent of the information set.

We finally note that the II solution can be transformed into the PI solution when the agent's information set is $\{z_t,\ x_t\}$. Choose just a subset of the information, $m_t = Jz_t$, such that JB is invertible. We then deduce from (18) that $P^A = BB'$ and hence $\tilde{z}_t = B\epsilon_t$. Substituting into (12) yields $z_{t+1,t} = Az_{t,t-1} + AB\epsilon_t = A(z_{t,t-1} + \tilde{z}_t) = Az_t$. Adding this to $\tilde{z}_{t+1} = B\epsilon_{t+1}$ yields $z_{t+1} = Az_t + B\epsilon_{t+1}$, the PI solution.[3]

2.3 From the Perspective of the Econometrician

This section shows how the econometrician's problem relates to the solution of the agents' problem presented in Subsection 2.2.

Informational Assumptions Throughout the paper, we assume under II that the agents have the same information set for the aggregate economy as the aggregate information set available to the econometricians; thus $m_t^A = m_t^E$.

A-invertibility: When II Replicates PI It is evident that II introduces non-trivial additional dynamics into the responses to structural shocks - a contrast which is crucial to much of our later analysis. However, there is a special case of the general problem under II, which asymptotically replicates perfect information, and hence where $P^A = BB'$.

Definition 1. ***A-invertibility****: The RE solution is A-invertible if agents can infer the true values of the structural shocks ϵ_t (and hence ϵ_{it}) from the history of their observables, or equivalently $P^A = BB'$ is a stable fixed point of the agents' Riccati equation, (18).*

E-invertibility: The ABCD (and E) of VARs Corresponding to A-invertibility we now define the corresponding concept from the viewpoint of the econometrician:

[3]Under PI, we have that $M_1 = I$ and $M_2 = 0$ so $x_t = -Nz_t$ is also observed. Then $J = I$, but then the this information set is in general of higher dimension than the shocks, so we pick a linear combination $\bar{J}$ of the information set such that $\bar{J}B$ is invertible and $\bar{Q}^A = F(I - B(\bar{J}B)^{-1}\bar{J})$ has stable eigenvalues (which is possible if (F, B) is controllable). From (18) it follows that $P^A = BB'$, the covariance matrix of the structural shocks, and $\bar{Q}^A$ is as above. Hence $\bar{Q}^A B = 0$ and therefore $\tilde{z}_t = B\epsilon_t$. Finally, adding $\tilde{z}_{t+1}$ to both sides of (12) yields the result for PI.

Definition 2. ***E-invertibility**: The RE solution is E-invertible if the values of the shocks ϵ_t can be deduced from the history of the econometrician's observables, $\{m_s^E : s \leq t\}$.*

To see how the two concepts of A- and E-invertibility relate to each other, consider an econometrician's state-space representations of the aggregate economy of the type that arise naturally from our solution method in Section 2.2, of the general form:

$$s_t = \tilde{A}s_{t-1} + \tilde{B}\epsilon_t \qquad m_t^E = \tilde{E}s_t \equiv \tilde{C}s_{t-1} + \tilde{D}\epsilon_t \tag{19}$$

where $\tilde{C} \equiv \tilde{A E}\tilde{B}$ and $\tilde{D} \equiv \tilde{E}\tilde{B}$ and where the tildes over each of the matrices distinguish this state-space representation from the particular form (without tildes) under perfect information. It is straightforward to show that both the PI and II representations of the previous two sections are in the ABE form of (19).[4]

For the PI case, given the informational assumptions set out above, we have, straightforwardly, $s_t = z_t, \tilde{A} = A, \tilde{B} = B, \tilde{E} = E$. For the II case, we have

$$s_t = \begin{bmatrix} z_{t,t-1} \\ \tilde{z}_t \end{bmatrix} \tag{20}$$

$$\tilde{A} \equiv \begin{bmatrix} A & A\mathcal{K}J \\ 0 & Q^A \end{bmatrix} \tag{21}$$

$$\tilde{B} \equiv \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{22}$$

$$\tilde{E} \equiv \begin{bmatrix} E & E\mathcal{K}J \end{bmatrix} \tag{23}$$

where A, $\mathcal{K}$, J, Q^A and E are as defined after (12) to (15).

Theorem 2. ***Poor Man's Invertibility Conditions (PMIC)**: The conditions for the RE solution to be E-invertible which we exploit below in Theorem 3 is then an application of the "Poor Man's Invertibility Condition" of Fernandez-Villaverde* et al. *(2007),*[5] *The necessary and sufficient conditions are:*

Condition 1. *A 'square system' with $m = k$ (an assumption we relax when we consider the innovations representation and when we come to Section 2.7 on measures of approximate invertibility/fundamentalness).*

Condition 2. *$\tilde{D}$ (now a square matrix) is non-singular.*

Condition 3. *$\tilde{E}\tilde{B}$ is invertible and that $\tilde{A}(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E})$ has stable eigenvalues.*

Proof. See Appendix A. □

[4]The advantages of using the ABE state-space form in what follows are (i) the Riccati equation is simpler than for any of the other formulations, (ii) the solution under II is much simpler to express and, most usefully, (iii) the representation of the model using the innovations process has the same structure as the original model (see Levine *et al.*, 2022 for further discussion). Note also that the ABCD state-space form can be written as a VARMA process as follows: From (19) $s_t = (I - \tilde{A}L)^{-1}\tilde{B}\epsilon_t$ where L is the lag operator. Substituting into the expression for m_t^E, we then have $|I - \tilde{A}L|m_t^E = \tilde{C}(I - \tilde{A}L)^*\tilde{B}\epsilon_{t-1} + \tilde{D}\epsilon_t$ where $|X|$ and X^* denote the determinant and matrix of sub-determinants of matrix X respectively. This is of VARMA form $\Lambda(L)m_t^E = \Phi(L)\epsilon_t$.

[5]This result appears to date back at least to the work of Brockett and Mesarovic (1965). A slightly weaker condition than invertibility is fundamentalness which allows some eigenvalues to be on the unit circle. However we use the two terms interchangeably and in fact, if we restrict our models to have only stationary variables, then the two concepts are equivalent.

E-invertibility: When Agents Have PI The conditions for E-invertibility under PI are straightforward, and are identical to the original PMIC, derived from the ABCD representation, in Fernandez-Villaverde *et al.* (2007) with $\tilde{A} = A, \tilde{B} = B, \tilde{E} = E, s_t = z_t$. Hence we immediately have: if agents have PI, the conditions for E-invertibility (as in Definition 2) are: the square matrix EB is of full rank and $A(I - B(EB)^{-1}E)$ is a stable matrix.

E-invertibility: When Agents Have II We now consider the more general case of E-invertibility under II. The result is straightforward, but powerful:

Theorem 3. *Assume that the number of observables equals the number of shocks ($m = k$). Assume further that the PMIC conditions under PI hold (so the system would be E-invertible under PI), but agents do not have PI. Then E-invertibility under II holds if and only if A-invertibility holds, and this requires that the square matrix JB is of full rank, and $Q_A = F(I - B(JB)^{-1}J)$ is a stable matrix.*

Proof. See Appendix B. □

While there is a clear mathematical parallel between the condition for invertibility under PI, the crucial difference is that, for the II case, the conditions for E-invertibility under PI are *necessary, but not sufficient conditions* for E-invertibility.

2.4 Why VARs Fail in the Presence of Blaschke Factors

Given a fundamental Moving Average (MA) representation $D(L)$ (all roots greater than or equal to unity), one can always find a Blaschke matrix to construct a non-fundamental one with some roots less than unity (called 'root-flipping'). Root-flipping can go in the opposite direction: we can transform a non-fundamental into a fundamental representation. Lippi and Reichlin (1994) and Canova (2007), p114, show how to construct such matrices.

The text-book description (see, for example, Kilian and Lutkepohl, 2017) goes as follows. Consider a general fundamental (invertible) MA representation written as

$$y_t = D(L)\epsilon_t = \underbrace{D(L)B(L)}_{\hat{D}(L)}\underbrace{(B(L))^{-1}\epsilon_t}_{e_t} \tag{24}$$

Then $y_t = \hat{D}(L)e_t$ is non-fundamental (non-invertible) if we choose $B(L)$ to be Blaschke matrices with two properties: (i) all roots inside the complex unit circle and (ii) $B(L)^{-1} = B^*(L^{-1})$ where * denotes the conjugate transpose. e_t are then the one-period ahead prediction errors. An example of a scalar Blaschke factor that satisfies (i) and (ii) is $B(L) = \frac{L-a}{1-aL}$ which will convert an invertible MA process $y_t = (1 - aL)\epsilon_t$ with $a < 1$ into a non-invertible one. Generally, these become $y_t = D(L)\epsilon_t$ and $y_t = \hat{D}(L)e_t$. These are two MA representations of the same time series with the same first and second moments; but the impulse response functions for ϵ_t and e_t are quite different.

In our illustrative example in Section 2.6, the RE solution for the PI case is of the form $y_t = D(L)\epsilon_t$ where parameter values can be chosen to make this MA process fundamental but, for the II case, the process is $y_t = D(L)B(L)\epsilon_t$ where a Blaschke factor $B(L) = \frac{L-a}{1-aL}$ appears with $a < 1$ thus turning a fundamental into a non-fundamental process. Then A-non-invertibility under II in this example shows clearly how the information assumptions in the model are a source of non-fundamentalness-invertibility.

If A-invertibility Fails The conditions for A- (and hence E-) invertibility to be satisfied are stringent. The following result reveals the role of Blaschke factors and forms the basis for our remaining analysis of cases where E-invertibility conditions are not satisfied.

Theorem 4. *Under the assumptions of Theorem 3, if A- (and hence E-) invertibility fail, then the II solution for aggregate variables can never be identical to the PI case, and will incorporate Blaschke factors in the IRFs.*

Proof. See Appendix C. □

The first element of this theorem is unsurprising: if agents cannot correctly identify the true structural shocks then their responses are bound to differ from those under PI. But the key feature that the aggregate solution that results from these response must incorporate Blaschke factors is less straightforward to prove, but crucial for what follows in the rest of the paper.

2.5 Explaining Imperfect Information in a Representative Agent Model

While there has been a substantial literature that assumes imperfect information in a representative agent model, building on the foundations developed by Pearlman *et al.* (1986), any such model is subject to the critique that it cannot explain *why* information is imperfect. Drawing on the recent heterogeneous agent imperfect information literature outlined in Section 1.2, this question is addressed in Levine *et al.* (2022). There it is shown that if, in a heterogeneous agent framework, agent i observes a composite aggregate plus idiosyncratic shock and we solve the model for the limiting case of extreme heterogeneity, as a general result, the solution for the aggregate economy turns out to have the same finite state-space form as for a parallel economy with a representative agent with II. But the aggregate dynamics of this parallel economy are affected in important ways by the underlying heterogeneity. The RA-II solution can then be rationalized as partial equilibrium symmetric heterogeneous-agent economy where idiosyncratic far outweighs aggregate uncertainty (empirically plausible) and agent i fails to take into account the fact that she is a representative agent.

2.6 An Illustrative Example

We illustrate our analysis so far with reference to the informational implications of a simple and tractable RBC model with an inelastic labour supply. In linearized form we have

$$\begin{aligned}
\text{Capital} &: \; k_{t+1} = \lambda_1 k_t + \lambda_2 a_t + (1 - \lambda_1 - \lambda_2) c_t \\
\text{Consumption} &: \; \mathbb{E}_t c_{t+1} = c_t + \frac{1}{\sigma} r_t \\
\text{Output} &: \; y_t = (1-\alpha) a_t + \alpha k_t = c_y c_t + (1 - c_y) i_t \\
\text{Investment} &: \; i_t = (k_{t+1} - (1-\delta) k_t)/\delta \\
\text{Real Interest Rate} &: \; r_t = \mathbb{E}_t r^K_{t+1} \\
\text{Gross Return on Capital} &: \; r^K_t = (1 - \beta(1-\delta)) v_t \\
\text{Rental Rate (Measurement)} &: \; v_t = (1-\alpha)(a_t - k_t) \\
\text{TFP Shock Process} &: \; a_t = \epsilon_{a,t} \sim \text{n.i.i.d}(0, \sigma_a^2)
\end{aligned}$$

where $\lambda_1 = \frac{1}{\beta}$, $\lambda_2 = \frac{(1-\alpha)}{\alpha\beta}(1 - \beta(1-\delta))$, β is the discount factor, α is the capital share of output in a Cobb-Douglas production function, δ is the depreciation rate, σ is the risk aversion parameter in the single-period utility function and, for the II case, the rental rate v_t is assumed to be observed.

The details of the solution are provided below, but the saddle path properties are identical to those under PI, with the stable root given by μ_1. Using the RE solution procedures set out in Section 2 the solutions for rental rate v_t for the RA model are then given by

$$\mathbf{PI}: \quad v_t = \frac{(1-\alpha)\left(1-\frac{(\lambda_1+\lambda_2)\mu_1 L}{\lambda_1}L\right)}{(1-\mu_1 L)}a_t \tag{25}$$

$$\mathbf{II}: \quad v_t = \frac{(1-\alpha)\left(1-\frac{\mu_1 L}{(\lambda_1+\lambda_2)\lambda_1}\right)}{(1-\mu_1 L)} \underbrace{\frac{(1-(\lambda_1+\lambda_2)L)}{\left(1-\frac{L}{(\lambda_1+\lambda_2)}\right)}}_{\text{Blaschke Factor (almost)}} a_t \tag{26}$$

The final term in (26) is not quite a Blaschke factor. To arrive at one we need to redefine the measurement as $-(\lambda_1+\lambda_2)v_t \equiv \tilde{v}_t$. This does not change the signal extraction problem of the agents so we then have

$$\begin{aligned}\mathbf{II}: \quad \tilde{v}_t &= \frac{(1-\alpha)\left(1-\frac{\mu_1 L}{(\lambda_1+\lambda_2)\lambda_1}\right)}{(1-\mu_1 L)} \underbrace{\frac{\left(L-\frac{1}{(\lambda_1+\lambda_2)}\right)}{\left(1-\frac{L}{(\lambda_1+\lambda_2)}\right)}}_{\text{Blaschke Factor}} a_t \\ &= \frac{(1-\alpha)\left(1-\frac{\mu_1 L}{(\lambda_1+\lambda_2)\lambda_1}\right)}{(1-\mu_1 L)}e_t \end{aligned} \tag{27}$$

Parameter values can be chosen to make the PI MA process, $D(L)$, fundamental.[6] Since $(\lambda_1+\lambda_2)>1$, the introduction of II then introduces a root within the unit circle and therefore non-fundamentalness. For an econometrician observing this economy, the presence of a Blaschke factor in (27) is crucial and provides the link to the seminal paper in the econometrics literature on invertibility-fundamentalness by Lippi and Reichlin (1994). In the absence of a theoretical model, the VAR estimated will recover the innovations e_t, but not the structural shock $\epsilon_{a,t}$, leading to misleading comparisons of impulse response functions.

One measure of the non-invertibility problem we later employ is the difference between impulse responses for the PI and II cases. In Figure 1, a temporary technology shock $\epsilon_{a,t}$ raises the gross return r_t^K, the single observable. Comparing PI and II trajectories the agent with II then underestimates the technology shock with $\mathbb{E}_t a_t < a_t$ and confuses this with an underestimate of the capital stock ($\mathbb{E}_t k_t < k_t$). She therefore expects the return to increase in the future and therefore overestimates the real interest rate $r_t = \mathbb{E}_t r_{t+1}^K$. Consumption falls, savings increase thus crowding in more investment and capital stock under II.

2.7 Exact and Approximate Invertibility-Fundamentalness

But suppose that the PMIC invertibility conditions fail? Then there exists an invertible *innovations representation* for the one-period ahead prediction errors

$$e_t = m_t^E - \mathbb{E}_{t-1} m_t^E \tag{28}$$

[6]See Levine *et al.* (2022).

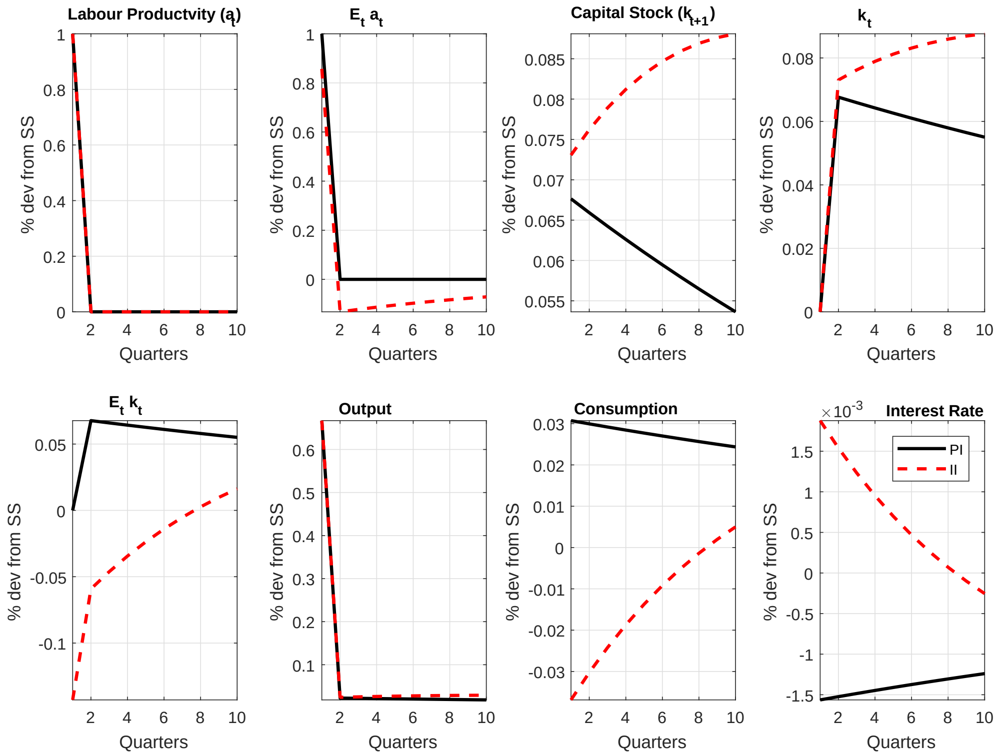


Figure 1: **Simple RBC Model. All Impulse Responses to a Temporary Technology Shock for PI and II. Parameter Values:** $r = 0.01$, $\alpha = 0.333$, $\delta = 0.025$, $\sigma = 2$, $\rho_a = 0$

where e_t is the innovation found by solving another filtering problem.[7] The resulting VAR in e_t is what the econometrician estimates so she does **not** recover the structural shocks ϵ_t. But when the system is invertible, $e_t = D\epsilon_t$, where D is a matrix of structural parameters in the ABC and D state-space representation of the model's RE solution so the two shock processes are perfectly correlated.

This leads to Forni *et al.* (2019) who suggest as a measure of non-fundamentalness regressing the structural shocks ϵ_t against the innovations e_t using the standard OLS measure of goodness of fit for shock i

$$\mathbb{F}_i = cov(\epsilon_{i,t}) - cov(\epsilon_{i,t}, e_t)cov(\epsilon_t)^{-1}cov(e_t, \epsilon_{i,t}) \tag{29}$$

$\mathbb{F}_i$ corresponds to a measure of goodness of fit of the innovations residuals to the fundamental shocks. In addition, the maximum eigenvalue of $\mathbb{F}_i$ then provides a measure of overall non-fundamentalness obtained from the models. If $m = k$, and if $\mathbb{F}_i = 0$ for all i, then since $\mathbb{F}^{PI}$ is by definition a positive definite matrix, it must be identically equal to 0. The more of the eigenvalues of $\mathbb{F}$ that are close to 0, the more one can trust that at least some of the residuals are good approximations to the fundamental shocks.[8]

Levine *et al.* (2022) provide a generalization of Forni *et al.* (2019) and develop measures of approximate fundamentalness for both perfect and imperfect information cases based on the

[7] This is the second result in Fernandez-Villaverde *et al.* (2007).
[8] This provides how well the VAR residuals correspond to the fundamentals. See Levine *et al.* (2022).

following measure of goodness of fit

$$\mathbb{F}^{PI} = V - B'E'(EP^E E')^{-1}EB \quad (30)$$
$$\mathbb{F}^{II} = V - B'J'(JP^A J')^{-1}JP^A E'(EZE')^{-1}EP^A J'(JP^A J')^{-1}JB \quad (31)$$

where the diagonal terms then correspond to the terms $\mathbb{F}_i$ of (29). In (30) we note that $EP^E E' = cov(\hat{e}_t)$, and $(EB)_i = cov(e_t, \epsilon_{i,t})$. Analogously to the perfect information case, $EZE' = cov(e_t)$, with $EP^A J'(JP^A J')^{-1}JB = cov(e_t, \epsilon_t)$. Z satisfies the Riccati solution

$$Z = AZA' - AZE'(EZE')^{-1}EZA' + P^A J'(JP^A J')^{-1}JP^A \quad (32)$$

where $E = M_1 + M_3 - (M_2 + M_4)N$, $A = G_{11} + H_{11} - (G_{12} + H_{12})N$, $J = M_1 - M_2 G_{22}^{-1} G_{21}$, $F = G_{11} - G_{12}G_{22}^{-1}G_{21}$, and V is the covariance matrix of shocks with diagonal terms $(= \sigma_i^2)$.

3 The Empirical Model

For our application, we use an industry standard DSGE model, Smets and Wouters (2007). The model has at its core our motivating RBC model example but with an elastic household labour supply. It features a number of nominal and real frictions in order to closely mimic the pattern of real aggregate variables such as output, inflation and interest rate. To save space, we refer to the original article for full details of the micro-foundations. The notation is consistent with the illustrative example of Subsection 2.6 and the Smets and Wouters (2007) - henceforth SW - paper.

3.1 The Linearized Model

$$\begin{aligned}
y_t &= C/Y c_t + I/Y i_t + R^k K/Y z_t + e_t^g \\
c_t &= c_1 c_{t-1} + (1 - c_1)\mathbb{E}_t c_{t+1} + c_2(h_t - \mathbb{E}_t[h_{t+1}]) - c_3(r_t - \mathbb{E}_t[\pi_{t+1}] + e_t^b) \\
i_t &= i_1 i_{t-1} + (1 - i_1)\mathbb{E}_t[i_{t+1}] + i_2 q_t + \epsilon_t^i \\
q_t &= q_1 \mathbb{E}_t[q_{t+1}] + (1 - q_1)\mathbb{E}_t[r_{t+1}^k] - (r_t - \mathbb{E}_t[\pi_{t+1}] + e_t^b) \\
y_t &= \alpha\phi_p k_t + (1 - \alpha)\phi_p h_t + \phi_p \epsilon_t^a \\
k_t^s &= k_{t-1} + z_t \\
z_t &= \psi/(1 - \psi) r_t^k \\
k_t &= k_1 k_{t-1} + (1 - k_1) i_t + k_2 \epsilon_t^i \\
mp_t &= \alpha(k_t^s - h_t) + e_t^a - w_t \\
\pi_t &= \pi_1 \pi_{t-1} + \pi_2 \mathbb{E}_t[\pi_{t+1}] - \pi_3 mp_t + e_t^p \\
r_t^k &= -(k_t - h_t) + w_t \\
mw_t &= w_t - \left(\sigma_n h_t + \frac{1}{1 + \lambda/\gamma}(c_t - \lambda/\gamma c_{t-1})\right) \\
w_t &= w_1 w_{t-1} + (1 - w_1)\mathbb{E}_t[(\pi_{t+1} + w_{t+1})] - w_2\pi_t + w_3\pi_{t-1} + mw_t + e_t^w \\
r_t &= \rho_r r_{t-1} + (1 - \rho_r)(\rho_\pi \pi_t + \rho_y(y_t - y_t^f) + \rho_{\Delta y}\Delta(y_t - y_t^f)) + e_t^r \\
&\quad + \ flexible\ economy\ equations
\end{aligned}$$

where variables with time subscript are variables from the original non-linear model expressed in log deviation from the steady state. The latter, variables without time subscript, are the corresponding balanced growth steady state with growth rate γ.[9] Flexible output is defined as the level of output that would prevail under flexible prices and wages in the absence of the two mark-up shocks. There are seven structural shocks. The model has five AR(1) processes for government spending, technology, preference, investment specific, monetary policy, and two ARMA(1,1) processes for price and wage mark-up. The process for 'government spending' includes a net exports demand effect which depends on the technology shock and in log-form is given by

$$e_t^g = \rho_g e_{t-1}^g + \epsilon_t^g + \rho_{ga} \epsilon_t^a \tag{33}$$

The nominal interest rate rule in the SW model differs from that used in the small-scale NK model in that the latter does not require knowledge of the output gap $y_t - y_t^f$ and is referred to as 'implementable' by Schmitt-Grohe and Uribe (2007). This is a more natural choice of rule in our imperfect information set-up. Indeed in the version of the SW model with measurement errors neither output nor inflation is directly observed so we introduce an implementable form of the monetary rule

$$r_t = \rho_r r_{t-1} + (1 - \rho_r)(\rho_\pi \pi_{t,t} + \rho_y y_{t,t} + \rho_{\Delta y} \Delta y_{t,t}) + e_t^r \tag{34}$$

Note that with agents' information PI (assumed in SW) we have $\pi_{t,t} = \pi_t$, $y_{t,t} = y_t$ and $\Delta y_{t,t} = \Delta y_t$.

3.2 Bayesian Estimation

The SW model is estimated by a Bayesian method based on seven quarterly macroeconomic time series: real output, consumption, investment, and real wage growth, hours, inflation, and the interest rate. The data sample is 1966Q1-2004Q4 which is the same as in Smets and Wouters (2007). The corresponding measurement equations for the 7 observables are

$$\begin{bmatrix} output\ growth \\ consumption\ growth \\ investment\ growth \\ real\ wage\ growth \\ hours \\ inflation \\ fed\ funds \end{bmatrix} = \begin{bmatrix} \bar{\gamma} + \Delta y_t \\ \bar{\gamma} + \Delta c_t \\ \bar{\gamma} + \Delta i_t \\ \bar{\gamma} + \Delta w_t \\ \bar{l} + l_t \\ \bar{\pi} + \pi_t \\ \bar{R} + R_t \end{bmatrix} \tag{35}$$

where all variables are measured in percent, $\bar{\pi}$ and $\bar{R}$ measure the steady state level of net inflation and short term nominal interest rates, respectively, $\bar{\gamma}$ captures the deterministic long growth rate of real variables, and $\bar{l}$ captures the mean of hours. Δy_t, Δc_t and Δi_t are the log growth of real GDP, personal consumption expenditure deflated by the GDP deflator and Fixed Private Domestic Investment, respectively. Hourly compensation is divided by the GDP price deflator in order to get the real wage variable (w_t). The aggregate real variables are expressed per capita by dividing with the population over 16. Inflation is the first difference of the log of the Implicit Price Deflator of GDP and the interest rate is the Federal Funds Rate divided by

[9] These are Y, C, I, R^k, K, W, H and, e.g., $y_t = \log\left(\frac{Y_t}{Y}\right)$, where Y_t is output from the non-linear equilibrium conditions.

four.

When we assume that this exactly coincides with the agents' imperfect information set so in effect the number of measurements is equal to the number of shocks and EB is non-singular. This we refer to as **Case 1**: the original SW with 7 shocks and 7 observables (the data). In the modified versions of the model, the only changes we make are that (1) we add an inflation target shock so the number of shocks exceeds the number of observables; (2) we further add measurement errors[10] to the observations of real variables and inflation. We refer to this non-square system as **Case 2**: SW with 13 shocks and again 7 observables.[11] Table 1 summarizes the full estimation results. In terms of fitting the model empirically, we show that, for the 7-shock case the perfect and imperfect information cases coincide. From Case 2, including the additional shocks under II leads to a small improvement in fitting the data including the second order empirical moments.[12]

3.3 Invertibility and Perfect vs Imperfect Information

Table 2 first presents the key invertibility results from the estimated models and the test for non-fundamentalness, based on Levine *et al.* (2022), as useful measures to show the (mis-)use of VARs to validate DSGEs. We find that the original system is completely invertible according to the eigenvalue measures and indeed produces exactly the same simulated moments (including the IRFs). The model is E- and A-invertible. When we add the additional shocks in Case 2, this introduces non-invertibility and non-fundamentalness into the model, drives a bigger wedge between PI and II, in the sense that the fundamentalness problem worsens for the performance of VARs (suggesting larger differences in IRFs). Figures in Online Appendix F.1 plot the posterior IRFs based on the estimates reported in Table 1 and compare them for Case 2 which is non-square and therefore has a non-invertible RE solution.

To illustrate the effectiveness of our measure of approximate fundamentalness-invertibility for individual shocks for Case 2, Figure 2 compares IRFs for the technology, monetary policy and preference shocks. For the former two, the relevant II measures are $\mathbb{F}_a^{II} = 0.0004$ and $\mathbb{F}_r^{II} = 0.0036$ which are close to indicating a good approximation of the structural shock to the innovation whereas for the latter, $\mathbb{F}_b^{II} = 0.9526$, indicating a poor approximation. From Section 2.3, invertibility requires A-invertibility and the ability of agents to back out the shocks. Then in this case, the PI and II equilibria coincide. Thus a wedge between PI and II IRFs is also a measure of non-fundamentalness. However, one needs to take into account that the estimated parameters for the PI and II cases are different so part of the wedge arises for this reason. So we also compare PI and II in the model where both simulations apply to the SW model estimated under our preferred II case (i.e., the latter generates a better fit in the Bayesian comparison in Table 1).

In the figures, the solid black lines are PI responses for the estimated model under PI. The dashed red lines are II responses for the estimated model under II. The dashed blue lines are PI responses with II estimated parameters. Hence the wedge between the blue and red lines arises solely from the different information assumptions (i.e., an indication of pure A-invertibility).

[10] These are equivalent to the noise in the signal in the "noisy news" papers by Blanchard *et al.* (2013) and Forni *et al.* (2017).

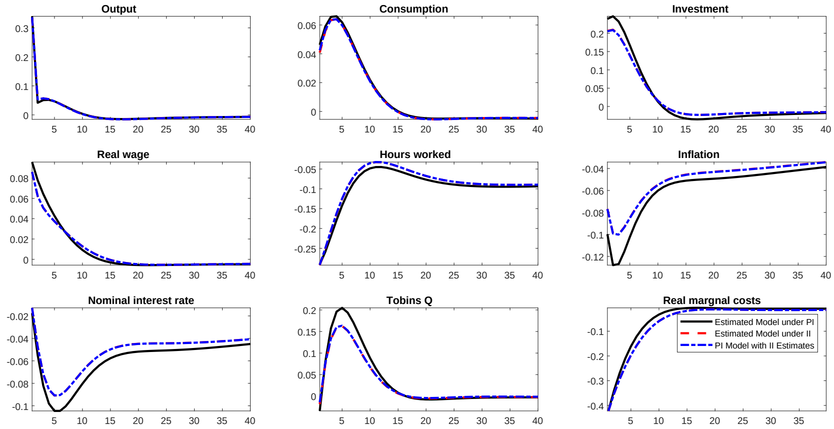
[11] According to the U.S. Bureau of Labor Statistics, national employment, hours and earnings statistics are surveyed and published very frequently (more so than GDP and CPI). The hours data is constructed based on these statistics. We do not assume a measurement error to the employment data and the reason for that is that the frequency in revising and publishing the employment data reduces measurement error, for hours to be observed.

[12] The complete set of empirical results is reported in Levine *et al.* (2020).

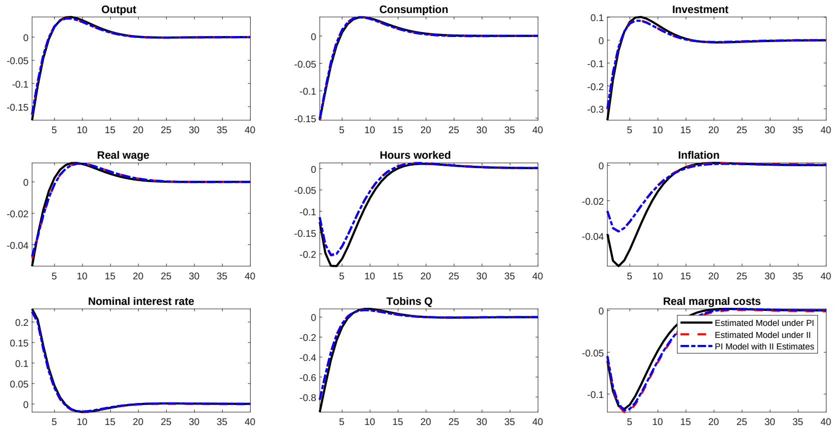
Parameter	Prior Mean	Post. Mean (PI)	Post. Mean (II)	Prior	Prior S.D.
ρ_a	0.5	0.9795	0.9786	beta	0.2
ρ_b	0.5	0.4241	0.4010	beta	0.2
ρ_g	0.5	0.8250	0.8972	beta	0.2
ρ_i	0.5	0.7231	0.7694	beta	0.2
ρ_r	0.5	0.2226	0.2501	beta	0.2
ρ_p	0.5	0.3646	0.8895	beta	0.2
ρ_w	0.5	0.9689	0.9091	beta	0.2
ρ_t	0.5	0.5093	0.5996	beta	0.2
μ_p	0.5	0.5552	0.5827	beta	0.2
μ_w	0.5	0.5325	0.7793	beta	0.2
ρ_{ga}	0.5	0.6002	0.6232	beta	0.25
ε_a	0.1	0.4792	0.4695	invg	2
ε_b	0.1	0.1304	0.1782	invg	2
ε_g	0.1	0.3986	0.4252	invg	2
ε_i	0.1	0.3809	0.3566	invg	2
ε_r	0.1	0.2581	0.2503	invg	2
ε_p	0.1	0.1686	0.0860	invg	2
ε_w	0.1	0.0790	0.2924	invg	2
ε_t	0.1	0.0814	0.0837	invg	2
γ_w	0.5	0.4897	0.6544	beta	0.15
γ_p	0.5	0.4507	0.4208	beta	0.15
ψ	0.5	0.4831	0.5125	beta	0.15
ϕ_p	1.25	1.5053	1.5561	norm	0.125
α	0.3	0.1970	0.2014	norm	0.05
ϕ	4	5.3027	5.1554	norm	1.5
σ_c	1.5	1.2731	1.4176	norm	0.375
σ_l	2	0.9992	0.9683	norm	0.75
ξ_w	0.5	0.6836	0.6789	beta	0.1
ξ_p	0.5	0.6122	0.6728	beta	0.1
λ	0.7	0.8305	0.7943	beta	0.1
ρ_π	1.5	1.6886	1.8027	norm	0.25
ρ_r	0.75	0.7714	0.7295	beta	0.1
ρ_y	0.125	0.0692	0.0708	norm	0.05
ρ_{dy}	0.125	0.2012	0.2005	norm	0.05
$\bar{\gamma}$	0.4	0.3800	0.3639	norm	0.1
$\bar{\pi}$	0.75	0.8747	0.9887	gamm	0.4
$\bar{\beta}$	0.25	0.2094	0.1878	gamm	0.1
$\bar{l}$	0	-0.2338	-1.0533	norm	2
Log Data Density (MHM)		-905.469531	-904.886754		

Table 1: **Posteriors Results for Model Parameters (Case 2)**

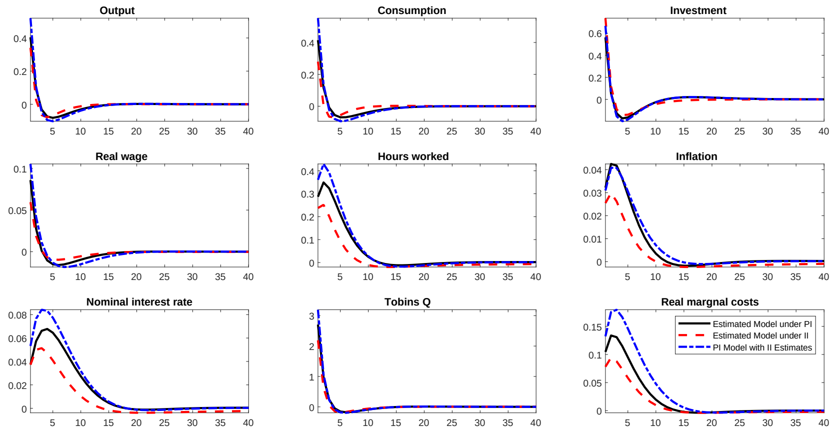
Notes: We report our results from Bayesian maximum-likelihood estimation. A sample from the posterior distribution is obtained with the Metropolis-Hastings (MH) algorithm using the inverse Hessian at the estimated posterior mode as the covariance matrix of the jumping distribution. Two parallel Markov chains of 250,000 runs each are run from the posterior kernel for the MH, sufficient to ensure convergence according to the indicators recommended by Brooks and Gelman (1998). The first 50,000 draws from each chain have been discarded.



(a) Technology: $\mathbb{F}^{II} = 0.0004$



(b) Monetary Policy: $\mathbb{F}^{II} = 0.0036$



(c) Preference: $\mathbb{F}^{II} = 0.9526$

Figure 2: **Estimated SW Model Non-invertible Case 2**

Notes: Solid black line PI responses. Dashed red line II responses. Dashed blue line PI responses with II estimated parameters. Where red lines are invisible they coincide with the blue lines and therefore PI is equivalent to II based on the same estimates. Each panel plots the mean response corresponding a positive one standard deviation of the shock's innovation. Each response is for a 40 period (10 years) horizon and is level deviation of a variable from its steady-state value.

	Case 1: Original SW Measurements = Shocks =7	Case 2: SW with MEs Measurements = 7 < 13 Shocks	
PMIC Conditions	Hold	Fail	
Goodness of Fit	$\mathbb{F}^{PI} = \mathbb{F}^{II} = 0$	$\mathbb{F}^{PI}_{(13\times13)}$	$\mathbb{F}^{II}_{(13\times13)}$
Diagonal Values	All Zero	0.0003 0.2904 0.2020 0.1211 0.0405 0.1680 0.0344 0.9904 0.9996 0.4551 1 0.9994 0.8671	0.0004 0.9526 0.0194 0.5085 0.0036 0.6655 0.0111 0.9989 1 0.1287 1 1 0.4968

Table 2: **Fundamentalness and Invertibility Measures for Estimated SW Model**

Notes: Order of shocks: technology, preference, government spending, investment specific, monetary policy, price and wage mark-up, inflation objective and measurement errors for output growth, consumption growth, investment growth, real wage growth and inflation. The simulation results in this table are based on the estimated posterior means.

From the figure for the approximately fundamental technology shock, these are impossible to discern whereas for the preference shock with a high non-fundamental $\mathbb{F}^{II}_i$ measure the wedge is considerable. Online Appendix F.1 shows similar plots for the other shocks. For example, for the monetary policy shock, $\mathbb{F}^{II}_r = 0.0036$, whereas for the investment specific shock, $\mathbb{F}^{II}_i = 0.5085$. For the former, IRFs diverge very little, but for the latter, this is not the case and there is substantial divergence and, for consumption, an opposite sign. Furthermore, Section 5.1 below reports the cumulative mean square distance that provides an additional measure for the wedge between the blue and red lines for the shocks that we focus on for the SVAR estimations in Section 4.

As noted, contrasting IRFs of II and PI depends not just on the information solutions, but also on the different estimated parameters which include different estimates of persistence. The latter in particular might drive the IRF differences (e.g., for the price mark-up shock). When we compare the wedge between the black and blue lines (i.e., the IRFs under PI computed with different posterior estimates) in Figure 2 and Online Appendix F.1, it is useful to know that the effect from the estimated parameters is very small for the investment specific, government spending and preference shocks, suggesting that the divergence in this case is almost entirely owing to non-A-invertibility.

Further insight into the differences between PI and II solutions can be obtained by comparing the agents' expectations of shock process from actual outcomes. Under PI (as an endowment), they are the same of course. But under II (the absence of A-invertibility), agents need to solve a signal extraction problem and learn about the shocks using the Kalman filter. Thus for each shock process x_t where $x_t = \{e^a_t, e^b_t, ...\}$ for the technology and preference AR(1) processes etc., $\mathbb{E}_t[e^a_t] = e^a_t$ under PI but not under II in the absence of A-invertibility. Impulse response functions give plots for each shock at a time, so with e^a_t we have $e^b_t = e^g_t = 0$, etc. But under II $\mathbb{E}_t[e^a_t] \neq e^a_t$ and nor are $e^b_t = e^g_t = 0$. Then the difference between $\mathbb{E}_t[x_t]$ and x_t is a measure of the imperfect information of the shock process.

Figure 3 shows this Kalman learning process about the shocks that do occur and the misperceptions regarding those that do not occur for the approximately fundamental technology shock and the very strongly non-fundamental preference shock.[13] For the approximately fundamental technology and monetary policy shocks, both types of misperception are very small with the exception of the government spending shock in the presence of only a technolgy shock. The reason for this is simple: namely, the inclusion of the latter in the AR(1) process (33) for government spending. Again Online Appendix F.1 compares the learning processes for the remaining shocks. As expected, for the technology shock ($\mathbb{F}_a^{II} = 0.0004$) and monetary policy shock again ($\mathbb{F}_r^{II} = 0.0036$), the responses between $\mathbb{E}_t[x_t]$ and x_t clearly overlap, showing no divergence driven by the learning process.[14]

4 Impulse Responses from Estimated SVAR and DSGE Models

In this section, we contrast the invertible Case 1 with the non-invertible Case 2 and compare IRFs from the RE solution of the estimated model with those of the SVAR estimated on artificial data simulated from RE solutions of the model under PI and II (the DGP). Our procedure for simulating the data under II is described in Appendix D.[15] We estimate and compare our SVAR using the following identification schemes: zero short-run restrictions, mixed sign and zero restrictions, sign restrictions with uniform prior, sign restrictions with distribution-free sets, and restrictions with bounded FEVD.

4.1 The SVAR(p) Approximation to the DGP

We first recall the ABC and D form of a RE solution:

$$\begin{aligned} \epsilon_t &= \tilde{D}^{-1} m_t^E - \tilde{D}^{-1}\tilde{C}\sum_{j=1}^{\infty}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^j \tilde{B}\tilde{D}^{-1} m_{t-j}^E \\ &\Rightarrow \; m_t^E = \tilde{C}\sum_{j=1}^{\infty}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^j \tilde{B}\tilde{D}^{-1} m_{t-j}^E + \tilde{D}\epsilon_t \end{aligned} \tag{36}$$

where, for $t = 1, \cdots, T$, m_t^E is a $n \times 1$ vector of endogenous observed variables (the data), ϵ_t is a $n \times 1$ vector of structural white noise processes and A_j, for $j = 0, 1 \cdots, p$, are matrices of estimated structural coefficients.

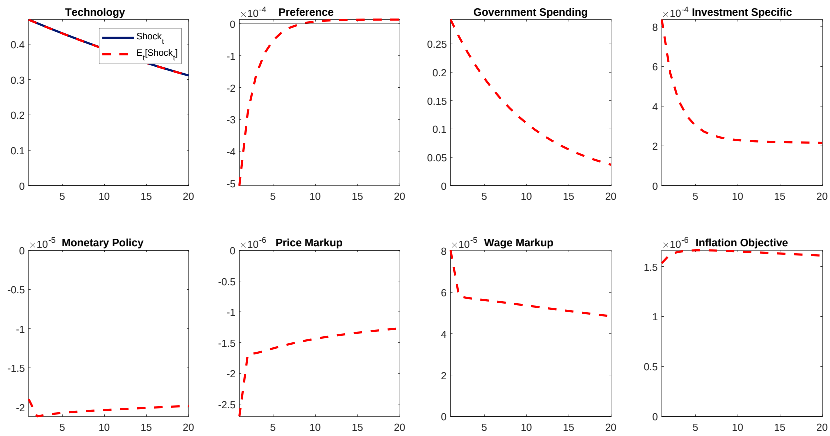
An invertible RE solution of a linearized model is of form (36) if the following PMIC hold: $\tilde{D}$ is non-singular and $(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})$ has stable eigenvalues. Both the state space s_t and the $\tilde{A}$, $\tilde{B}$ $\tilde{C}$ and $\tilde{D}$ matrices *differ for PI and II.*

For a possibly non-square system, the econometrician estimates an SVAR(p) model in struc-

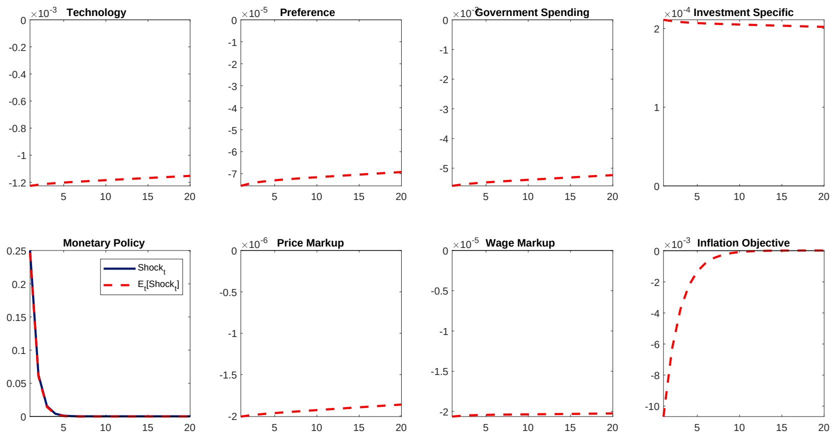
[13] Impulse response functions have a standard interpretation with leisure as a normal good.

[14] For this reason, in what follows, we exclude the technology shock from our exercise as it does not display any invertibility issue and is completely fundamental in our example but include the monetary shock for completeness. Furthermore, results from Cholesky identification do not include the inflation objective shock as in the literature this is not identified with short-run zero restrictions for obvious reasons.

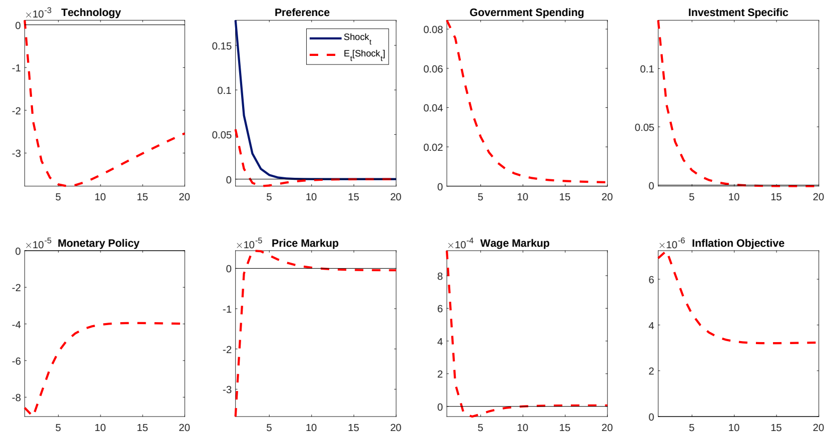
[15] Simulated using the DSGE posterior means, our artificial dataset consists of 1,000 periods (discarding the initial conditions), meaning that, in practice, there is no sample bias. This implies that the uncertainty around the VAR estimates is mostly identification uncertainty which we address systematically in the following sections with set-identification. However, all the results shown here are robust once sample bias is taken into account.



(a) Technology



(b) Monetary Policy



(c) Preference

Figure 3: **Estimated SW Model Non-invertible Case 2: Misperceptions About the Shocks under II. The graphs compare the actual structural unobserved shock process x_t with the agents belief $\mathbb{E}_t[x_t]$ for the technology, monetary policy and preference shocks in turn.**

tural shocks

$$m_t^E = \sum_{j=1}^{p} A_j m_{t-j}^E + P\epsilon_t \tag{37}$$

where A_j and P are the dynamic and impact matrices, respectively, in the SVAR. (37) is the reduced form VAR representation where the reduced form coefficients A_j are nonlinear functions of A_j and the vector of reduced form errors $u_t = P\epsilon_t$.

The IRFs stem from the MA representation:

$$m_t^E = \sum_{j=0}^{\infty} C_j P\epsilon_{t-j} \tag{38}$$

where each C_j is a matrix of $(I_n - \sum_{j=1}^{p} A_j L^j)^{-1}$, with its i-th column $c_{i,j}$ multiplying the i-th shock. **Identification** then comes down to the choice of matrix P that satisfies $\Sigma_u = PP'$.

In the absence of any identifying restrictions with an invertible system,

$$\Sigma_u \equiv \mathbb{E}[u_t u_t'] = \Sigma_{tr}\Sigma_{tr}' = P\Sigma_\epsilon\Sigma_\epsilon' P' = PP' = \Sigma_{tr}QQ'\Sigma_{tr}' \tag{39}$$

$$P = \Sigma_{tr}Q \tag{40}$$

where Σ_{tr} is lower triangular of Cholesky factor of $\hat{\Sigma}$ and Q (the 'rotation matrix') is an orthonormal matrix.

But if the PMIC fails and the model RE solution is not A-invertible, then the a-theoretical econometrician may think that the reduced form VAR representation of the DGP is (37) whereas in fact it is given by

$$m_t^E = \sum_{j=1}^{p} \tilde{A}_j m_{t-j}^E + \hat{P}e_t \tag{41}$$

where, we recall from (28), e_t is an $n \times k$ vector of one-period ahead prediction errors and *not* the structural shocks. If the model RE solution is invertible then e_t is a linear transformation of ϵ_t, and then estimating and identifying (41) becomes equivalent to estimating (37).

4.2 Zero Short-Run Restrictions

Our first experiment is the classical recursive identification scheme which typically imposes equality restrictions such as zero (short-run) restrictions on the off-diagonal elements of A_0. There are many A_0 matrices with the given pattern of zeros: one simple solution is a lower triangular A_0 with positive diagonal values obtained by orthogonalizing Σ_e (Cholesky decomposition). As a result, this can exactly identify the system with the exact number of equality restrictions on each structural shock satisfying the condition for exact (point-)identification. However, due to its simplicity, clearly this requires the sequence of causation in the model. We identify and estimate the SVAR with the following most common ordering for y_t: output, consumption, investment, real wages, hours, inflation, interest rates (for example, Christiano *et al.*, 2005).[16] Given the sequence of causation, the restrictions can be written as linear constraints on the columns of Q as a function of the reduced form parameters.

The obvious problem of making the system recursive is that restrictions imposed regarding the rotation matrix Q can be inconsistent with those imposed on the theoretical model (e.g., the

[16] Our first robustness check looks at the alternative sequences of y_t. The main result is robust to the ordering of variables in the SVAR.

DSGE model). Nevertheless, we estimate our SVAR identified by the simple Cholesky scheme using the artificial data from the estimated Cases 1 and 2. Online Appendix F.2 reports the estimated IRFs. These results show a clear message. The Cholesky scheme fails to recover the DSGE IRFs, predicting the opposite sign on impact in many cases, even when there is perfect information and the system is E- and A-invertible. For example, the bottom panels of Figure 4 below highlight how the IRFs from the VAR are less able to recover the DGP, compared with Case 1, and generate the wrong sign for the impact mean responses (*vis-à-vis* the assumed DGP responses). This is a well-known fact as these timing restrictions implied by a recursive structure typically do not hold in DSGE models.

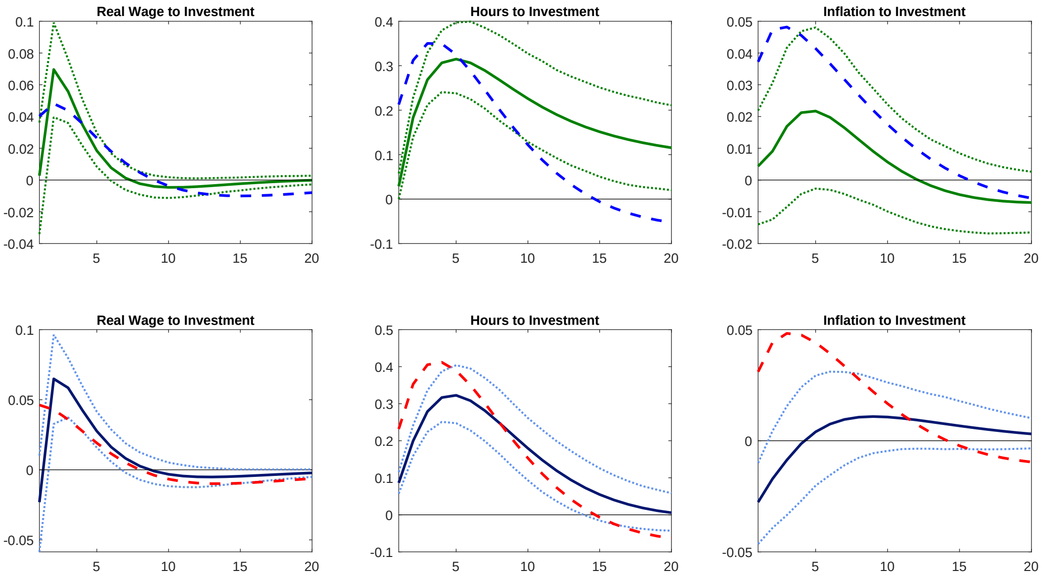


Figure 4: **Responses to Investment Specific Shock (Cholesky Decomposition)**
Notes: Case 1 (top panels) and Case 2 (bottom panels). The three variables are the real wage (left), hours (centre) and inflation (right). In each panel, the solid lines plot the posterior means of the VAR responses with the corresponding 95% band of the point estimates (dotted). The dashed red lines are the SW-II responses for Case 2 and the dashed blue lines are the SW-PI responses for Case 1.

4.3 Sign Restrictions with a Robust-Prior Inference

In order to further address the uncertainty about our identifying assumptions within set-identified models, in this section, we revisit the sign-restricted SVAR estimated in Online Appendix E.3, in which the shocks are identified using the algorithm proposed by Uhlig (2005). The Bayesian sampling algorithm generates a posterior distribution of IRFs to reflect uncertainty about the reduced form parameters and the rotation matrix Q by specifying a commonly used uniform prior on Q.[17] The problem is that this prior choice does not imply a uniform distribution over the identified set of the structural parameters which are a function of both reduced form parameters and Q (Baumeister and Hamilton, 2015). Baumeister and Hamilton (2015) and Giacomini and Kitagawa (2021) have also highlighted that this approach induces prior information on the IRFs that cannot be updated by data even asymptotically because Q is not identified and the likelihood is flat over the space of the admissible $Q's$, and substantially affects the posterior estimation of IRFs. This issue of having a posterior that is proportional

[17] Details of the algorithm designed to implement the sign restrictions are set out in Online Appendix E.3. Our results are also appended to the paper.

to the prior, even asymptotically, is clearly relevant to us as our focus here is to separate the impact of identification from invertibility.

To resolve this issue, this section estimates the sign-restricted models with a robust-prior algorithm over the bounds of the identified set through a numerical optimization procedure set out by Volpicella (2021) where the identified set is distribution-free and does not depend on a specific prior over the Q matrix.[18] The algorithm triggers an iterative procedure that solves a constrained optimization problem consisting of a given objective function and, in general, the user-specified linear, nonlinear inequality and equality constraints, to produce a distribution-free identified set.[19]

Algorithm 1 Robust-Prior Procedure

Here we briefly describe the numerical algorithm that we use to robustly compute the impulse response identified sets with a linear objective and inequality constraint. For each variable $i = 1, ..., n$ and for each horizon H:

1. Draw the estimate $\tilde{A}_i$ of A_i in (37) and $\hat{\Sigma}_e$ from the posterior distribution of the reduced form parameters.

2. Compute the bounds of the identified set by solving the following problem:

$$\begin{aligned} \min_q \quad \textit{and} \quad \max_q \quad & c'_{ih}(\tilde{A}, \Sigma_e)q \\ \text{s.t.} \quad & S(\tilde{A}, \Sigma_e)q \geq 0 \\ & \|q\| = 1 \end{aligned} \tag{42}$$

where q is the column of Q that corresponds to the shock of interest. $c'_{ih}(\cdot)$ represents the i-th row vector of $C_h\Sigma_{tr}$ for computing the impulse vectors. $\|\cdot\|$ denotes the Euclidean norm. $S(\tilde{A}, \Sigma_e)q \geq 0$ collects all the sign restrictions.

3. Repeat Steps 1-2 N times. Save the results as upper and lower bounds.

In Online Appendix F.4, we present the estimated IRFs from the SVAR using PI-simulated artificial data and SW model (PI). Online Appendix F.4 also depicts the IRFs from the estimated SVAR Model (II), SW Model (II) and SW Model (PI) simulated with the II estimates for Case 2. The prior-robust IRFs implied by the sign restrictions are now displayed with the posterior means of the set bounds (solid) and the corresponding 95% band of the set (dotted). Table 3 summarizes the sign of the first period responses of the seven variables to Υ_t under PI and II obtained from simulating the estimated models (calibrated at their posterior means). We use (D.11) from Appendix D which is the DSGE solution capturing the shocks' impact effect on observables given RE. Sign restrictions are therefore necessary to identify the DSGE shocks.

It is not surprising to see that they are significantly more effective in recovering the DGP relative to the Cholesky VARs, getting closer to those of the DGP, and can pin down the sign of the SW IRFs in a decent number of cases, but still deliver very large set-estimates (imprecise estimation) as there is high identification uncertainty (as is indeed the case in Section E.3). However, what is relevant to our application is that, in some cases, the introduction of II on

[18] In effect, the optimizer computes [inf, sup] over all admissible rotation matrices $Q's$.

[19] We apply Algorithm 1 for 1000 draws from the reduced form posterior specification, where a flat Normal Inverse Wishart distribution is employed as prior. In our figures involving set-identified responses, we accordingly report the posterior mean for upper and lower bound of the impulse responses identified sets and the corresponding 95% Bayesian credibility region. The latter is defined as Equal-Tailed Interval (ETI). For the case of our sign-restricted models, this is in practice equivalent to the solutions in Amir-Ahmadi and Drautzburg (2021) and Giacomini and Kitagawa (2021).

	$dlGDP_t$	$dlCON_t$	$dlINV_t$	$dlWAG_t$	HOU_t	$dlCPI_t$	FED_t
	Case 1 Perfect Info. (IRFs based on PI case of (D.11))						
GovExp	+	−	−	+	+	+	+
MonPol	−	−	−	−	−	−	+
Preference	+	+	+	+	+	+	+
Investment	+	−	+	+	+	+	+
PMarkup	−	−	−	−	−	+	+
	Case 2 Perfect Info. (IRFs based on PI case of (D.11))						
GovExp	+	+	−	+	+	+	+
MonPol	−	−	−	−	−	−	+
Preference	+	+	+	+	+	+	+
Investment	+	−	+	+	+	+	+
PMarkup	−	−	−	−	−	+	+
InfObj	+	+	+	+	+	+	−
	Case 2 Imperfect Info. (IRFs based on (D.11))						
GovExp ($\mathbb{F}_i^{II} = 0.0194$)	+	+	−	+	+	+	+
MonPol ($\mathbb{F}_i^{II} = 0.0036$)	−	−	−	−	−	−	+
Preference ($\mathbb{F}_i^{II} = 0.9526$)	+	+	+	+	+	+	+
Investment ($\mathbb{F}_i^{II} = 0.5085$)	+	+	+	+	+	+	+
PMarkup ($\mathbb{F}_i^{II} = 0.6655$)	−	−	−	−	−	+	+
InfObj ($\mathbb{F}_i^{II} = 0.9989$)	+	+	+	+	+	+	−

Table 3: **Sign Restrictions for Smets and Wouters (2007) Shocks**

Notes: The non-fundamental measures, $\mathbb{F}_i^{II}$, are in parentheses. The IRF of Y_t can be obtained based on PI case of (D.11), that is, the Wold representation of (37). The IRFs can be derived as the conditional moments depending on the history of shocks. We also check the signs of the impact impulse responses computed from simulating the model using 10,000 draws of the posterior estimates.

agents' information set makes the sign-restricted VAR less able to recover the assumed DGP. Thus, at least partially, the informational frictions from agents can play a role, and this is evident from Online Appendix F.4 in a straight comparison of PI and II solutions for the preference and investment specific shocks. To focus the presentation, we compare the effectiveness of both PI and II in recovering the DGP based on several selected figures in Section 4.5.

4.4 Restrictions on Bounds of the Forecast Error Variance Decomposition

To help further shrink the set of admissible structural parameters in our sign-SVAR, we utilize Volpicella (2021) and impose bounds on the FEVD implied by the estimated DGP as an additional strategy of appropriating the impulse vectors and to eliminate any uncertainty about the specific values used for bounding the IRFs. In other words, we identify and estimate the SVAR restricted with both the sign restrictions and bounds on the variances of the forecast errors (FEV) implied by the SW model. This approach, by complementing sign restrictions with a novel methodology, aims to further improve the estimation precision of our sign-restricted model and deliver informative inference.

Both the sign restrictions and FEV bounds are imposed at $h = 0$.[20] We generate the bounds by randomly drawing DSGE parameter vectors from the posterior distribution. FEVD decomposes the variation in each endogenous variable into each shock to the system, thus providing information on the relative importance of each disturbance as a source of variation for each variable. In particular, it decomposes the FEV for the target $\mathsf{Y}_{z,t+h}$ using information

[20] As a robustness check, we extend the sign restrictions and bounds on the FEV up to 4 quarters. This does not change the results.

at time t into the percentage explained by each of the shocks s

$$FEVD_s^z(h) \equiv \frac{FEV_s^z(h)}{FEV^z(h)} \tag{43}$$

where $FEV_s^z(h)$ is the FEV of variable z due to shock s at h, $FEV^z(h)$ the total FEV of variable z at h, and $0 \leq FEVD_s^z(h) \leq 1$. Using the notations in Volpicella (2021), we can write (43) as

$$FEVD_s^z(h) = q_s' \Gamma_h^z(\tilde{A}, \Sigma_e) q_s = q_s' \frac{\sum_0^h c_{zh}(\tilde{A}, \Sigma_e) c_{zh}'(\tilde{A}, \Sigma_e)}{\sum_0^h c_{zh}'(\tilde{A}, \Sigma_e) c_{zh}(\tilde{A}, \Sigma_e)} q_s \tag{44}$$

We define the set of bounds on the FEVD for Y_z at h from shock s

$$lb_s^{zh} \leq q_s' \Gamma_h^z(\tilde{A}, \Sigma_e) q_s \leq ub_s^{zh} \tag{45}$$

then we simply add to (42) the above quadratic inequality constraints on the columns of Q when searching for the set of $Q's$ that satisfy the both linear and quadratic constraints that are now more restrictive. Step 2 of Algorithm 1 becomes

$$\begin{aligned} \min_{q_s} \quad and \quad \max_{q_s} \quad & c_{ih}'(\tilde{A}, \Sigma_e) q_s \\ \text{s.t.} \quad & S(\tilde{A}, \Sigma_e) q_s \geq 0 \\ & lb_s^{zh} \leq q_s' \Gamma_h^z(\tilde{A}, \Sigma_e) q_s \leq ub_s^{zh} \\ & \|q_s\| = 1 \end{aligned} \tag{46}$$

To derive the theory-driven restrictions using bounds on the FEVD, we simply compute the 90% intervals or the maximum and minimum value of the FEVDs simulated by the 10,000 draws. Solutions to problem (46) then allow to compute bounds of the identified sets of the impulse responses. Among the restrictions, we have sign restrictions and bounds on the FEVD. Table 4 displays these FEV bounds at horizon $h = 0$ generated by the posterior estimation of the DGP for the seven variables of the SW model. These bounds show relatively small intervals for many variables. For instance, in the short run, a monetary shock explain a large share of unexpected movements in the interest rate and over 50% of the fluctuations of inflation can be attributed to the price mark-up shock. We therefore anticipate that these bounds, in conjunction with the impact sign constraints, can be very informative in terms of tightening the estimation precision and removing implausible effects of shocks from our set-identified IRFs. Having been able to maximise the ability of identifying assumptions to recover the DGP responses, we can turn our focus to invertibility of the DGP.

4.5 Assessment

A clear message emerges from the results in this section is that, with respect to the sign restrictions (set out in Table 3), the identification uncertainty decreases because of the additional restrictions (set out in Table 4), significantly improving the precision of our estimated IRFs in line with the theoretical SW model. Clearly, identification is what mostly matters when the system/shocks are exactly fundamental but the information frictions can still impact on the recoverability of the SW IRFs which is conditional on the RE solution for agents being consistent with A-invertibility or not.

First, we can actually report some good news for the estimated SW model. For the original square Case 1 there is no invertibility problem so the divergence between estimated model and

	$dlGDP_t$	$dlCON_t$	$dlINV_t$	$dlWAG_t$	HOU_t	$dlCPI_t$	FED_t
	Case 1 Perfect Info., $[lb_s^z, ub_s^z]$, $h = 0$						
GovExp	[0.23,0.50]	[0.00,0.07]	[0.00,0.02]	[0.00,0.01]	[0.21,0.47]	[0.00,0.01]	[0.00,0.06]
MonPol	[0.02,0.12]	[0.04,0.26]	[0.01,0.09]	[0.00,0.04]	[0.02,0.12]	[0.00,0.07]	[0.35,0.79]
Preference	[0.15,0.41]	[0.51,0.94]	[0.01,0.21]	[0.00,0.09]	[0.14,0.39]	[0.00,0.04]	[0.07,0.39]
Investment	[0.04,0.26]	[0.00,0.07]	[0.65,0.93]	[0.00,0.02]	[0.04,0.25]	[0.00,0.09]	[0.00,0.06]
PMarkup	[0.00,0.06]	[0.00,0.05]	[0.00,0.07]	[0.13,0.43]	[0.00,0.04]	[0.51,0.95]	[0.03,0.17]
	Case 2 Perfect Info., $[lb_s^z, ub_s^z]$, $h = 0$						
GovExp	[0.17,0.46]	[0.00,0.02]	[0.00,0.03]	[0.00,0.01]	[0.15,0.47]	[0.00,0.02]	[0.00,0.06]
MonPol	[0.02,0.13]	[0.01,0.17]	[0.01,0.19]	[0.00,0.04]	[0.02,0.12]	[0.00,0.17]	[0.66,0.96]
Preference	[0.08,0.51]	[0.16,0.68]	[0.01,0.62]	[0.00,0.15]	[0.08,0.50]	[0.00,0.12]	[0.00,0.13]
Investment	[0.03,0.28]	[0.00,0.02]	[0.18,0.94]	[0.00,0.01]	[0.03,0.32]	[0.00,0.10]	[0.00,0.06]
PMarkup	[0.00,0.04]	[0.00,0.02]	[0.00,0.05]	[0.00,0.25]	[0.00,0.02]	[0.07,0.89]	[0.00,0.17]
InfObj	[0.00,0.05]	[0.00,0.03]	[0.00,0.07]	[0.00,0.02]	[0.00,0.05]	[0.00,0.14]	[0.00,0.06]

Table 4: **Estimated FEV Bounds for Smets and Wouters (2007) Shocks**

Notes: The bounds of the FEVD are computed as the maximum and minimum value of the estimated FEVDs simulated using the 10,000 parameter draws.

SVAR are entirely due to a combination of the finite VAR assumption and the choice of the mapping matrix (the traditional identification problem). For example, if we focus on the real effect of IRFs to a monetary policy shock for Case 1 and compare the outputs from the three identification schemes (Cholesky vs Sign vs BoundsFEV) in Figure 5, it is very clear that our latest identification approach delivers the best estimation precision, removing the implausible responses and outperforming the Cholesky- and sign-VARs in replicating the responses in the assumed DGP (i.e., in terms of the median responses).

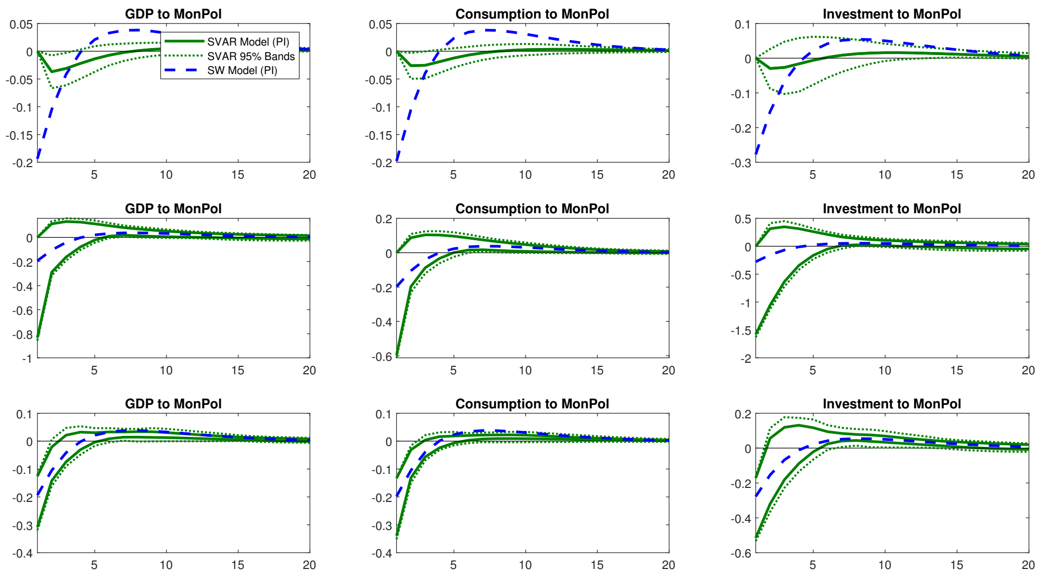


Figure 5: **The Real Effect of Monetary Policy Shock for Invertible Case 1 (Cholesky vs Sign vs BoundsFEV)**
Notes: Cholesky (top panels), Sign (middle panels) and BoundsFEV (bottom panels). The real variables are GDP (left), consumption (centre) and investment (right). The solid lines plot the posterior means of the VAR response set bounds for Sign and BoundsFEV with the corresponding 95% band of the set (dotted). The solid lines plot the mean responses for Cholesky with the corresponding 95% band of the point estimates (dotted). The dashed blue lines are the SW-PI responses.

Our second finding reports more good news even for the non-square non-invertible Case 2. Namely, the monetary policy and government spending shocks seem to be approximately fundamental as indicated by the IRF comparisons and the $\mathbb{F}_i$ measures for the two shocks. This is encouraging as many researchers only focus on these two shocks in the empirical literature. These results are very robust to the alternative identification strategy implemented in Section 4.3, i.e., when we impose only the sign restriction to achieve the identified set. Indeed, our IRF figures show that, for the monetary policy shock (with the smallest $\mathbb{F}_r^{II} = 0.0036$) for example, most of the SW-II posterior responses capture the empirical responses very well, with most of them lying inside the 95% uncertainty bands and the means of the identified sets (Figure 6).

There is more evidence where the divergence in IRFs starts to appear and the IRFs from the SVAR may be badly misspecified for one particular set of IRF if we just focus on comparing the investment responses in Figure 7 below which highlight an IRF comparison between the two cases from a government spending shock. Note that $\mathbb{F}_g^{II} = 0.0194$ which is slightly higher than $\mathbb{F}_r^{II}$. For Case 2 assuming II, there is a clear impact on the recoverability of the SW IRFs as there is considerable divergence in IRFs between the DGP and the posterior mean of the upper bound of the sets (and the dotted 95% credibility bands of the sets).

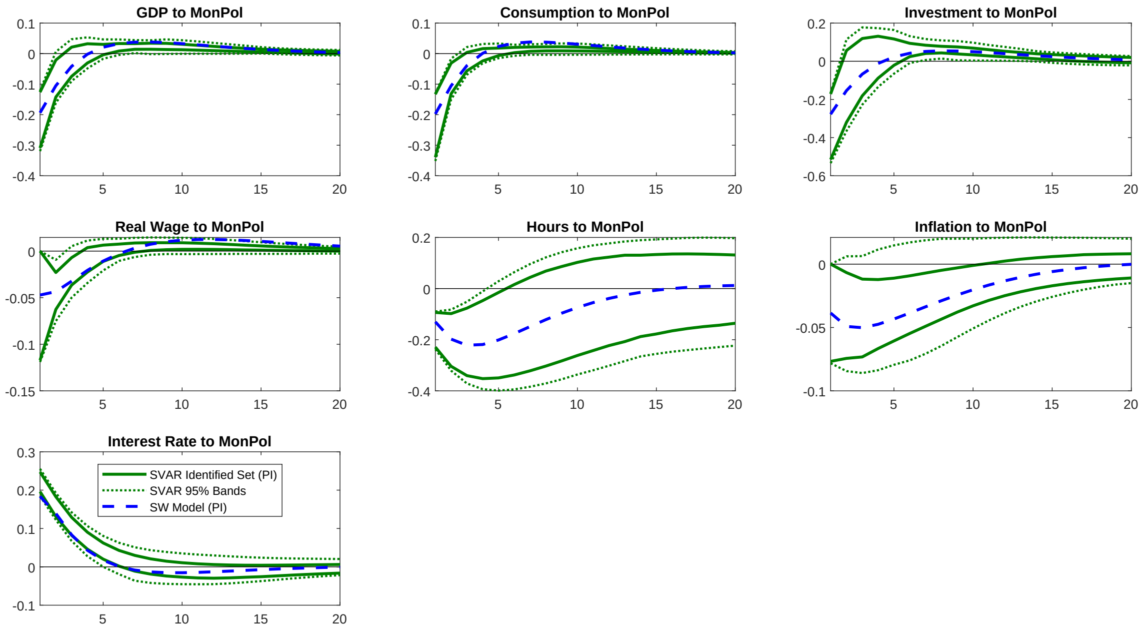
Now we turn to the remaining shocks that are not approximately fundamental based on our $\mathbb{F}_i$ indicators. There is a mixed outcome from matching IRFs and the $\mathbb{F}_i$ measures. The first interesting case to examine is the investment specific shock ($\mathbb{F}_i^{II} = 0.5085$). As before, we compare the results in Figure 8 between the invertible (top panels) and non-invertible (bottom panels) models. In addition, if we impose the same parameter estimates in simulating the SW model, we find that II introduces more persistence compared with PI with the longer drawn-out responses following this particular shock (we refer to Appendix F.1 panel (b) for details). This implies that persistence is endogenously generated which should lead to a better fit of the data without relying on other persistence mechanisms (e.g., AR(1) shock processes). This is a well-known finding and has been extensively discussed in Collard *et al.* (2009) and Levine *et al.* (2012).

However, if we focus on Figure 8 below, our main finding of this paper can be clearly revealed again. In particular, the bottom panels show how the IRFs from the SVAR may be badly misspecified and are therefore less able to recover the DGP which does generate more hump-shaped responses and endogenous persistence in the IRFs (especially comparing the left panels). When it comes to matching the higher order responses, the evidence is even clearer, with the Case 1 SVAR generally fitting better the dynamics seen in the DSGE model, while the implied VAR responses produced by Case 2 match very poorly the DGP counterparts towards the end of the horizon.

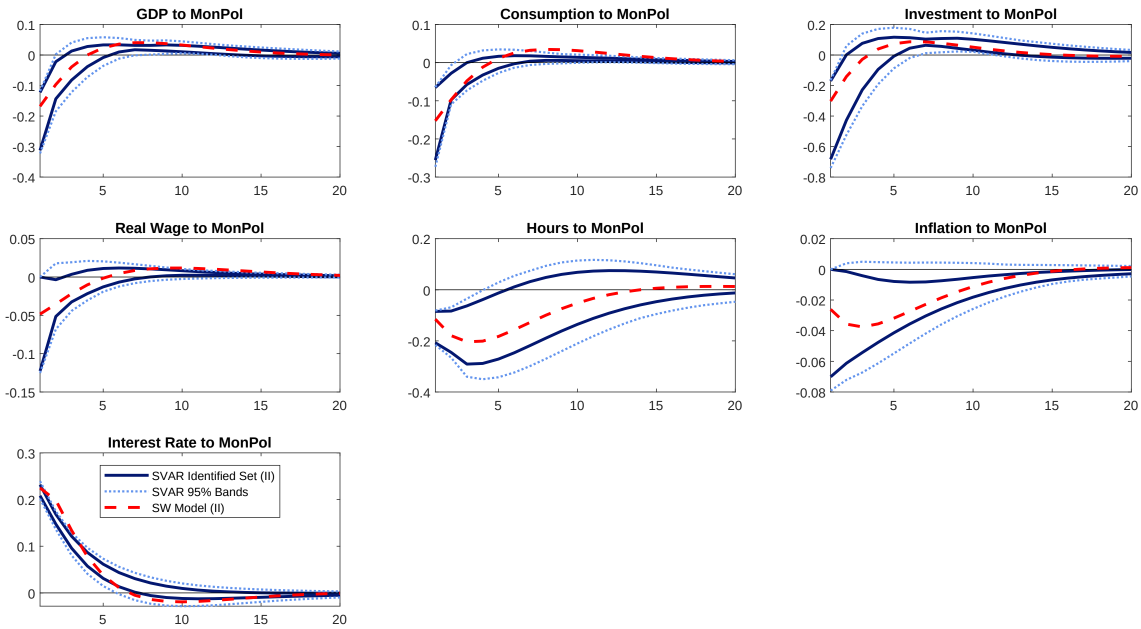
Finally, our findings are consistent across the different shocks that we identify for the VAR and DSGE models but are not E- and A-invertible in the latter. For example, the same conclusions above can be drawn for the same variables we have seen (i.e., inflation and the interest rate) if we look at the preference shock ($\mathbb{F}_b^{II} = 0.9526$) in Figure 9 below. In other words, the ability of identifying assumptions for the SVAR to recover the DGP response worsens with a non-invertible non-fundamental system between Case 1-PI and Case 2-II.

In addition, as the final structural shock identified by the DSGE Case 2 and SVAR, the bounds on the FEV in Table 4 are much more restrictive for the inflation objective shock suggesting the narrow identified sets reported in the empirical IRFs. Despite the narrower sets because of the added information and that the SVAR can successfully pin down the sign of all the SW IRFs, the performance of VARs in matching the DGP seems to be much worse compared with the other figures due to the fundamentalness problem associated with this shock. Appendix F.5 shows similar results for the other shocks and individual IRFs.

Overall, our main findings on the potential (in)ability of an SVAR to match IRFs of a DGP,

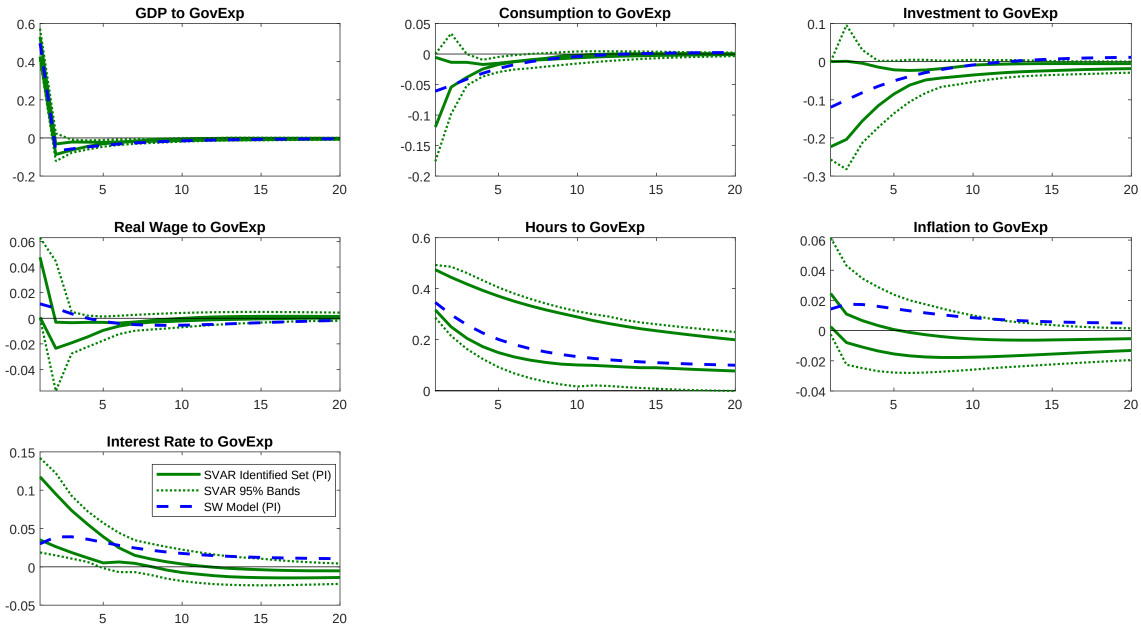


(a) Case 1 (PI)

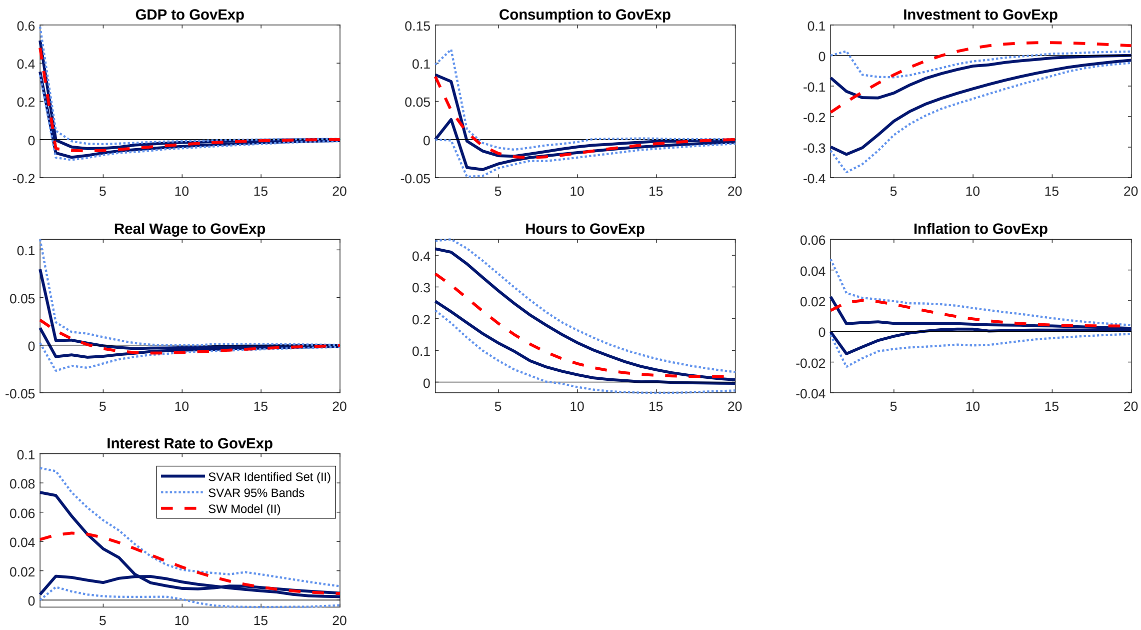


(b) Case 2 (II)

Figure 6: **Responses to Monetary Policy Shock (BoundsFEV)**
Notes: In each panel, the solid lines plot the posterior means of the VAR response set bounds with the corresponding 95% band of the set (dotted). The dashed blue/red lines are the SW-PI/SW-II responses for Case 1/Case 2. $\mathbb{F}_r^{II} = 0.0036$.



(a) Case 1 (PI)



(b) Case 2 (II)

Figure 7: **Responses to Government Spending Shock (BoundsFEV)**

Notes: In each panel, the solid lines plot the posterior means of the VAR response set bounds with the corresponding 95% band of the set (dotted). The dashed blue/red lines are the SW-PI/SW-II responses for Case 1/Case 2. $\mathbb{F}_g^{II} = 0.0194$.

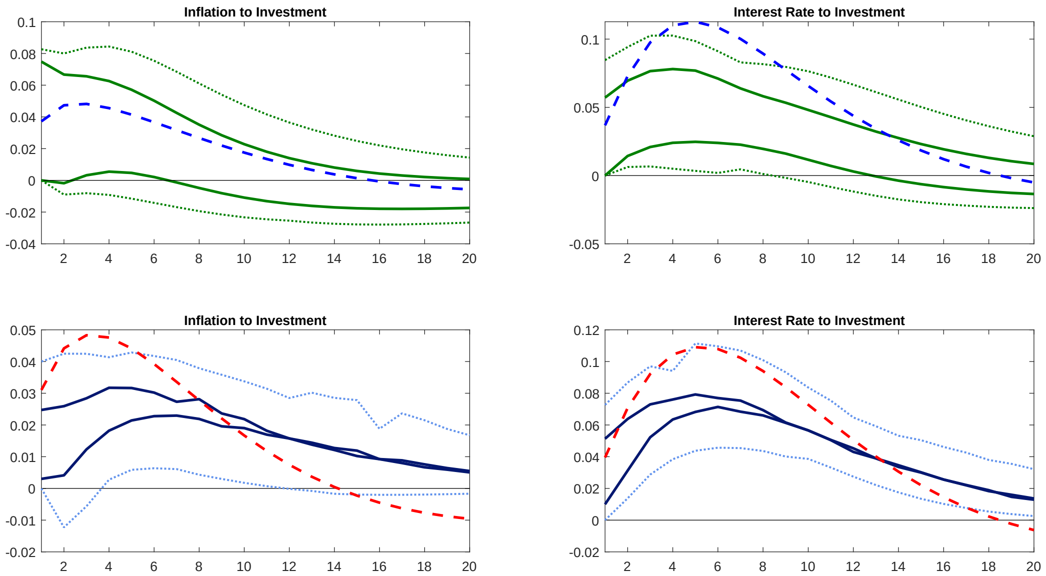


Figure 8: **Responses to Investment Specific Shock (BoundsFEV)**
Notes: Case 1 (top panels) and Case 2 (bottom panels). The two variables are inflation (left) and the interest rate (right). In each panel, the solid lines plot the posterior means of the VAR response set bounds with the corresponding 95% band of the set (dotted). The dashed red lines are the SW-II responses for Case 2 and the dashed blue line are the SW-PI responses for Case 1. $\mathbb{F}_i^{II} = 0.5085$.

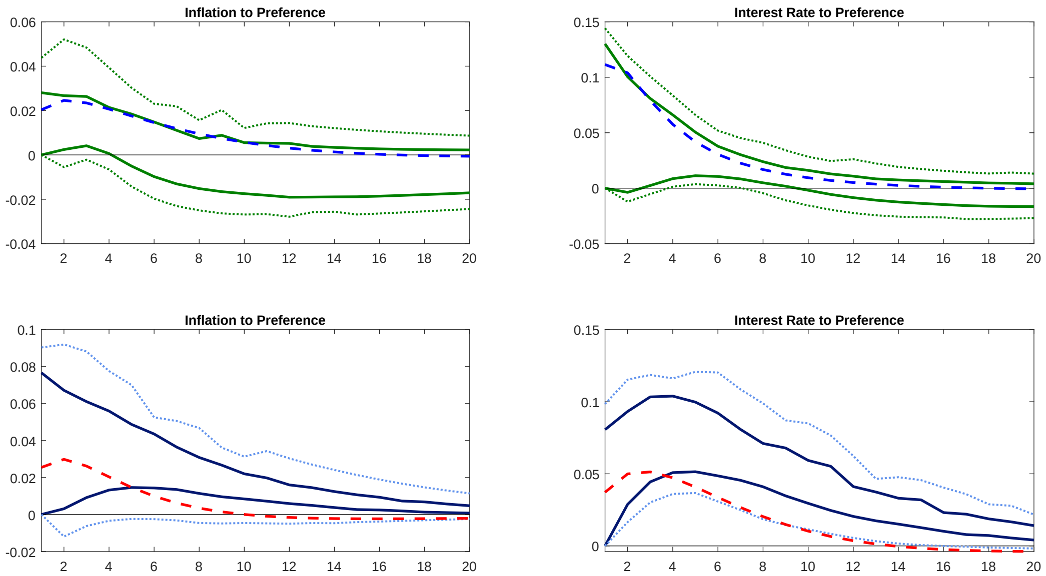


Figure 9: **Responses to Preference Shock (BoundsFEV)**
Notes: Case 1 (top panels) and Case 2 (bottom panels). The two variables are inflation (left) and the interest rate (right). In each panel, the solid lines plot the posterior means of the VAR response set bounds with the corresponding 95% band of the set (dotted). The dashed red lines are the SW-II responses for Case 2 and the dashed blue line are the SW-PI responses for Case 1. $\mathbb{F}_b^{II} = 0.9526$.

based on a well-established estimated DSGE model, are robust to a variety of identification schemes that we use to address the identification problem: zero restrictions, mixed sign and zero restrictions, sign restrictions with uniform prior, sign restrictions with distribution-free sets, and sign restrictions with theory-driven bounded FEV. We also carry out a number of checks and find that the results are robust to the choice of variables in the SVAR (Cholesky), lag order (see Section 5.4 below), different horizons for the sign and FEV constraints, prior specification, and a large number of random parameter draws from the DGP posterior distributions.

4.6 Can the Econometrician By-pass SVARs and Estimate IRFs Directly?

Using the method of local projections (LP) of Jorda (2005), we can indeed bypass the intervening step of a VAR. The LP approach uses "external instruments" which are variables correlated with a particular shock of interest, but not with the other shocks. External instruments can then be used to directly estimate causal effects by direct instrumental variables (IV) regressions. This method does not require invertibility, but *does* require good instruments which, for many shocks, may not be available to the econometrician.

Stock and Watson (2018) compares the LP-IV approach with a more efficient SVAR-IV approach and proposes a new test for invertibility which is applied to the study of Gertler and Karadi (2015). Plagborg-Moller and Wolf (2021), building on Stock and Watson (2018), show that the addition of an instrumental variable, whether external or internal, to the econometrician's information set may enable estimation of at least a scaling of the true IRF even when structural shocks are non-invertible. However, in the context of our paper which stresses the information problem of agents in the model, this then begs the question why agents are not able to observe the additional information as well. What are the consequences of agents having this additional source of information?[21]

A second alternative that bypasses SVARs is to follow the traditional route of the RBC model literature and simply stick with the estimation of the structural model by Bayesian (as in this paper) or GMM methods. Validation then proceeds by comparing second moments of the estimated model with those of the data.

5 Cumulative Mean Square Distance

As we have established that the potential reasons for the IRF differences could be due to the problems of (1) approximate invertibility; (2) with the identification; and (3) the lag length of the SVAR fitted to a large number of variables. In this section, we address each of these points in turn in order to gain further insights into the IRF estimations. We do so by defining a metric to measure the square distance from the true responses. We also examine the performance from the several different identification schemes using the above analysis by focusing on the retrievable shocks (ε_r and ε_g). The results can be informative for the empirical researchers about the reliability of identification schemes.

5.1 SW Model Case 2: Perfect vs Imperfect Information

We first compute a measure of the cumulative difference that corresponds to the analysis in Section 3.3 and Figure 2. For the main shocks that we have identified for the SVAR estimation including the technology shock again, we focus on comparing the responses based on the same

[21]See Levine *et al.* (2022) for more discussion and analysis of this point.

parameterization (i.e., II simulations with II estimates vs. PI simulations with II estimates) to isolate the effects of information on IRFs. The cumulative difference is given by

$$d_H^m = \sum_{h=0}^{H} |[IRF_{SW}^{m=PI}(h,\theta)] - [IRF_{SW}^{m=II}(h,\theta)]| \tag{47}$$

where d_H^m measures the distance between the impulse responses accumulated from $h = 0$ to H. m is the information assumption index ($m = PI, II$). $|\cdot|$ stands for the Euclidean norm which we take to be the mean square distance between two trajectories. Table 5 compares the results as an additional indication of pure A-invertibility and of the wedge between the IRFs arisen solely from the different information assumptions.

	$dlGDP_t$	$dlCON_t$	$dlINV_t$	$dlWAG_t$	HOU_t	$dlCPI_t$	FED_t	q_t	mc_t	d_H Total
Technology ($\mathbb{F}_i^{II} = 0.0004$)	0.0003	0.0004	0.0004	0.0001	0.0005	0.0002	0.0001	0.0014	0.0003	0.0037
MonPol ($\mathbb{F}_i^{II} = 0.0036$)	0.0005	0.0005	0.0011	0.0003	0.0009	0.0003	0.0010	0.0026	0.0016	0.0087
GovExp ($\mathbb{F}_i^{II} = 0.0194$)	0.0107	0.0166	0.0090	0.0029	0.0197	0.0026	0.0042	0.0612	0.0109	0.1378
Investment ($\mathbb{F}_i^{II} = 0.5085$)	0.0185	0.0295	0.0199	0.0053	0.0328	0.0028	0.0070	0.1166	0.0185	0.2509
PMarkup ($\mathbb{F}_i^{II} = 0.6655$)	0.0256	0.0183	0.0637	0.0175	0.0611	0.0203	0.0244	0.0744	0.0634	0.3686
Preference ($\mathbb{F}_i^{II} = 0.9526$)	0.0487	0.0747	0.0365	0.0131	0.0942	0.0089	0.0222	0.2496	0.0521	0.6001

Table 5: **Cumulative Mean Square Distance – Comparison of PI and II solutions in Estimated SW Model**

It clearly shows that the divergence in IRFs (measured by d_H Total) is consistent with the measure of approximate fundamentalness-invertibility ($\mathbb{F}_i^{II}$) from those VAR-identified shocks. Quantitatively, the table reaffirms the usefulness of our $\mathbb{F}_i^{II}$ measure of approximate invertibility-fundamentalness for each shock: the good approximation of the structural shock to the innovation for the technology and monetary policy shocks produces a cumulative difference very close to 0 for every individual IRF. The intuition of the IRF results for the estimated SW model has been discussed in detail in Section 3.3.

5.2 SVAR(1) and SW Model Cases 1 and 2

We further compute a measure of the cumulative difference between the responses to VAR and SW model structural shocks

$$d_H^m = \sum_{h=0}^{H} |[IRF_{VAR}^{m}(h,\theta)] - [IRF_{SW}^{m}(h,\theta)]| \tag{48}$$

for m =PI or II.

We present the cumulative of the Euclidean norm for the two cases under PI and II as the horizon increases ($H = 20$). We can gain further understanding on (i) the difference of responses to shocks; (ii) how the cumulative of the Euclidean distance changes over time with the horizon after the shock hits the system. We begin with the estimated SVAR(1) identified by sign restrictions with theory-driven bounded FEV. Table 6 reports the cumulative mean square distance (CMSD) for the two cases when $p = 1$ and $H = 20$.

d_H^m measures the cumulative distance between the posterior means of the VAR responses and SW model responses. The benefit of this exercise is allow us to quantitatively study the degree and effects of invertibility and/or identification in response to a shock. Based on the statistics reported, not surprisingly, Table 6 shows that the cumulative differences between the VAR and DSGE responses are the smallest for the monetary policy and government spending shocks, the

	$dlGDP_t$	$dlCON_t$	$dlINV_t$	$dlWAG_t$	HOU_t	$dlCPI_t$	FED_t	d_H Total
	Case 1 d_H^{PI} $(H = 20)$							
MonPol	0.011	0.016	0.020	0.007	0.014	0.004	0.015	0.089
GovExp	0.008	0.006	0.016	0.007	0.054	0.018	0.021	0.131
Investment	0.020	0.032	0.156	0.010	0.157	0.010	0.036	0.420
PMarkup	0.032	0.029	0.056	0.030	0.239	0.052	0.073	0.511
Preference	0.024	0.048	0.021	0.009	0.108	0.010	0.019	0.239
	Case 2 d_H^{II} $(H = 20)$							
MonPol ($\mathbb{F}_i^{II} = 0.0036$)	0.014	0.016	0.037	0.006	0.022	0.005	0.016	0.115
GovExp ($\mathbb{F}_i^{II} = 0.0194$)	0.011	0.012	0.082	0.007	0.014	0.011	0.010	0.147
Investment ($\mathbb{F}_i^{II} = 0.5085$)	0.017	0.024	0.071	0.013	0.033	0.015	0.021	0.194
PMarkup ($\mathbb{F}_i^{II} = 0.6655$)	0.012	0.020	0.046	0.031	0.093	0.029	0.031	0.263
Preference ($\mathbb{F}_i^{II} = 0.9526$)	0.043	0.047	0.076	0.019	0.107	0.013	0.027	0.331

Table 6: **Cumulative Mean Square Distance (BoundsFEV)**

shocks that are approximately fundamental according to the $\mathbb{F}_i^{II}$ measures, even for the non-square Case 2. These values are close to being economically insignificant in terms of the square distance in their cumulative responses of all the observable variables (d_H in the last column is close to 10 percentage points for ε_r for example). The preference shock ($\mathbb{F}_i^{II} = 0.9526$), on the other hand, reports the largest cumulative distance generated by Case 2 estimated under II, which is again consistent with our results above based on the IRF figures and invertibility table. In particular, Table 6 clearly indicates, for Case 2 solved and simulated under the relevant II assumption, a monotonic relationship between our measures of approximate invertibility-fundamentalness, $\mathbb{F}_i^{II}$, and d_H Total.

Based on the estimated VAR responses using the bounds FEV identification restrictions, nearly all the CMSD measures to each shock (for $H = 20$ periods) are close to being economically insignificant (i.e., $d_H < 10$ percentage points). In most cases, the introduction of II makes the identifying restrictions less able to recover the DGP. The two exceptions are the responses of hours to the investment specific (with a large $d_H = 0.157$) and mark-up shocks ($d_H = 0.239$) for the invertible Case 1. One potential explanation for this is because $p = 1$ lag may not enough for fitting the SVAR to this observable. We turn to Section 5.4 for examining the increased lags of the SVAR for all the seven variables.

5.3 Identification of Retrievable Shocks

It is also useful to compare these statistics generated by the various identification schemes but only for the retrievable shocks in the SVAR: the monetary policy and government spending shocks. For the non-retrievable ones, the results are non-informative since no identification schemes can be successful. Table 7 can help make a clear recommendation about the performance of our most reliable identification scheme. BoundsFEV produces the smallest and most acceptable d_H across the different assumptions and for all the observable variables whereas the Cholesky scheme fails considerably in recovering most of the IRFs especially for the PI Case 1 which is invertible (for example, d_H Total $= 1.057$).

It is interesting to note that, even for these retrievable shocks identified by the superior BoundsFEV scheme, the fundamentalness problem worsens for the overall performance of VARs under II when adding the additional shocks in the non-invertible Case 2. This is clearly evident that, when we are able to minimise the identification uncertainty, the informational assumption plays a key role in our model's (in)ability to recover the DGP responses (i.e., d_H Total has risen to 0.115 and 0.147 for the monetary policy and government spending shocks, respectively). The

	$dlGDP_t$	$dlCON_t$	$dlINV_t$	$dlWAG_t$	HOU_t	$dlCPI_t$	FED_t	d_H Total
	Case 1 d_H^{PI} $(H = 20)$							
	BoundsFEV							
MonPol	0.011	0.016	0.020	0.007	0.014	0.004	0.015	0.089
GovExp	0.008	0.006	0.016	0.007	0.054	0.018	0.021	0.131
	Sign Restrictions							
MonPol	0.051	0.029	0.127	0.052	0.107	0.038	0.056	0.459
GovExp	0.051	0.038	0.130	0.059	0.267	0.048	0.047	0.641
	Zero Restrictions							
MonPol	0.085	0.062	0.412	0.159	0.247	0.034	0.058	1.057
GovExp	0.358	0.027	0.038	0.024	0.163	0.144	0.031	0.784
	Case 2 d_H^{II} $(H = 20)$							
	BoundsFEV							
MonPol	0.014	0.016	0.037	0.006	0.022	0.005	0.016	0.115
GovExp	0.011	0.012	0.082	0.007	0.014	0.011	0.010	0.147
	Sign Restrictions							
MonPol	0.052	0.036	0.109	0.055	0.091	0.043	0.077	0.463
GovExp	0.045	0.053	0.111	0.061	0.193	0.048	0.048	0.561
	Zero Restrictions							
MonPol	0.150	0.051	0.289	0.146	0.239	0.030	0.077	0.982
GovExp	0.307	0.046	0.067	0.024	0.143	0.091	0.039	0.717

Table 7: **Cumulative Mean Square Distance (BoundsFEV vs SR vs ZR)**

sign-VARs also deliver very large set estimates implying that the uncertainty around these estimates is mostly identification uncertainty (similar to the Cholesky case). In addition to the informational assumptions, not surprisingly, identification uncertainty also plays a key role in recovering the DGP responses. The key result here is that, further to our brief discussion in Figure 5, Table 7 is able to quantify and measure the superiority of the BoundsFEV identification scheme that we apply to tackle the information/invertibility issue when using SVARs for validation of a theoretical model.

5.4 Lags of the SVAR

This is our final check. In this section, we evaluate two additional specifications to test the robustness of the main results. As we simulate and work on quarterly data, we check the VAR responses with $p > 1$ as some may argue that the responses of some observables (e.g., the real GDP) to these shocks may be sluggish. While the information criteria such as AIC or BIC suggest that the optimal number of lags is between 1 and 2, we allow for sufficiently long lags (up to 5 lags) and report the CMSD results with $p = 2$ and $p = 3$ in Table 8. Along with a number of robustness checks that have been conducted in the previous section, we need to check if our major finding in the paper withstands the choices of lag length when we re-estimate the same SVAR identified by the BoundsFEV scheme.

The results are broadly consistent with those in Section 5.2. In particular, for the non-square Case 2 estimated under II, the preference shock shows the largest cumulative difference whereas the two retrievable shocks report the smallest cumulative differences between the VAR and DSGE responses. If we look closely at the d_H Total column, note that increasing $p > 1$ appears to worsen the VAR performance for the least fundamental shock (Preference) and the most fundamental shock (MonPol), suggesting a larger deviation from the DGP responses compared

	$dlGDP_t$	$dlCON_t$	$dlINV_t$	$dlWAG_t$	HOU_t	$dlCPI_t$	FED_t	d_H Total
$p = 2$								
	Case 1 d_H^{PI} $(H = 20)$							
MonPol	0.011	0.011	0.029	0.015	0.019	0.006	0.017	0.108
GovExp	0.009	0.010	0.021	0.007	0.023	0.012	0.020	0.101
Investment	0.023	0.031	0.146	0.006	0.162	0.006	0.038	0.413
PMarkup	0.027	0.025	0.048	0.026	0.189	0.051	0.060	0.425
Preference	0.032	0.054	0.018	0.013	0.122	0.007	0.022	0.268
	Case 2 d_H^{II} $(H = 20)$							
MonPol ($\mathbb{F}_i^{II} = 0.0036$)	0.018	0.008	0.058	0.006	0.034	0.007	0.006	0.137
GovExp ($\mathbb{F}_i^{II} = 0.0194$)	0.009	0.010	0.075	0.012	0.011	0.002	0.004	0.123
Investment ($\mathbb{F}_i^{II} = 0.5085$)	0.021	0.019	0.079	0.011	0.051	0.016	0.024	0.221
PMarkup ($\mathbb{F}_i^{II} = 0.6655$)	0.010	0.019	0.021	0.031	0.077	0.030	0.028	0.216
Preference ($\mathbb{F}_i^{II} = 0.9526$)	0.044	0.050	0.077	0.020	0.118	0.019	0.032	0.359
$p = 3$								
	Case 1 d_H^{PI} $(H = 20)$							
MonPol	0.014	0.011	0.034	0.015	0.031	0.009	0.016	0.130
GovExp	0.014	0.011	0.029	0.008	0.036	0.012	0.020	0.131
Investment	0.023	0.028	0.138	0.009	0.147	0.009	0.038	0.392
PMarkup	0.025	0.021	0.047	0.025	0.148	0.048	0.057	0.371
Preference	0.041	0.060	0.019	0.015	0.176	0.009	0.025	0.345
	Case 2 d_H^{II} $(H = 20)$							
MonPol ($\mathbb{F}_i^{II} = 0.0036$)	0.018	0.009	0.057	0.010	0.021	0.006	0.012	0.132
GovExp ($\mathbb{F}_i^{II} = 0.0194$)	0.011	0.010	0.076	0.013	0.015	0.004	0.005	0.134
Investment ($\mathbb{F}_i^{II} = 0.5085$)	0.022	0.022	0.085	0.012	0.056	0.014	0.020	0.230
PMarkup ($\mathbb{F}_i^{II} = 0.6655$)	0.015	0.015	0.039	0.028	0.060	0.027	0.025	0.209
Preference ($\mathbb{F}_i^{II} = 0.9526$)	0.041	0.049	0.067	0.019	0.116	0.018	0.031	0.341

Table 8: **Cumulative Mean Square Distance (BoundsFEV) Based on SVAR(2) and SVAR(3)**

to the case when $p = 1$.

However, another result is worth noting here. As mentioned, the responses of hours to the investment specific (with a large $d_H = 0.157$) and price mark-up shocks ($d_H = 0.239$) for the invertible Case 1 generate a large CMSD between the VAR and DSGE IRFs. Sufficiently long lags may be needed for estimating the effects of these two shocks on hours even though linear information criteria such as AIC or BIC often result in more parsimonious models. Indeed, in the case of SVAR(2), d_H for the mark-up shock has decreased to 0.189, and when estimating the SVAR(3), these statistics have become 0.147 and 0.148 for the investment and mark-up shocks, respectively.[22] As a consequence, Table 8 reports an overall pattern of reduction in d_H Total for these two shocks as p increases.

6 Conclusions and Future Research

Can indeed SVAR methods be employed to recover the structural shocks and impulse response functions if the data generating process is a DSGE model? In this paper, we tackled this question by addressing both the invertibility and identification issues, thus providing a novel procedure to uncover the potential (in)ability of an SVAR to match the structural IRFs of DSGE models. The source of non-invertibility in our paper is the imperfect information of the agents in the DSGE model, the assumed DGP.

By generating artificial data using the appropriate DSGE model assumptions and estimating

[22]To save space, we only report the CMSD results from estimating the SVAR(2) and SVAR(3) – our main results hold up to 5 lags.

several identified SVARs, we studied and revealed two sources of potential misspecification for the each shock we identified for the SVAR in turn: (1) invertibility for each shock using our $\mathbb{F}_i^{II}$ measure of approximate invertibility-fundamentalness and (2) inappropriate identification restrictions highlighting our contribution using the identifying scheme employing theory-driven bounds on the forecast error variance. Our application based on an industry standard DSGE model yielded very strong results that withstood a wide array of tests and checks and provided a clear-cut answer to the research question. In doing so, we provide a methodology which is completely general for the macro-econometrics literature and should precede any SVAR validation of a particular model using actual data.

There is some good news to report on both absolute and fundamental invertibility summarized in the main results reported in Section 1.1. However, for some shocks, the results indicated that SVARs cannot be used to compare IRFs with those of a DSGE model. It is important to stress that our computational results are only specific to a well-established medium-sized NK model and more (or less) severe invertibility and identification problems could well emerge with other examples. In particular, one feature of DSGE models that might prove important in this respect are uncertainty shocks which have driven an important literature on business cycles in recent years for which Fernandez-Villaverde and Guerron-Quintana (2020) provides a very useful review. They show how stochastic volatility can be conveniently modelled in linear models by adding time-varying standard deviations as a ARMA (possibly an AR(1)) process. If we allow for such volatility for every shock process in our model this then doubles the number of shocks and accentuates the non-invertibility problem. However pursuing this research objective would require an II solution that captures non-linearity and goes beyond the linear Kalman Filter utilized in our paper.

Another area for future research is in models with heterogeneous agents and dispersed information alluded to in the Subsections 1.2 and 2.5. Providing *general* solutions to the solution of dynamic RE models with dispersed information in a heterogeneous agents setting remains a major challenge.[23]

Finally, NK models with financial frictions often include a large number of financial shocks, not necessarily matched with data, thus moving further away from the (possibly) invertible square structure. This and other sources of non-invertibility, coupled with our measure of approximate invertibility and the identification method of Volpicella (2021), suggest possible areas for future research into the relationship between SVARs and DSGE models.

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[23] As alluded to in Section 2.5, Levine *et al.* (2022) provide a first step by solving a general class of heterogeneous agents models in the empirically realistic case where idiosyncratic far exceeds aggregate uncertainty. The empirical implications for medium-sized NK models therefore remain unexplored.

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Appendix

A Proof of the PMIC (Theorem 2)

From (19) we have $\epsilon_t = \tilde{D}^{-1}(m_t^E - \tilde{C}Ls_t)$ where L is the lag operator and $(I - \tilde{A}L)s_t = \tilde{B}\epsilon_t = \tilde{B}\tilde{D}^{-1}(m_t^E - \tilde{C}Ls_t)$ from which we obtain $s_t = [I - (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})L]^{-1}\tilde{B}\tilde{D}^{-1}m_t^E$ and hence

$$\epsilon_t = \tilde{D}^{-1}(m_t^E - \tilde{C}s_{t-1}) = \tilde{D}^{-1}(m_t^E - \tilde{C}[I - (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})L]^{-1}\tilde{B}\tilde{D}^{-1}m_{t-1}^E) \tag{A.1}$$

Expanding $(I - X)^{-1} = I + X + X^2 + \cdots$ we then have

$$\epsilon_t = \tilde{D}^{-1}\left(m_t^E - \tilde{C}\sum_{j=1}^{\infty}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^j\tilde{B}\tilde{D}^{-1}m_{t-j}^E\right) \tag{A.2}$$

A necessary and sufficient condition for the summation to converge is that $\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}$ has stable eigenvalues (eigenvalues within the unit circle in the complex plane).

In ABE form in the Theorem we have:

$$\tilde{C}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^j = \tilde{E}\tilde{A}(\tilde{A} - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}\tilde{A})^j = \tilde{E}\tilde{A}(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E})\tilde{A})^j \tag{A.3}$$

Define $X = (I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E})$. Then we use

$$A(XA)^j = (AX)^jA \tag{A.4}$$

This can be easily proved by induction: Suppose it is true for $j = n$. Then $A(XA)^{n+1} = AXA(XA)^n = AX(AX)^nA = (AX)^{n+1}A$. Hence it is true for $j - n + 1$. But the result is true for $j = 1$ (scalars). Hence true for $j = 2, j = 3, ...$

Using (A.4) we then have

$$\tilde{C}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^j = \tilde{E}\tilde{A}(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E})\tilde{A})^j = \tilde{E}(\tilde{A}(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}))^j\tilde{A} \tag{A.5}$$

so that the PMIC requirements are that $\tilde{E}\tilde{B}$ is invertible and that $\tilde{A}(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E})$ has stable eigenvalues as in the main text.

B Proof of Theorem 3

Using the expressions (20)–(23) for II, and the invertibility requirement that $\tilde{A} - \tilde{A}\tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}$ has stable eigenvalues, we calculate the latter as the matrix

$$\begin{bmatrix} A - AP^AJ'(EP^AJ')^{-1}E & 0 \\ -F(I - P^AJ'(JP^AJ')^{-1}J)(JB)^{-1}JP^AJ'(EP^AJ')^{-1}E & F(I - B(JB)^{-1}J) \end{bmatrix} \tag{B.6}$$

If $F(I - B(JB)^{-1}J)$ has eigenvalues outside the unit circle, it immediately follows that II is not E-invertible. If its the eigenvalues are inside the unit circle, it follows that the solution to (18) is $P^A = BB'$; this is because the Convergence Condition for P^A is that $F - FP^AJ'(JP^AJ')^{-1}J = F(I - B(JB)^{-1}J)$ is a stable matrix. Furthermore it follows that $A - AP^AJ'(EP^AJ')^{-1}E = A(I - B(EB)^{-1}E)$, so that (B.6) is a stable matrix as required for invertibility.

To show that invertibility implies that II and PI are equivalent, we note that (13) now implies that $\tilde{z}_t = B\epsilon_t + (F(I - B(JB)^{-1}J))^t \tilde{z}_0$, which in dynamic equilibrium implies $\tilde{z}_t = B\epsilon_t$. This implies that $z_{t+1,t} = Az_{t,t-1} + AB\epsilon_t$, and hence that $z_{t+1} = \tilde{z}_{t+1} + z_{t+1,t} = Az_{t,t-1} + AB\epsilon_t + B\epsilon_{t+1} = Az_t + B\epsilon_{t+1}$ as in the PI case. In addition, from (15), $m_t^A = Ez_{t,t-1} + E\tilde{z}_t = Ez_t$, also as in the PI case. If $F(I - B(JB)^{-1}J)$ is not a stable matrix, then $P^A \neq BB'$, and the overall dynamics of (12)–(15) are of a higher dimension than under PI.

C Proof of Theorem 4

If F is a stable matrix, consider the stochastic process y_t represented by

$$x_t = Fx_{t-1} + B\epsilon_t \qquad y_t = Jx_t \qquad cov(\epsilon_t) = I$$

where y_t and ϵ_t have equal dimension. This has AR roots given by the eigenvalues of F and MA roots that are the eigenvalues of $F(I - B(JB)^{-1}J)$. Given the Riccati matrix P^A, there exists a process $\tilde{y}_t$ with identical spectrum to y_t of the form

$$\tilde{x}_t = F\tilde{x}_{t-1} + P^A J'(JP^A J')^{-1}\eta_t \qquad \tilde{y}_t = J\tilde{x}_t \qquad cov(\eta_t) = JP^A J'$$

It follows from the formula above that the MA roots of this process are given by the eigenvalues of $F(I - P^A J'(JP^A J')^{-1}J)$.

Noting that the spectrum of a process in L-operator form will contain terms of the form $(1-aL)(1-aL^{-1})$, it follows that the roots of the MA part of the spectrum are pairwise inverses of one another. From this it automatically follows since y_t and $\tilde{y}_t$ have identical spectra, that the roots of $F(I - B(JB)^{-1}J)$ and $F(I - P^A J'(JP^A J')^{-1}J)$ are either identical or else reciprocals of one another.

Returning to the representation of the II solution, from the proof of Theorem 3, we have seen that the MA roots of the VARMA process include the eigenvalues of $F(I - B(JB)^{-1}J)$, while from (12)–(13), the AR roots include the eigenvalues of $F(I - P^A J'(JP^A J')^{-1}J)$. One or more of these are reciprocals of one another, as we have shown above. Hence the transfer function from shocks to observables incorporates at least one Blaschke factor. It follows that IRFs of structural shocks from the latter cannot be linear combinations of IRFs from VAR residuals, which will only mimic the IRFs from the innovations process.

D Impulse Response Functions and Generating Artificial Data

First, we rewrite the system (12) and (13) with a one-period lead

$$\begin{bmatrix} z_{t+1,t} \\ \tilde{z}_{t+1} \end{bmatrix} = \begin{bmatrix} A & A\left[P^A J'(JP^A J')^{-1}J - I\right] \\ 0 & F[I - P^A J'(JP^A J')^{-1}J] \end{bmatrix} \begin{bmatrix} z_{t,t-1} \\ \tilde{z}_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \epsilon_{t+1} \tag{D.7}$$

To obtain the impulse response for the underlying variables Y_t we use the relationship

$$Y_t = V_1 x_t + V_2 s_t \tag{D.8}$$

Recalling that $z_{t+1} = [\epsilon_{t+1}, s_t, x_t]'$, it follows that $s_t = [0 \;\; I \;\; 0]z_{t+1}$, and we may write

$$Y_t = V_1 x_t + \begin{bmatrix} 0 & V_2 & 0 \end{bmatrix} \left(Az_t + A\left[P^A J'(JP^A J')^{-1}J - I\right]\tilde{z}_t \right) \tag{D.9}$$

or more simply

$$Y_t = \begin{bmatrix} 0 & V_2 & V_1 \end{bmatrix} z_{t+1} = \begin{bmatrix} 0 & V_2 & V_1 \end{bmatrix} \begin{bmatrix} \epsilon_{t+1} \\ s_t \\ x_t \end{bmatrix} \tag{D.10}$$

To calculate the IRFs of observable states s_t, we know that, at time t, the first period response, using (D.7), is

$$I_{s,1} = \begin{bmatrix} A & A\left[P^A J'(JP^A J')^{-1} J - I\right] \\ 0 & F[I - P^A J'(JP^A J')^{-1} J] \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} \sigma_\epsilon \tag{D.11}$$

where σ_ϵ is the standard error of ϵ_t. So the first period IRF of Y_t can be obtained using (D.11) after a one-time shock.

To obtain a simulation with shocks happening every single period, we use the same strategy as above for simulating data. The only thing that is different is that we compute the sum of the IRFs from all of the past shocks when at each point in time a new random shock hits the above system.

E Online Appendix: Identification by Sign Restrictions

E.1 Literature Background

In our macroeconomic application, clearly, we cannot determine the sequence of causation in the model. Nevertheless, understandably zero contemporaneous recursive structures are hard to find within the DSGE context (even for identifying monetary disturbances). It is clear that we can draw on the information in (D.11) about the pattern of responses: e.g., restricting P in (37) to match a set of a priori sign restrictions in (D.11) would allow the structural shocks in the VAR to be interpreted by theoretical models. While (37) does not produce any zero restrictions or recursive structures, the model can produce a large number of non-parametric sign restrictions that can be used for the identification process (Canova, 2007). However, unlike the exact identification achieved using the zero restrictions (e.g., Cholesky identification), the sign-rotation process requires an unidentified system (37), making the zero-case impossible to achieve by rotation only.

From the early work of Uhlig (2005), the generation of the impulse vector is based on a Givens rotation for the monetary shocks (partial identification). In the case of Rubio-Ramirez *et al.* (2010) with pure sign restrictions, the candidate impact matrix is generated using a QR-decomposition (via additional linear restrictions in the Householder transformation matrix). Mountford and Uhlig (2009) provide the first important contribution and devise a penalty function approach that imposes restrictions on the model using numerical optimization methods to draw the structural parameters that is able to cope with multiple shocks simultaneously.

In terms of combining both sign and zero restrictions in SVAR models, Benati (2013) and Binning (2013) propose alternative ways to implement multiple zero restrictions in the context of sign identification scheme. For example, Benati and Lubik (2012) is one example of a series of papers that implement both sets of restrictions using special rotation matrices to generate candidate impact matrices, i.e., a combination of Householder transformations and Givens rotation matrices. In a Bayesian setting, Arias *et al.* (2018) provide an efficient algorithm that can correctly draw posterior of the structural parameters in order to impose sign and zero restrictions in SVAR models, involving Householder transformations and additional orthogonality restrictions.

E.2 Implementing the Algorithm

Implementing the identification algorithm involves the following general steps iteratively:

1. Estimate (37) to obtain $\hat{\hat{A}}_j$ and $\hat{\Sigma}_e$ (using either OLS or Step 2).

2. Draw $\hat{\hat{A}}_j$ and $\hat{\Sigma}_e$ from the posterior distribution of the reduced form parameters (e.g. Uhlig (1994), Uhlig (2005) and Arias *et al.* (2018), using a Bayesian approach).

3. Before starting the iterative stages, consider a Cholesky decomposition to orthogonalise shocks that do not necessarily satisfy the sign (and zero) restrictions: i.e. compute $P_{tr} = chol(\Sigma_e)$ where P_{tr} is the lower Cholesky factor of Σ_e.

4. At each iteration, draw a random orthonormal matrix Q (i.e. $QQ' = I_n$), so that

$$\Sigma_e = P_{tr}P_{tr}' = P_{tr}QQ'P_{tr}' = PP'$$

5. Keep the Q-draw if the transformed orthogonal impulse vector associated with the candidate impact matrix $P_{tr}Q$ fulfills a set of a priori restrictions, over a specific horizon (Table 3), and such that it still holds that the orthogonal shocks have the same variance-covariance matrix of the reduced form residuals

$$\Sigma_\epsilon = \mathbb{E}(Q'P_{tr}^{-1}e_te_t'P_{tr}'^{-1}Q) = I$$

6. Otherwise repeat Steps 4 and 5 until we obtain N replications. With each retained Q-transformation, the structural impulse responses are saved.

As noted, the type of decomposition for Q involved in Steps 4 and 5 differs slightly between Uhlig (2005) and Rubio-Ramirez *et al.* (2010). The way to parameterize Q in order to include orthonormality restrictions is using the Givens rotation matrices in Uhlig (2005) and using a QR decomposition of a random Normal matrix by Rubio-Ramirez *et al.* (2010). In Arias *et al.* (2018), the selection matrix that introduces zeros to Q is based on the QR decomposition and the Householder transformation matrix per model draw via some additional linear restrictions on each column of Q.

E.3 Estimation with Pure Sign-Based Restrictions

We estimate the SVAR(1) of (37) on data simulated from the SW model under PI and II. Following Uhlig (2005), and as described in Section E.2, in order to identify the shocks from the VAR errors we use sign restrictions implied by the $h = 0$ impact IRFs from the estimated SW model summarized in Table 3.

< Table 3 >

Online Appendix F.3 compares the IRFs of the estimated SVAR(1) model assuming PI and II for the 7 structural shocks with those from the estimated model for the invertible Case 1 and non-invertible Case 2. As in the previous section, we also provide a comparison between PI and II DGSE IRFs based on the same II estimated parameters. The reason for this additional check is that, in addition to the contrasting information solutions, the different estimated parameters including different estimates of persistence might drive the IRF differences – this is clearly evident in Figure 33 in Appendix F.3 for the price mark-up shock.[24] Given non-A-invertibility, agents cannot recover the shock processes, and our $\mathbb{F}^{II}$ values provide a useful measure of the divergence between the VAR IRFs and those implied by the DGP.

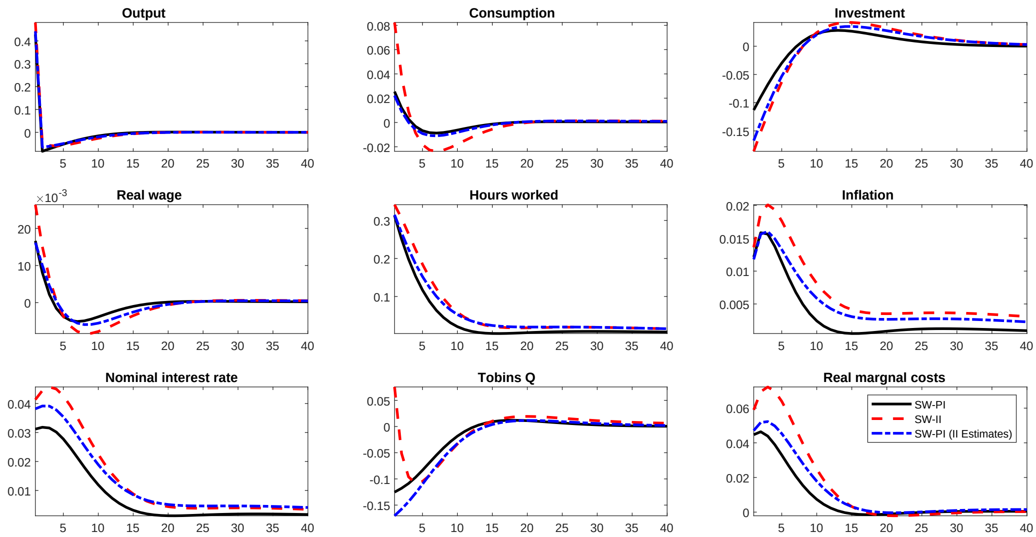
Our second robustness check follows the agnostic identification procedure in Arias *et al.* (2019) and estimates the SVAR by imposing mixed sign and zero restrictions for the identification of the monetary policy shock. The motivation for this additional check is that the set of admissible structural parameters satisfying the sign restrictions may produce very different or implausible implications for IRFs and other implied second moments (see, for example, Arias *et al.*, 2019). Again, our focus here is on the invertibility condition which does not seem to play a role across the II and PI estimations, but this is expected because the monetary policy shock is approximately invertible based on Table 2 ($\mathbb{F}_r^{II} = 0.0036$). In the literature, we are not aware of any meaningful mix of zero and sign restrictions for the shocks that, in our model, are not approximately fundamental.

If we now look closely at the invertible Case 1 from, for example, the responses to the government spending and investment specific shocks in Online Appendix F.3, some of the IRFs are not entirely successful in recovering the SW estimated IRFs even when there is PI, suggesting that the shortcomings of the estimated SVAR in terms of reproducing the IRFs of the DSGE model can also be due to the possible poor choice of the transformation between the reduced form errors and structural shocks even when the RE solution of model is invertible or at least approximately fundamental (i.e., the traditional identification problem highlighted in Section 4.1). Therefore, in Section 4.3, we turn to examining additional identification strategies to address the identification and estimation uncertainties in our SVAR and we aim to explicitly disentangle the potential effect of non-invertibility on validating DSGE models using empirical SVARs.

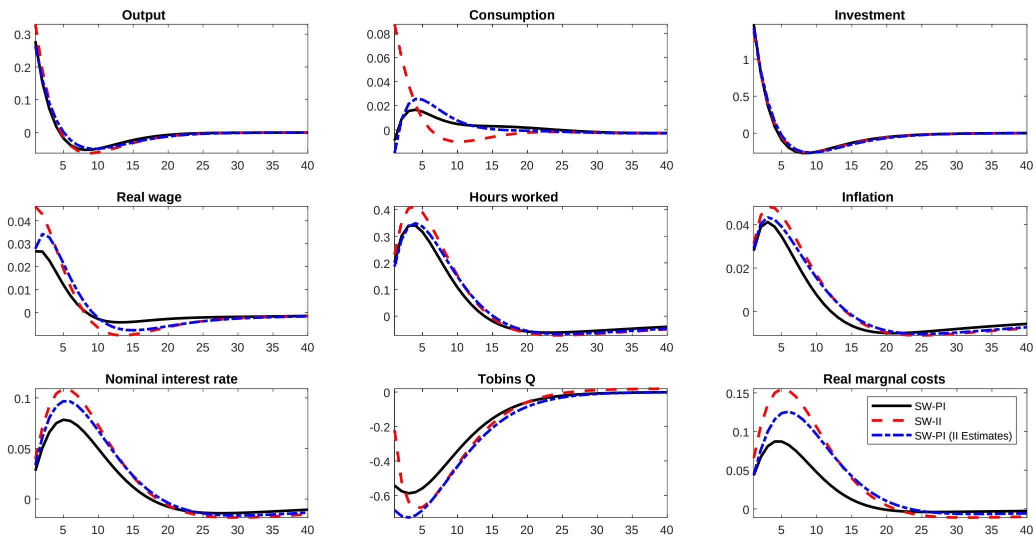
[24] Our complete analysis provides the results using PI simulated with II estimates for Case 2 and carries out the additional checks using the methods described in Section 5. We have already established that, for Case 2 (without A-invertibility and therefore E-invertibility), the valid informational equilibrium is II. Thus, to save space, in Sections 4.5 and 5, we only report the results comparing PI and II responses for Case 2.

F Online Appendix: Additional Impulse Response Figures

F.1 Comparison of PI and II solutions in Estimated SW Model



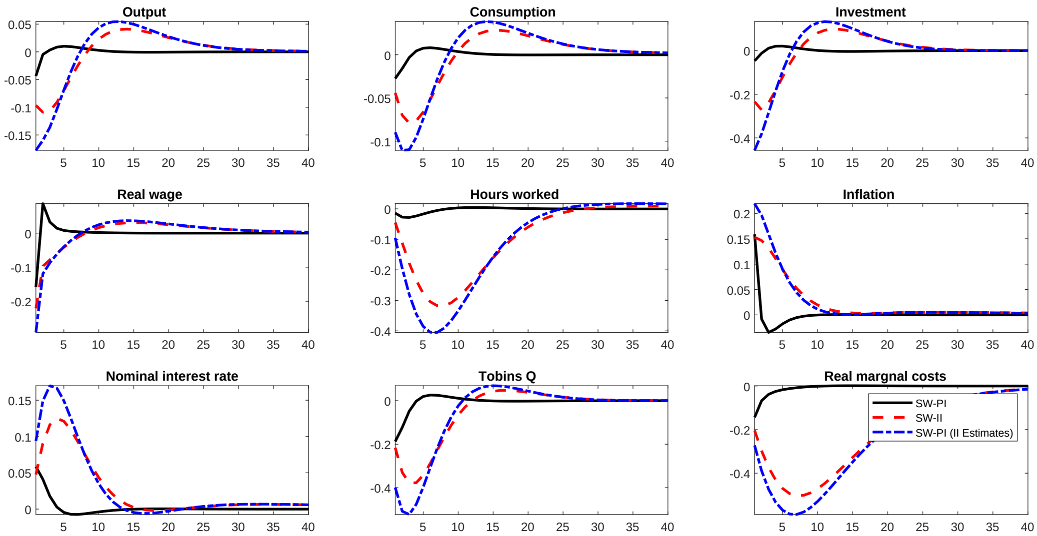
(a) Government Spending $\mathbb{F}^{II} = 0.0194$



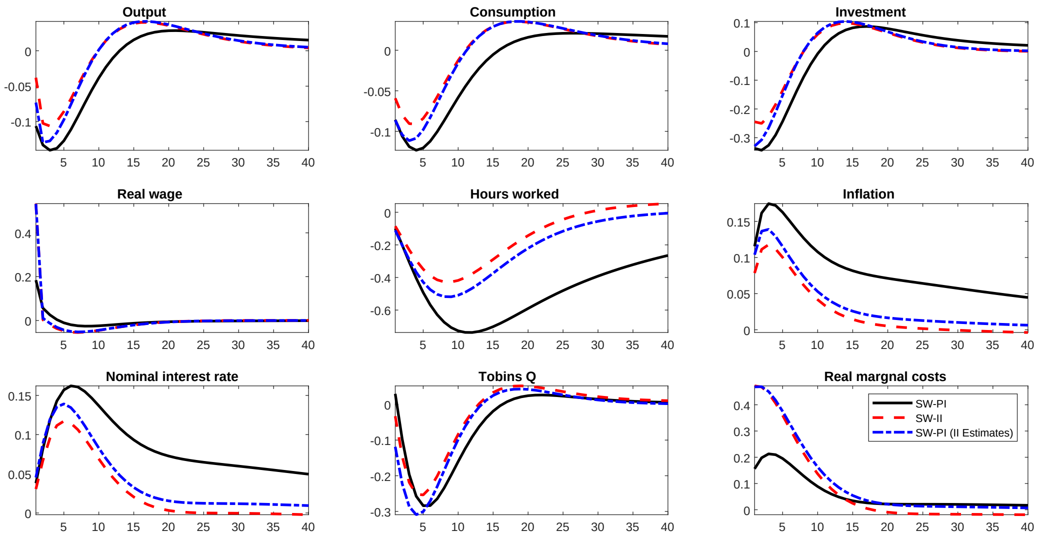
(b) Investment Specific $\mathbb{F}^{II} = 0.5085$

Figure 10: **Estimated SW Model Non-invertible Case 2**

Notes: Solid black line PI responses. Dashed red line II responses. Dashed blue line PI responses with II estimated parameters. Each panel plots the mean response corresponding a positive one standard deviation of the shock's innovation. Each response is for a 40 period (10 years) horizon and is level deviation of a variable from its steady-state value.



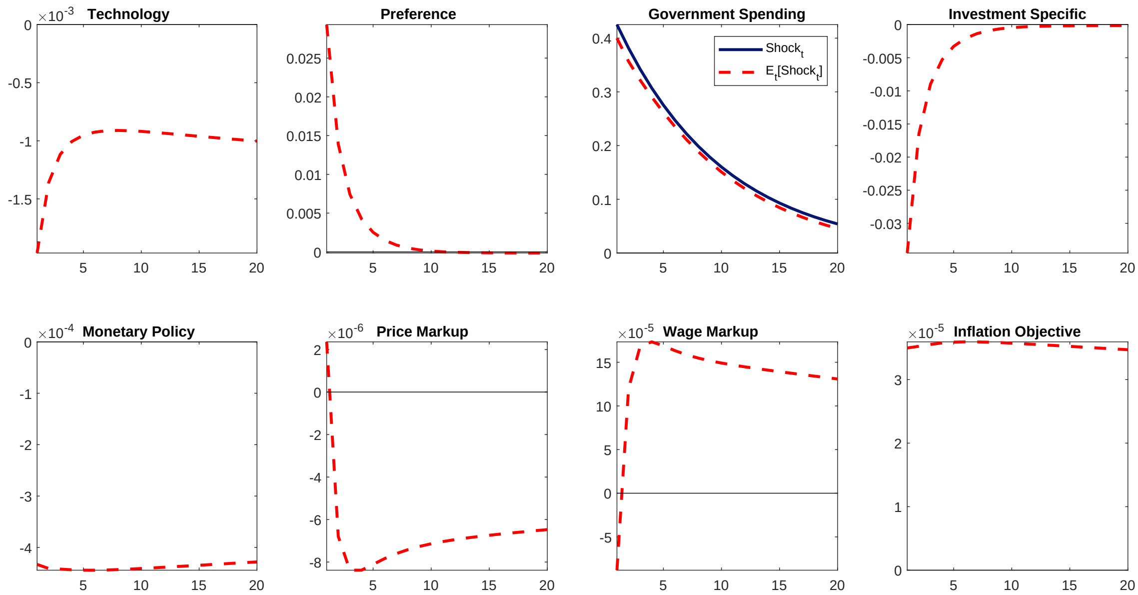
(a) Price Markup: $\mathbb{F}^{II} = 0.6655$



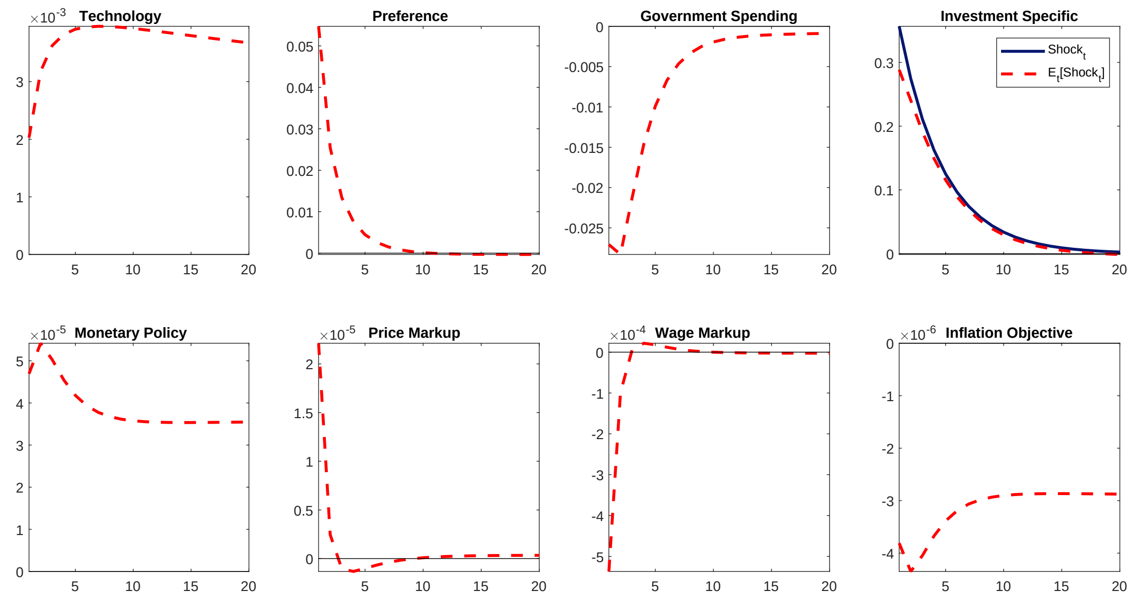
(b) Wage Markup: $\mathbb{F}^{II} = 0.0111$

Figure 11: **Estimated SW Model Non-invertible Case 2**

Notes: Solid black line PI responses. Dashed red line II responses. Dashed blue line PI responses with II estimated parameters. Each panel plots the mean response corresponding a positive one standard deviation of the shock's innovation. Each response is for a 40 period (10 years) horizon and is level deviation of a variable from its steady-state value.

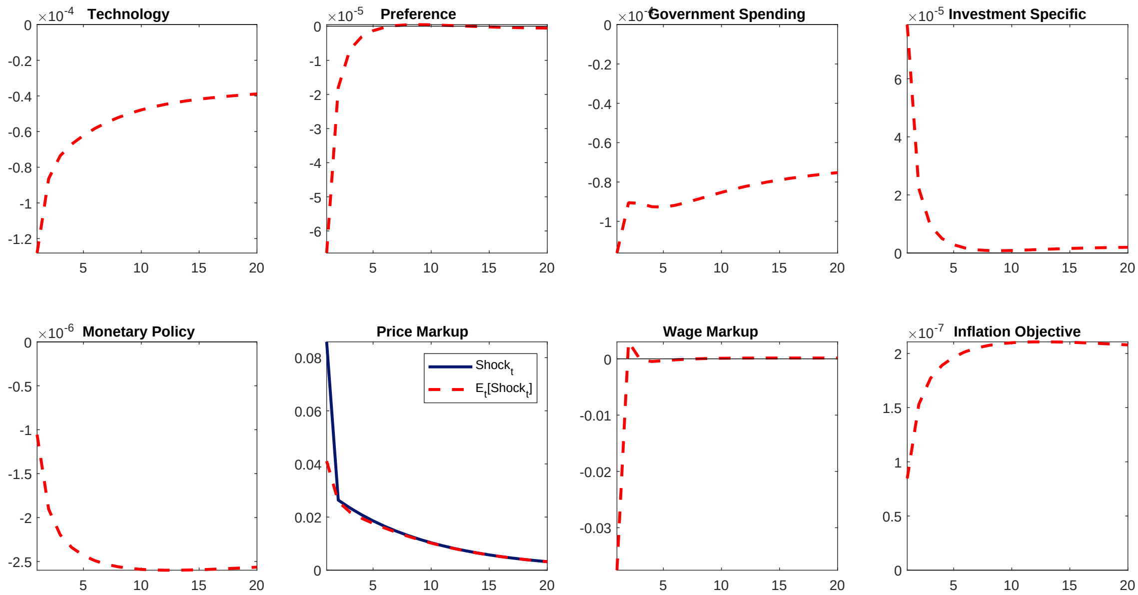


(a) Government Spending

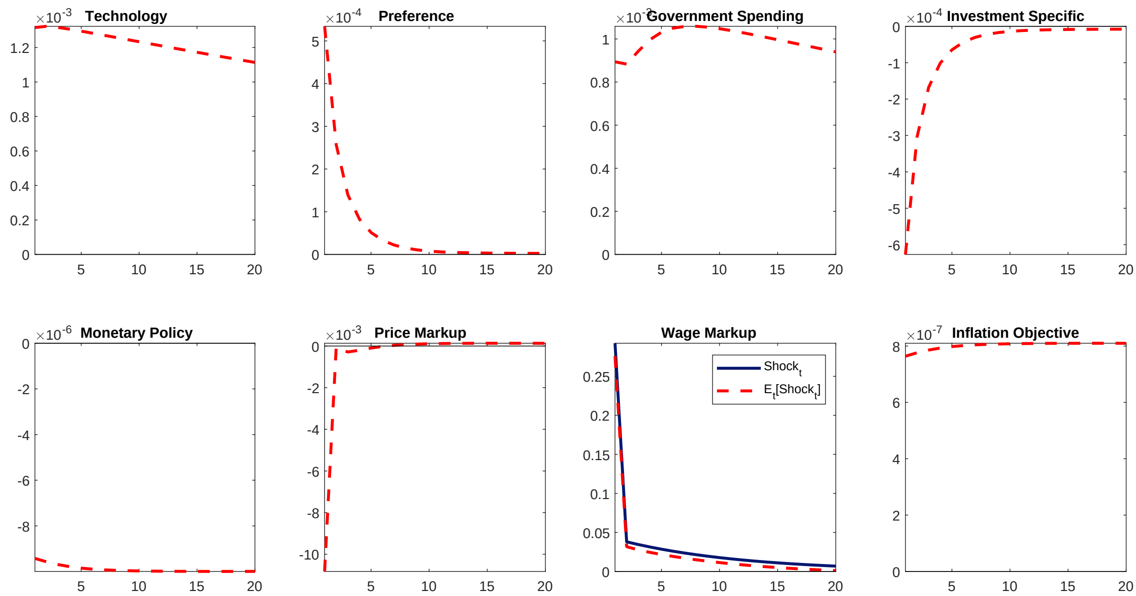


(b) Investment Specific

Figure 12: **Estimated SW Model Non-invertible Case 2: Learning About the Shocks**



(a) Price Markup



(b) Wage Markup

Figure 13: **Estimated SW Model Non-invertible Case 2: Learning About the Shocks**

F.2 Comparison of SVAR Identified using Zero Short-Run Cholesky Restrictions and Estimated SW Model

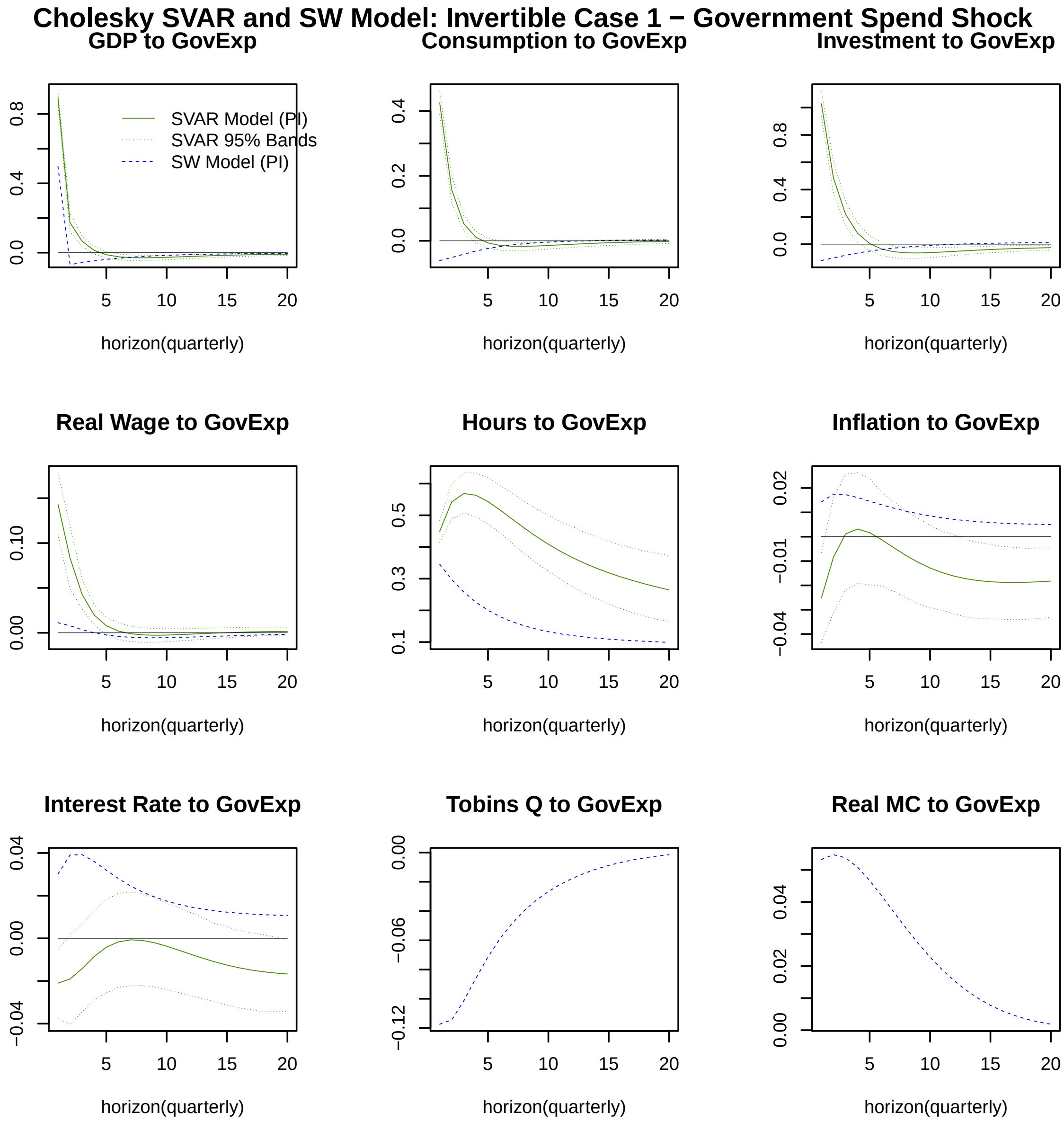


Figure 14: **Estimated SVAR(1) Model using Identified using Zero Short-Run Cholesky Restrictions (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Government Spending Shock**

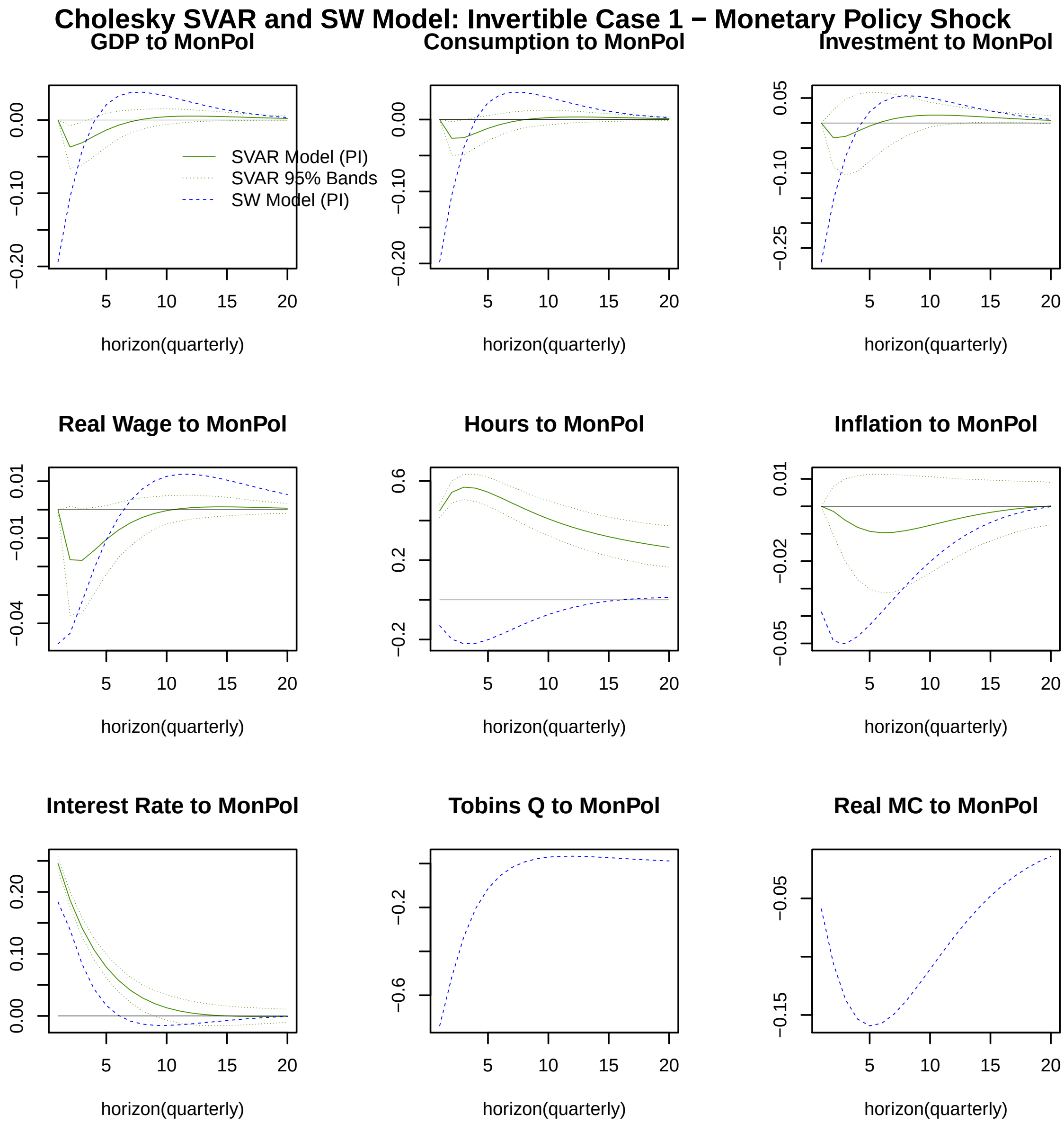


Figure 15: **Estimated SVAR(1) Model using Identified using Zero Short-Run Cholesky Restrictions (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Monetary Policy Shock**

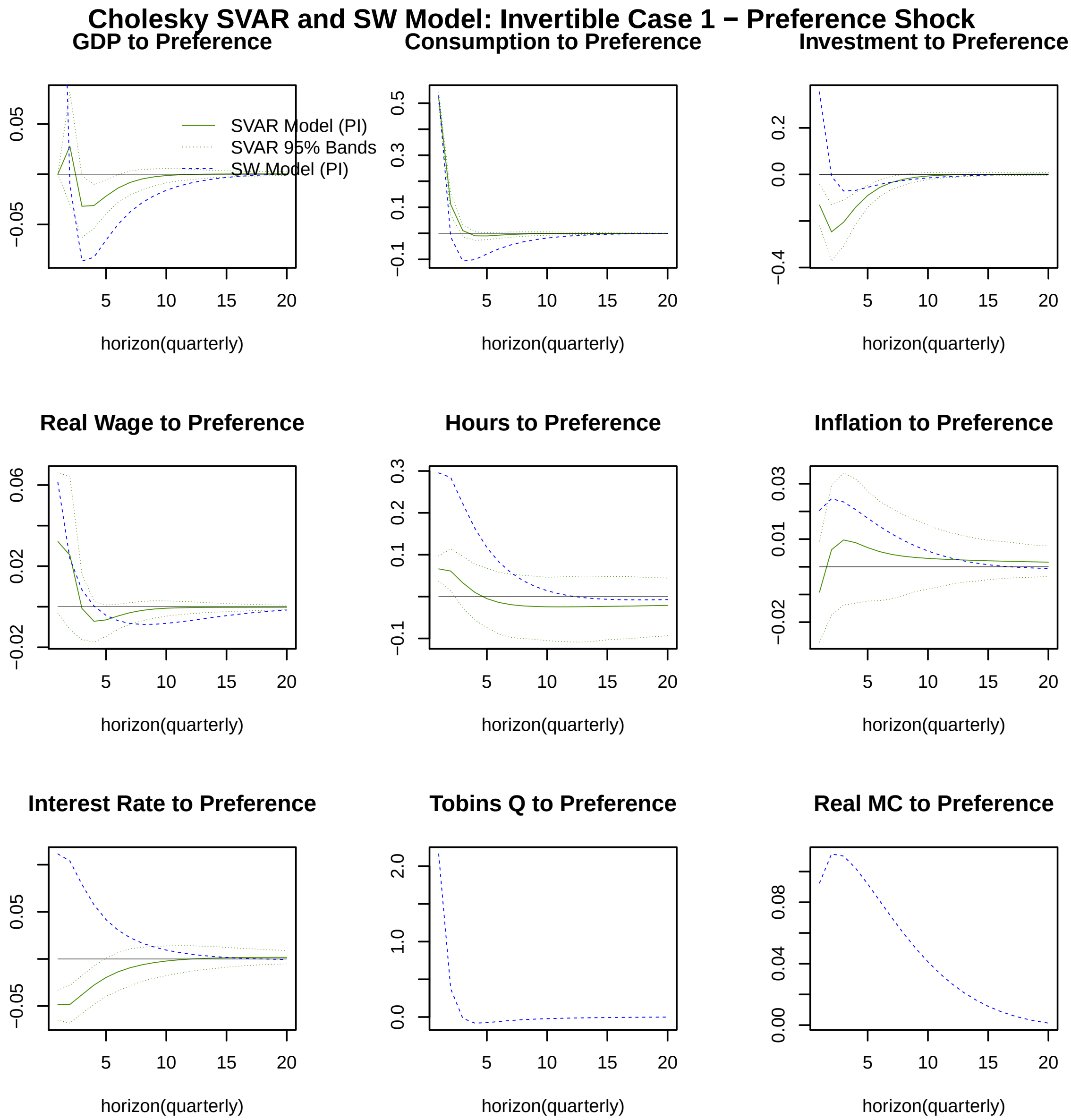


Figure 16: **Estimated SVAR(1) Model using Identified using Zero Short-Run Cholesky Restrictions (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Preference Shock**

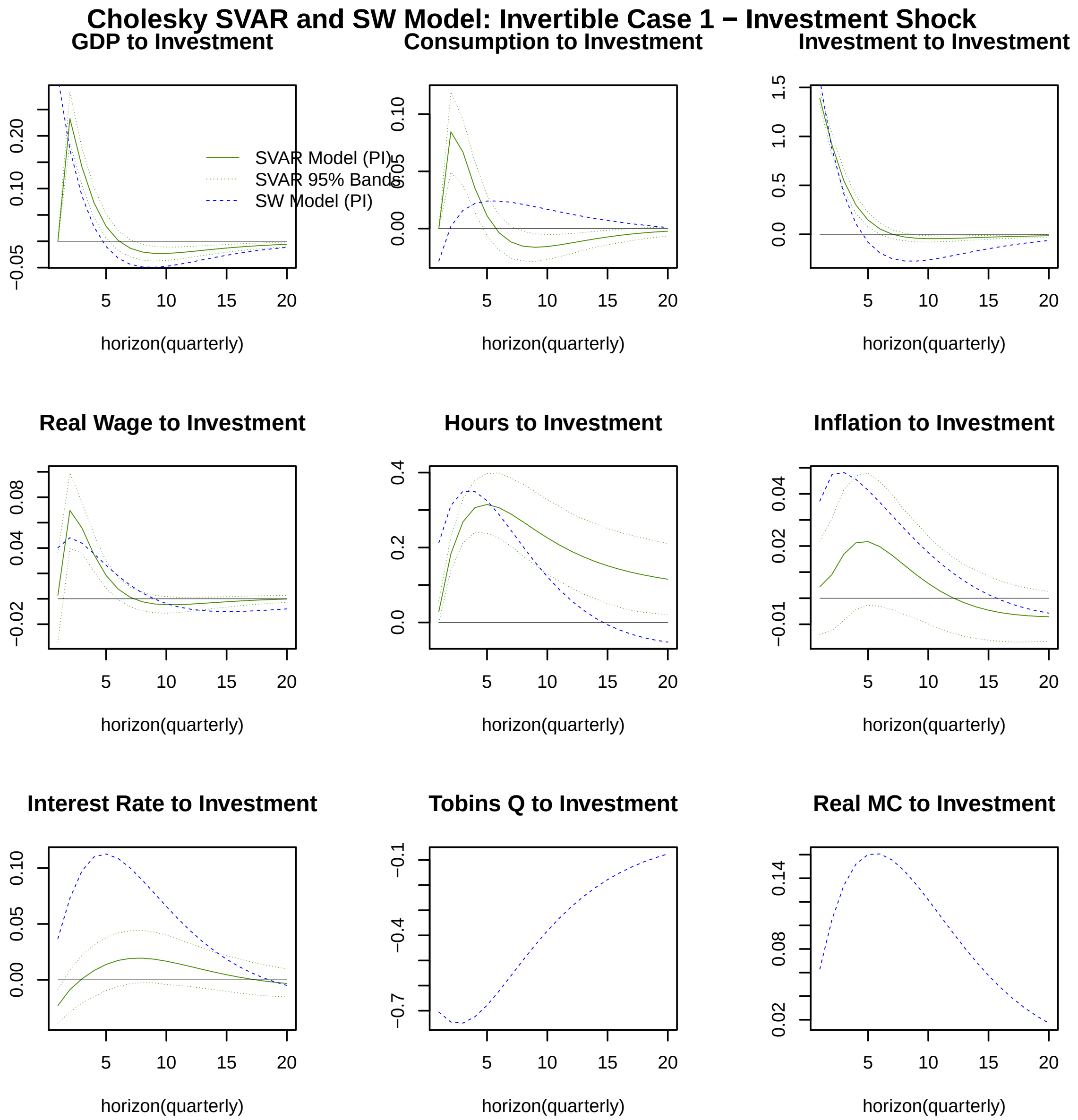


Figure 17: **Estimated SVAR(1) Model using Identified using Zero Short-Run Cholesky Restrictions (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Investment Shock**

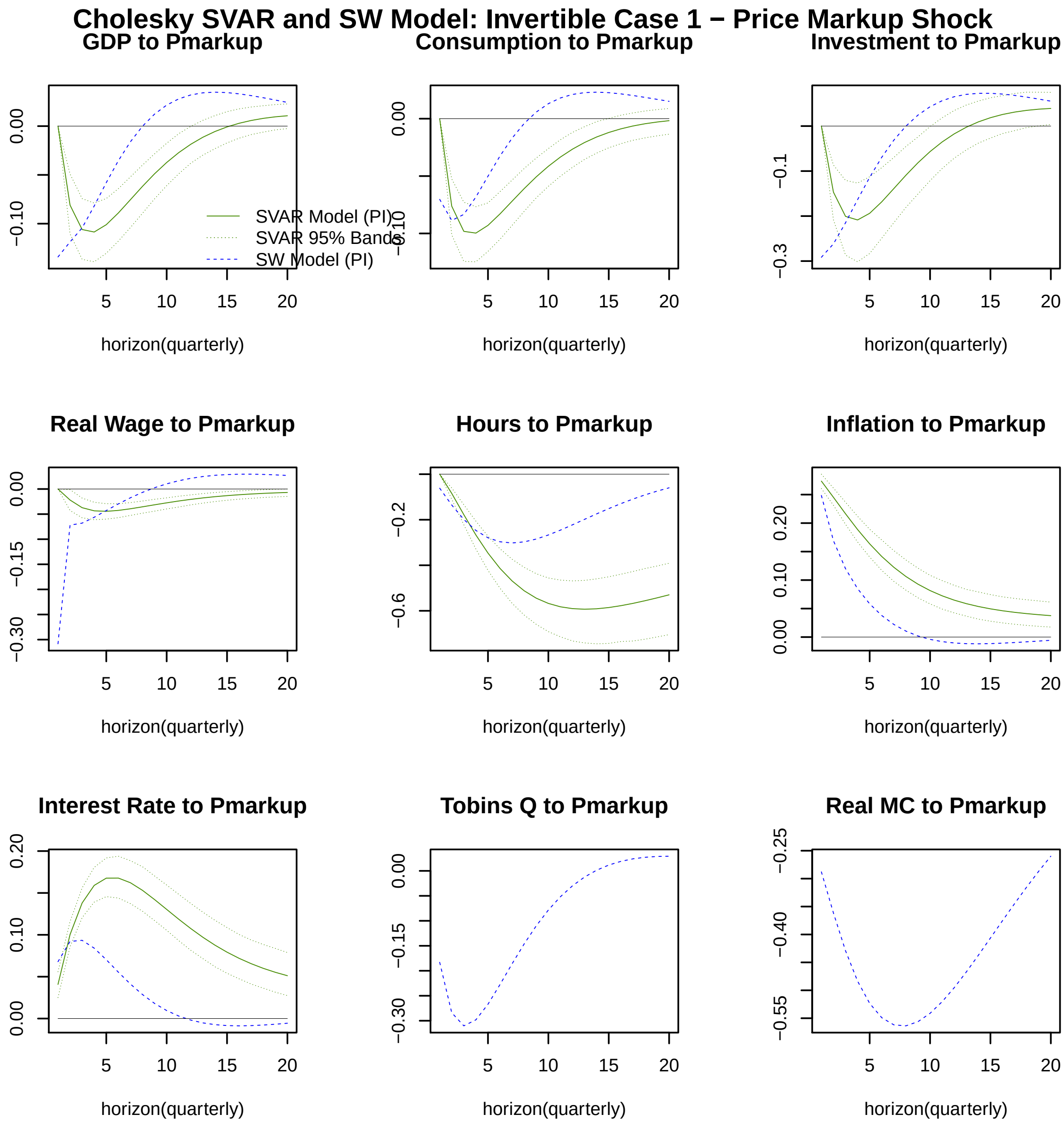


Figure 18: **Estimated SVAR(1) Model using Identified using Zero Short-Run Cholesky Restrictions (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Price Markup Shock**

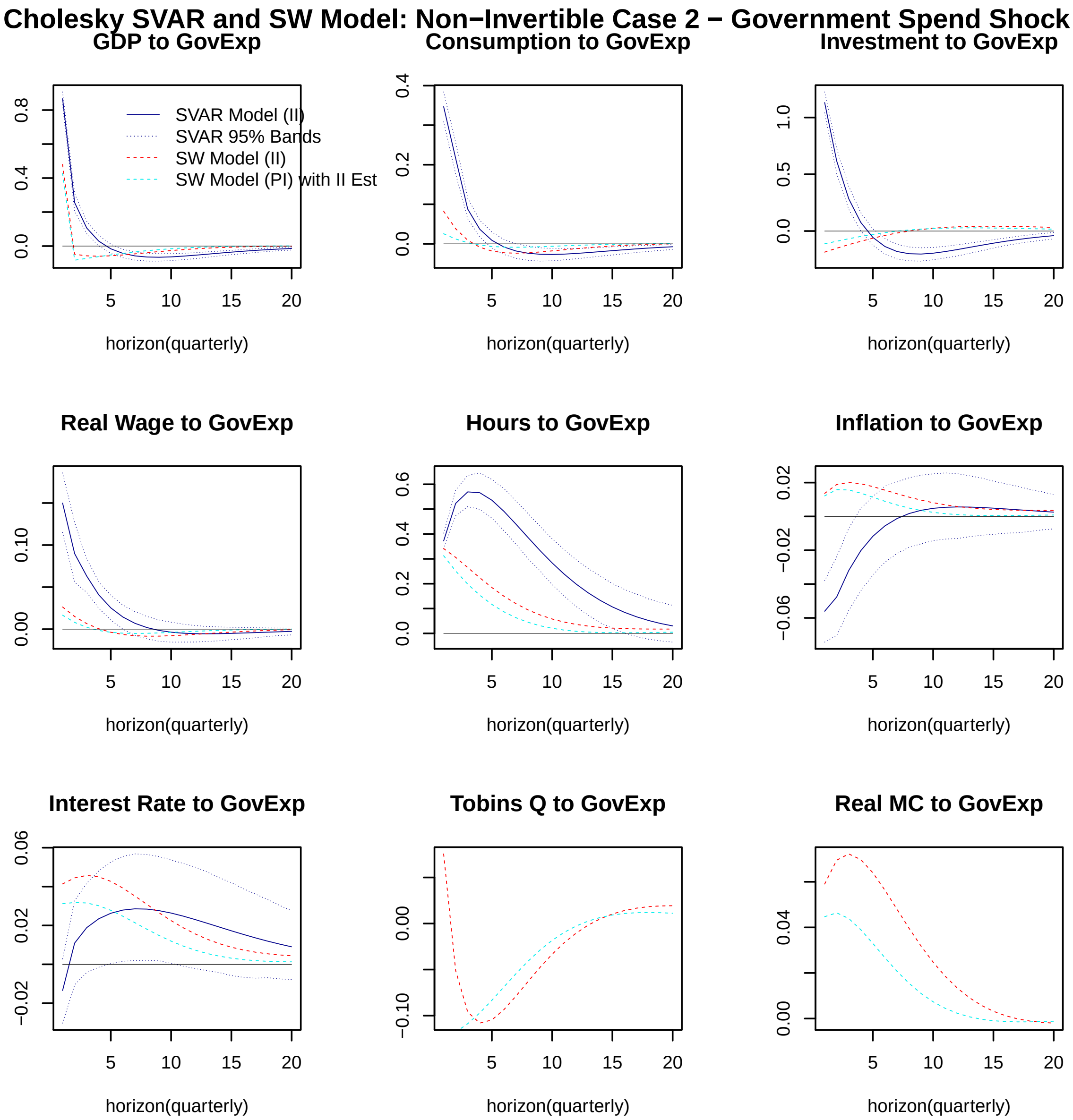


Figure 19: **Estimated SVAR(1) Model using Identified using Zero Short-Run Cholesky Restrictions (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Government Spending Shock** ($\mathbb{F}_g^{II} = 0.0194$, $\rho_g = 0.90$, $\varepsilon_g = 0.43$)

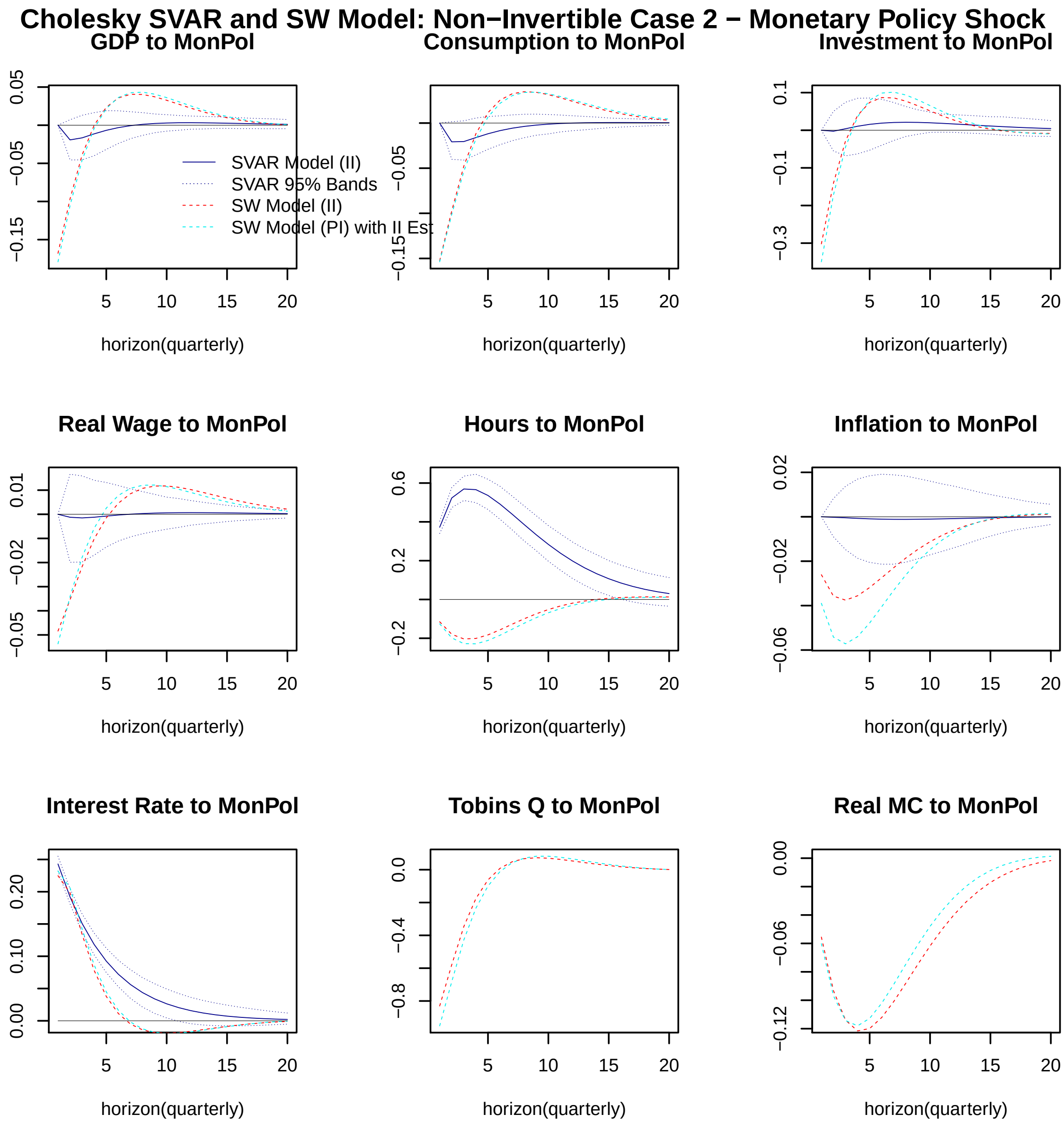


Figure 20: **Estimated SVAR(1) Model using Identified using Zero Short-Run Cholesky Restrictions (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Monetary Policy Shock** ($\mathbb{F}_r^{II} = 0.0036$, $\rho_r = 0.25$, $\varepsilon_r = 0.25$)

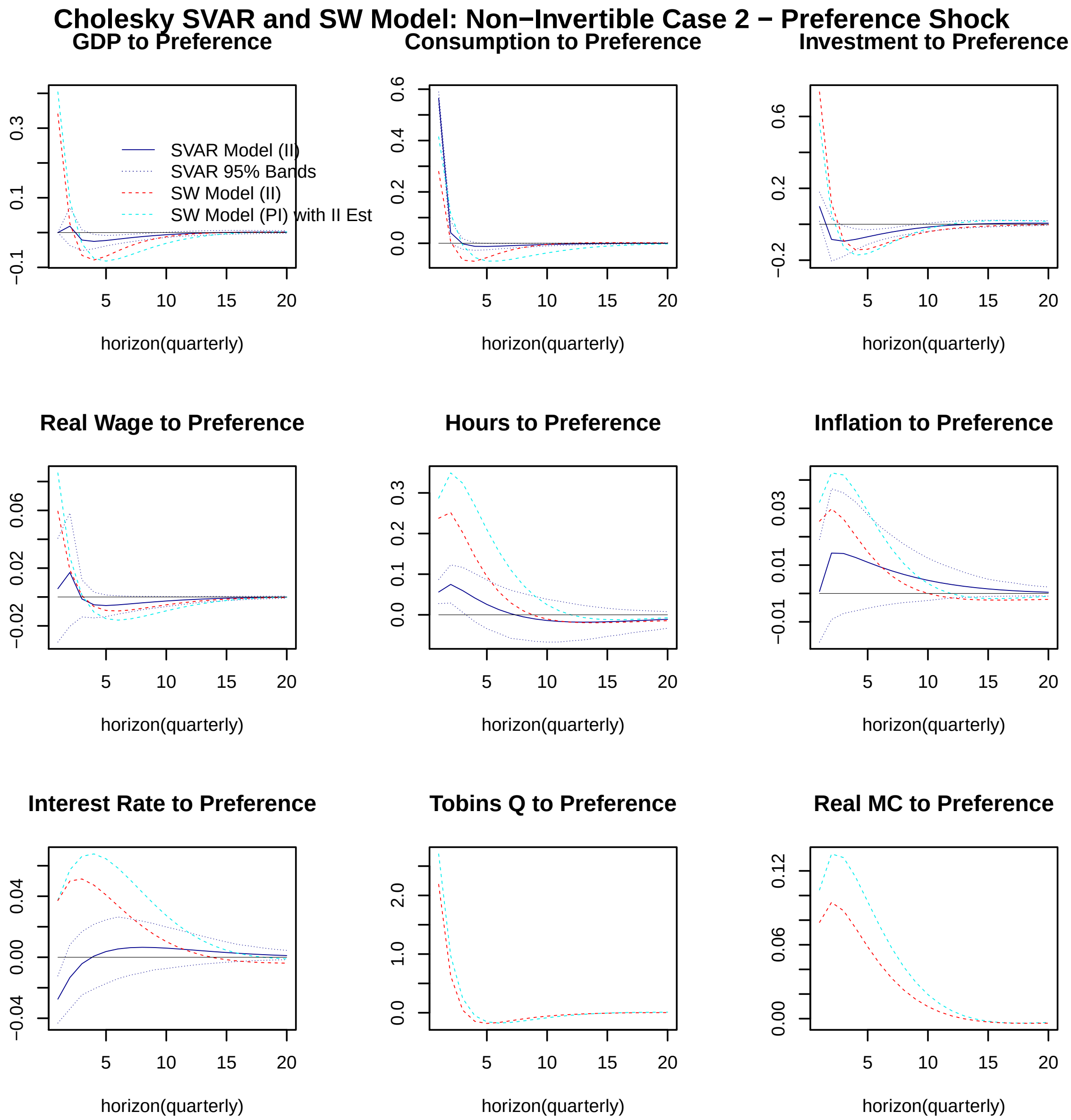


Figure 21: **Estimated SVAR(1) Model using Identified using Zero Short-Run Cholesky Restrictions (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Preference Shock** ($\mathbb{F}_b^{II} = 0.9526$, $\rho_b = 0.40$, $\varepsilon_b = 0.18$)

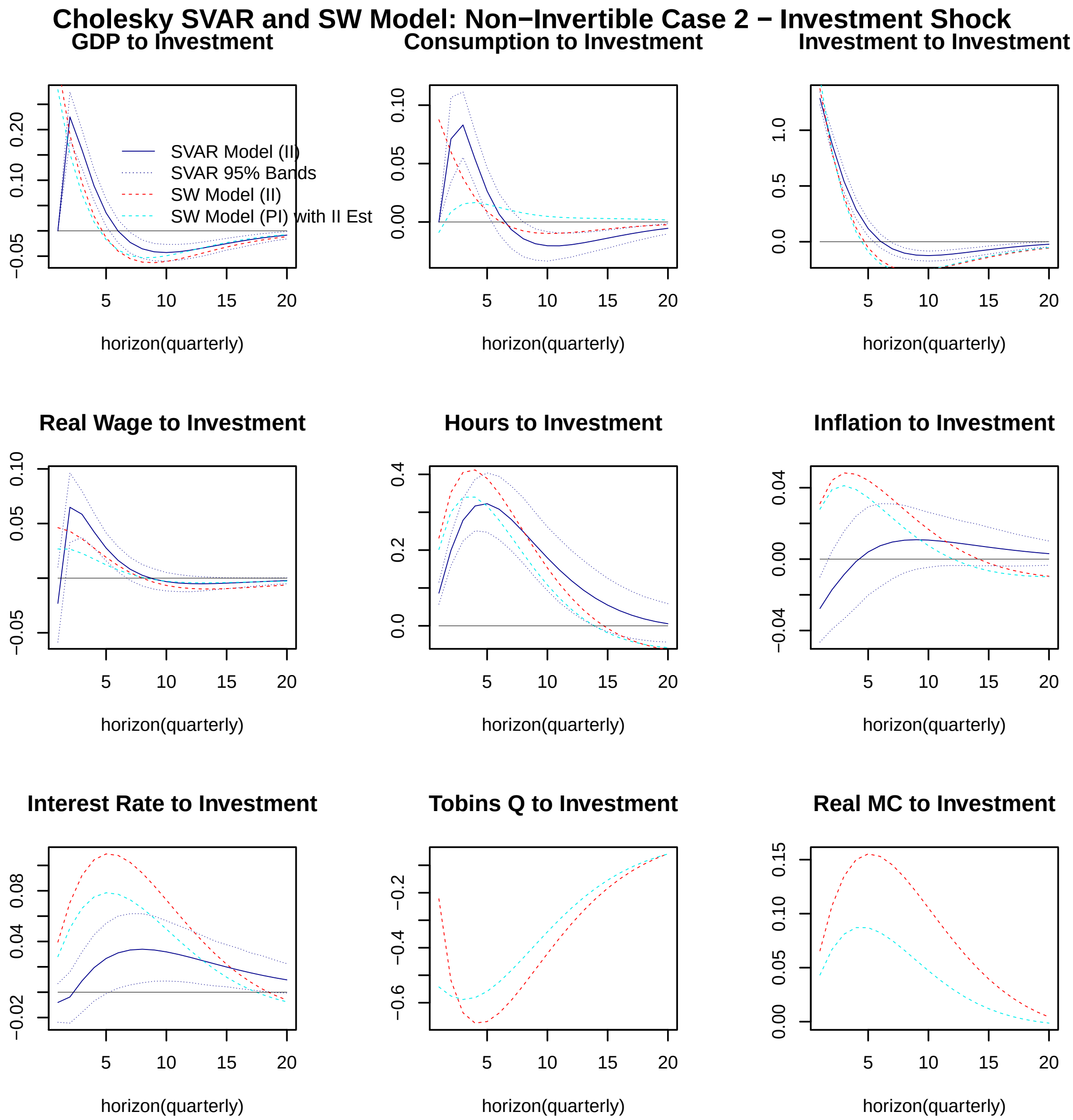


Figure 22: **Estimated SVAR(1) Model using Identified using Zero Short-Run Cholesky Restrictions (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Investment Specific Shock** (($\mathbb{F}_i^{II} = 0.5085$, $\rho_i = 0.77$, $\varepsilon_i = 0.37$)

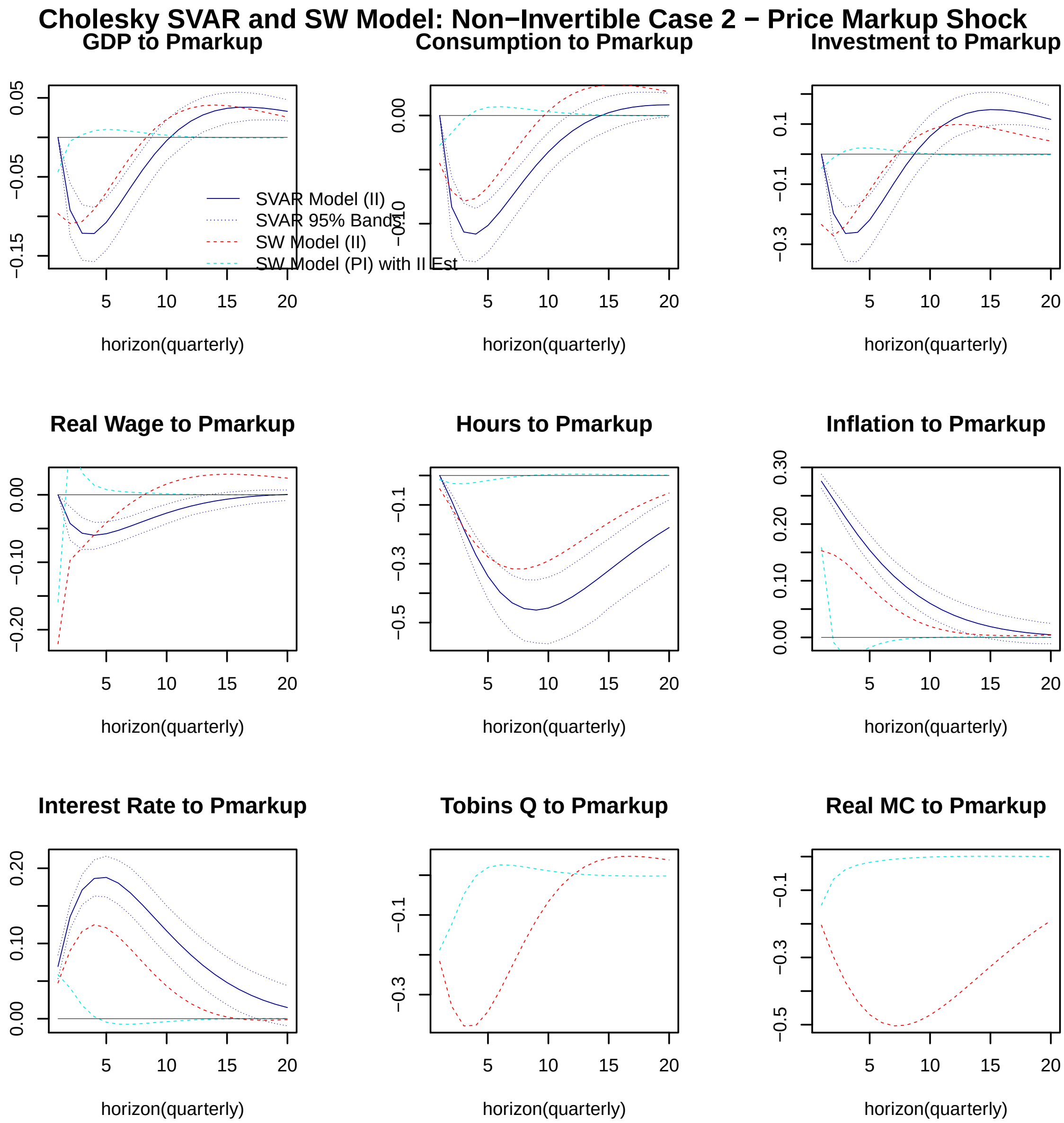


Figure 23: **Estimated SVAR(1) Model using Identified using Zero Short-Run Cholesky Restrictions (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Price Markup Shock** ($\mathbb{F}_p^{II} = 0.6655$, $\rho_p = 0.89$, $\varepsilon_p = 0.09$)

F.3 Comparison of SVAR Identified using Sign Restrictions and Estimated SW Model

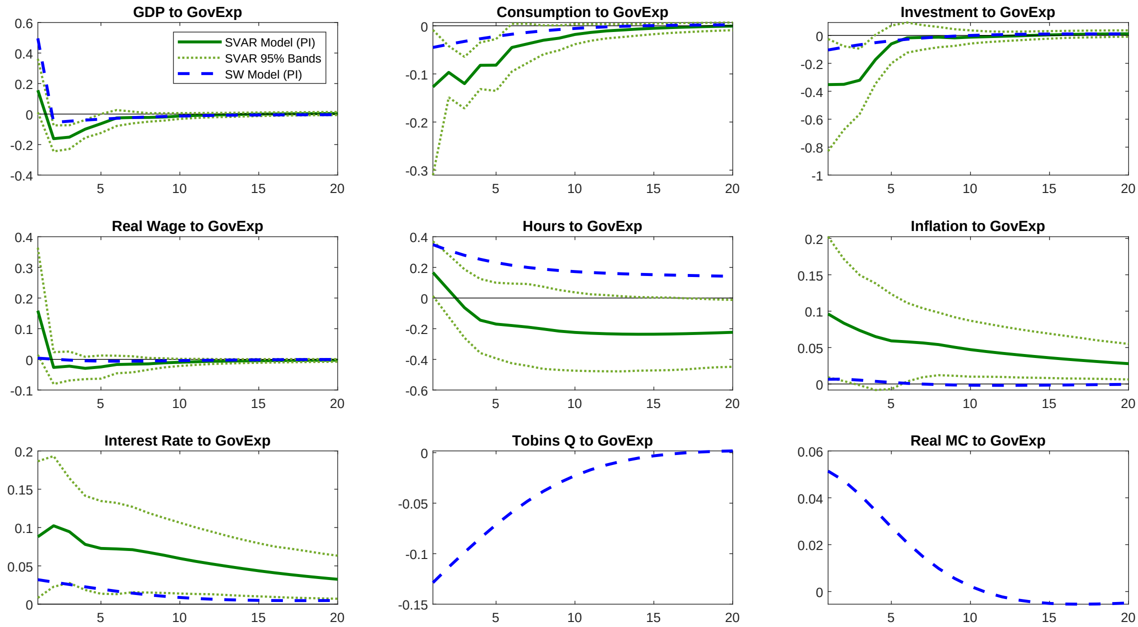


Figure 24: **Estimated SVAR(1) Model using Artificial Data (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Government Spending Shock**

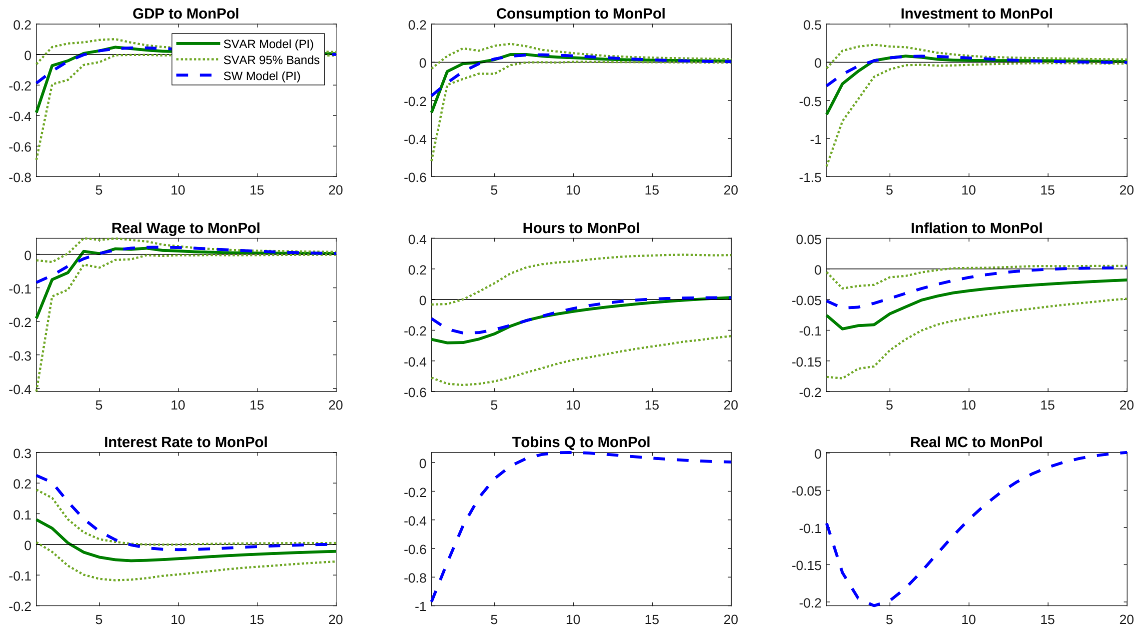


Figure 25: **Estimated SVAR(1) Model Identified using Sign Restrictions (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Monetary Policy Shock**

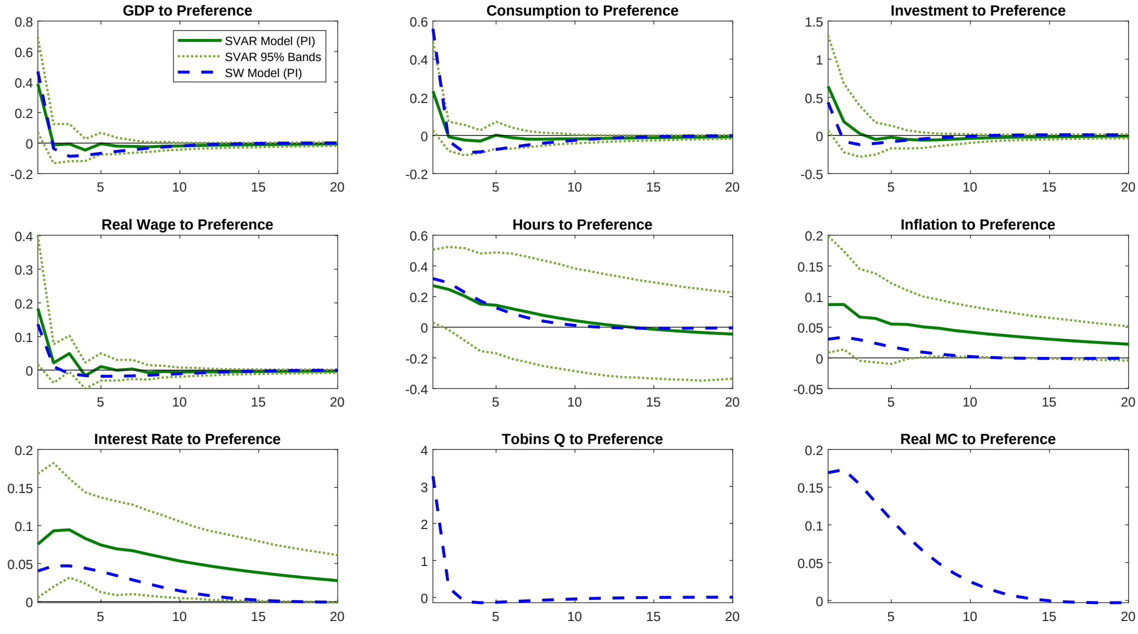


Figure 26: **Estimated SVAR(1) Model Identified using Sign Restrictions (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Preference Shock**

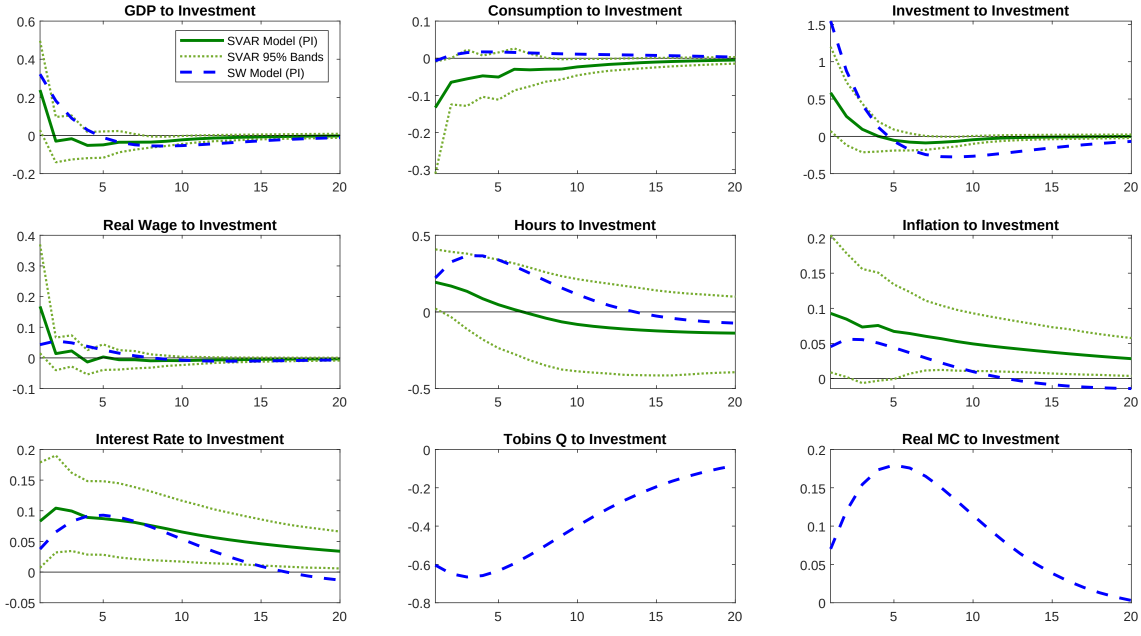


Figure 27: **Estimated SVAR(1) Model Identified using Sign Restrictions (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Investment Shock**

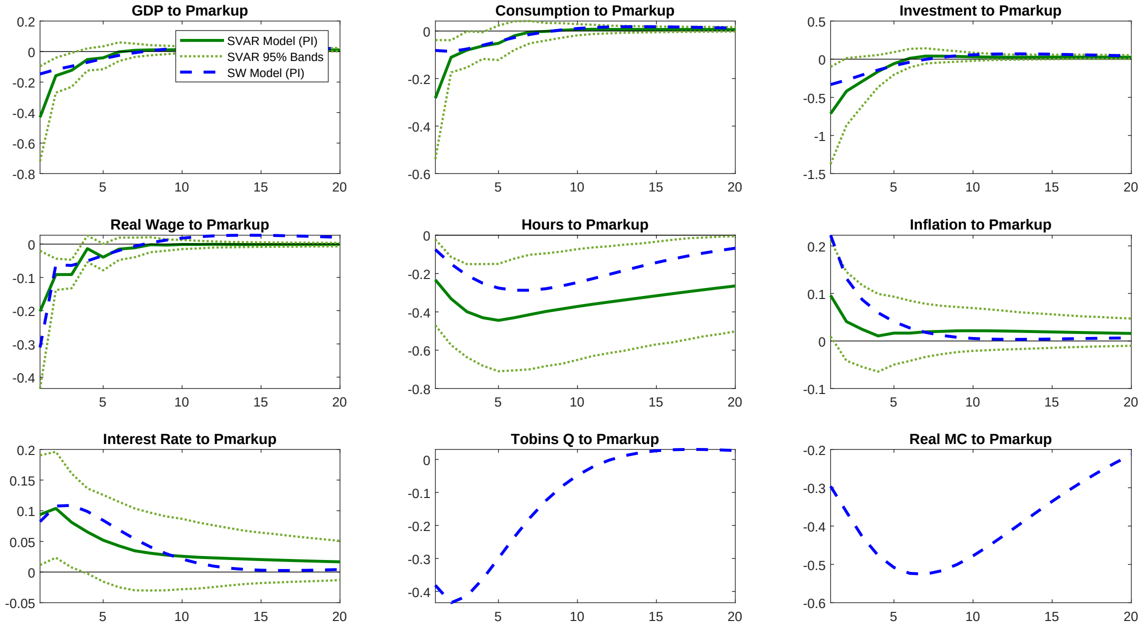


Figure 28: **Estimated SVAR(1) Model Identified using Sign Restrictions (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Price Markup Shock**

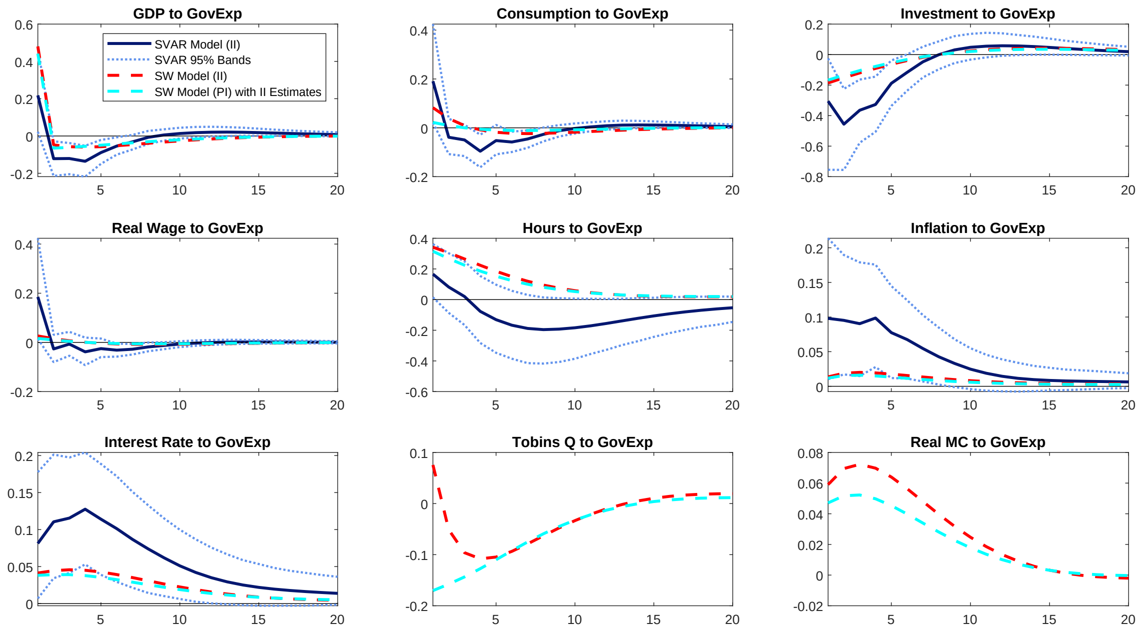


Figure 29: **Estimated SVAR(1) Model Identified using Sign Restrictions (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Government Spending Shock** ($\mathbb{F}_g^{II} = 0.0194$, $\rho_g = 0.90$, $\varepsilon_g = 0.43$)

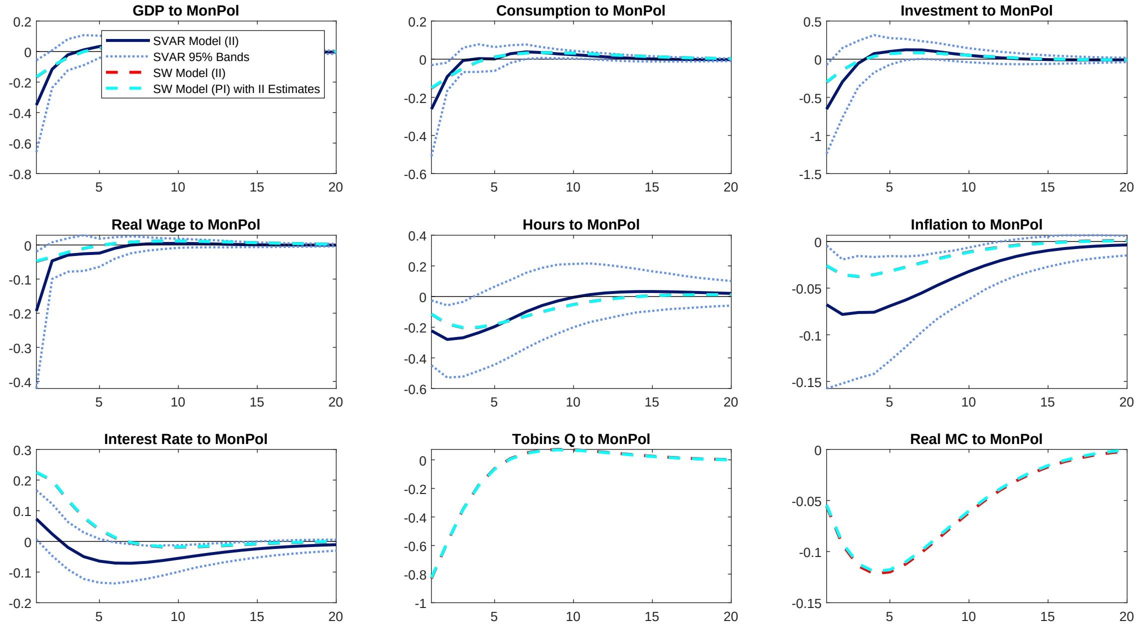


Figure 30: **Estimated SVAR(1) Model Identified using Sign Restrictions (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Monetary Policy Shock** ($\mathbb{F}_r^{II} = 0.0036,\ \rho_r = 0.25,\ \varepsilon_r = 0.25$)

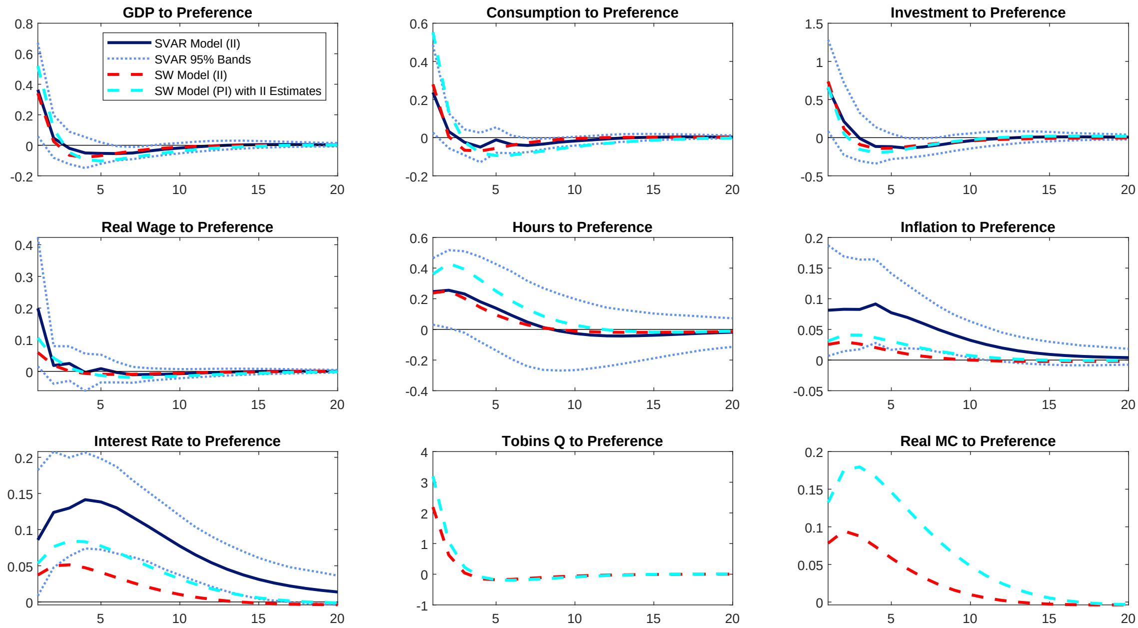


Figure 31: **Estimated SVAR(1) Model Identified using Sign Restrictions (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Preference Shock** ($\mathbb{F}_b^{II} = 0.9526,\ \rho_b = 0.40,\ \varepsilon_b = 0.18$)

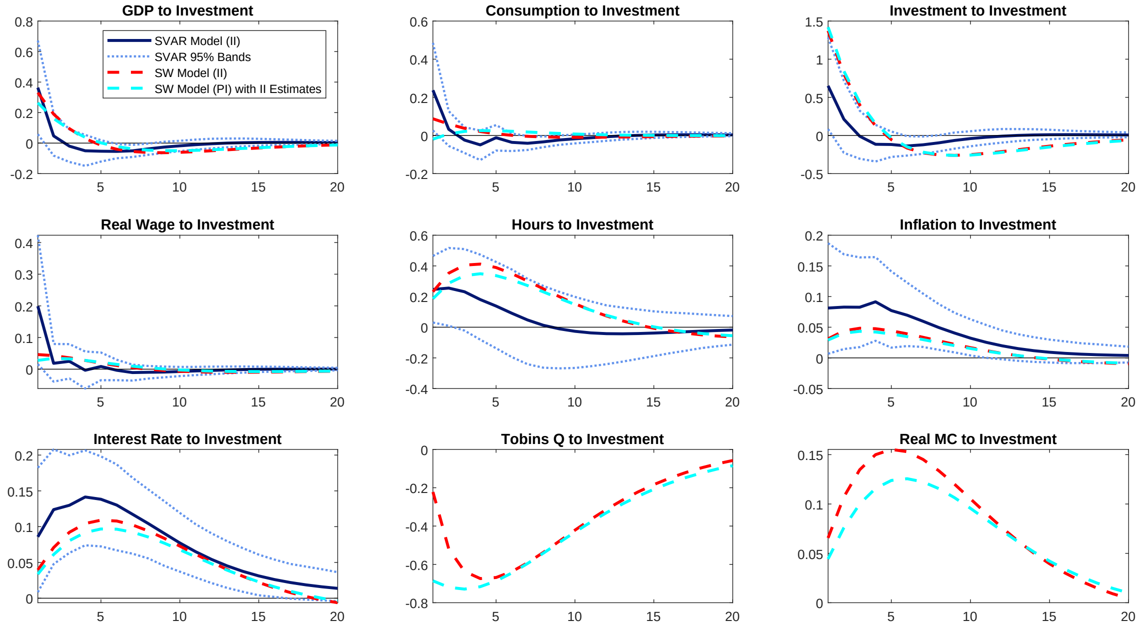


Figure 32: **Estimated SVAR(1) Model Identified using Sign Restrictions (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Investment Specific Shock** ($\mathbb{F}_i^{II} = 0.5085,\ \rho_i = 0.77,\ \varepsilon_i = 0.37$)

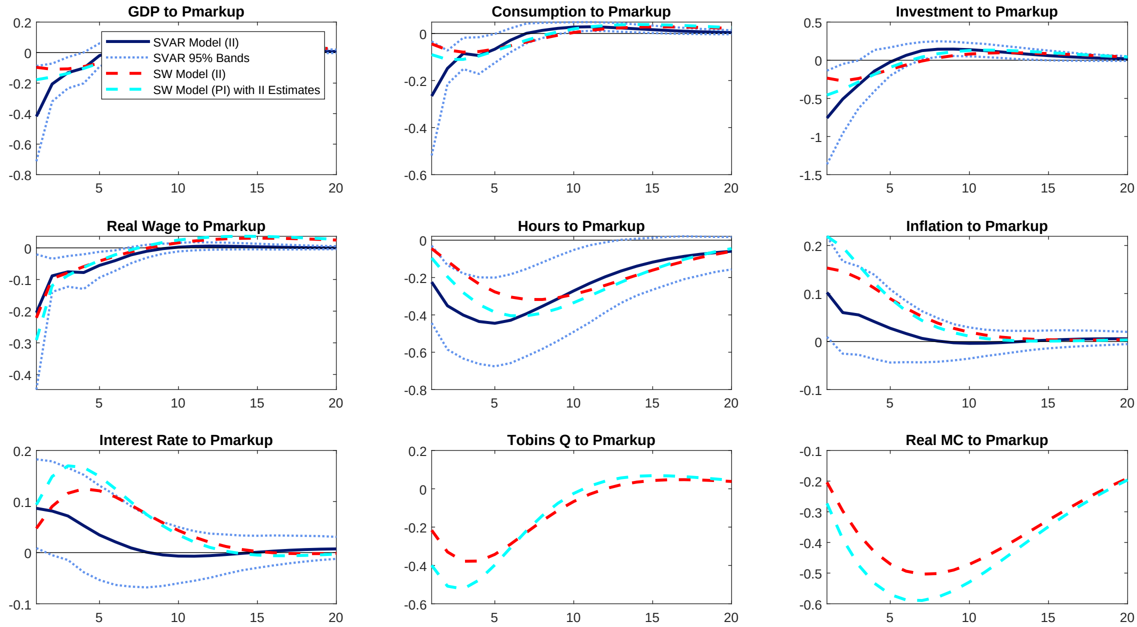


Figure 33: **Estimated SVAR(1) Model Identified using Sign Restrictions (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Price Markup Shock** ($\mathbb{F}_p^{II} = 0.6655,\ \rho_p = 0.89,\ \varepsilon_p = 0.09$)

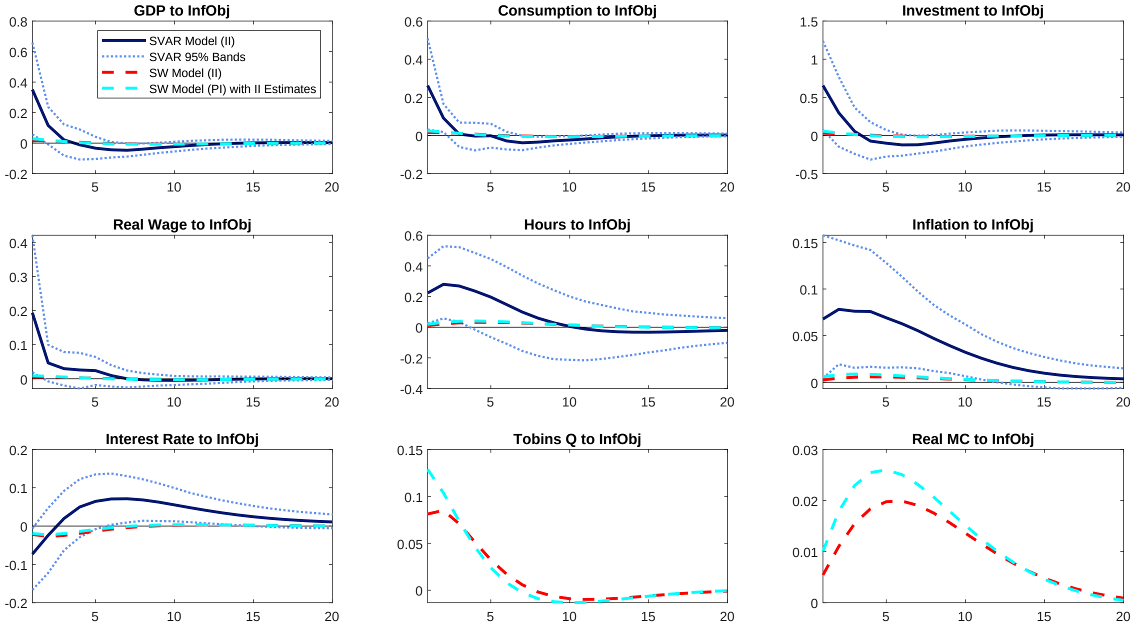


Figure 34: **Estimated SVAR(1) Model Identified using Sign Restrictions (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Inflation Objective Shock** ($\mathbb{F}_t^{II} = 0.9989,\ \rho_t = 0.60,\ \varepsilon_t = 0.08$)

F.4 Comparison of SVAR Identified using Sign-Restricted Robust Prior and Estimated SW Model

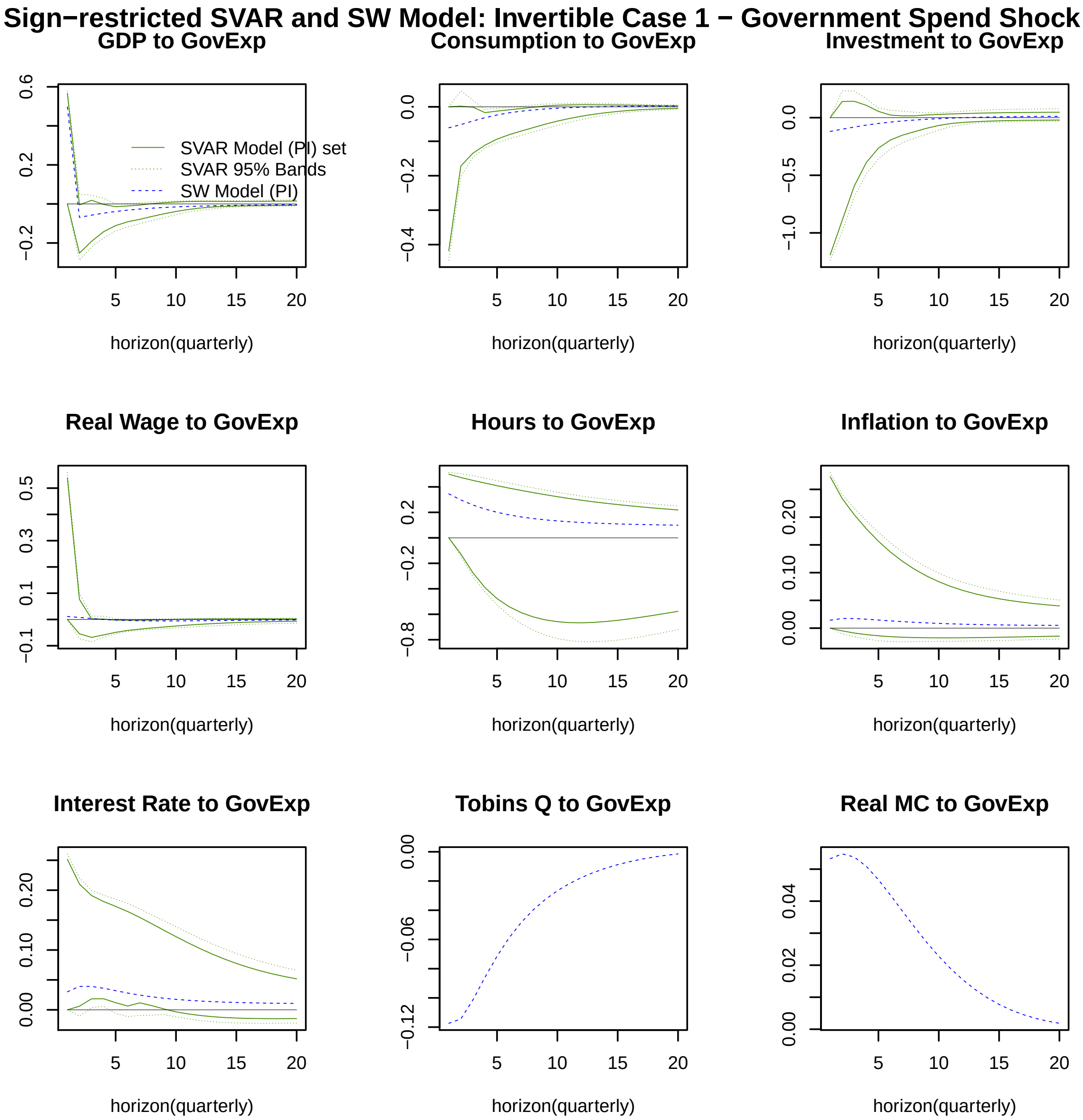


Figure 35: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using Sign-Restricted Robust Prior (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Government Spending Shock**

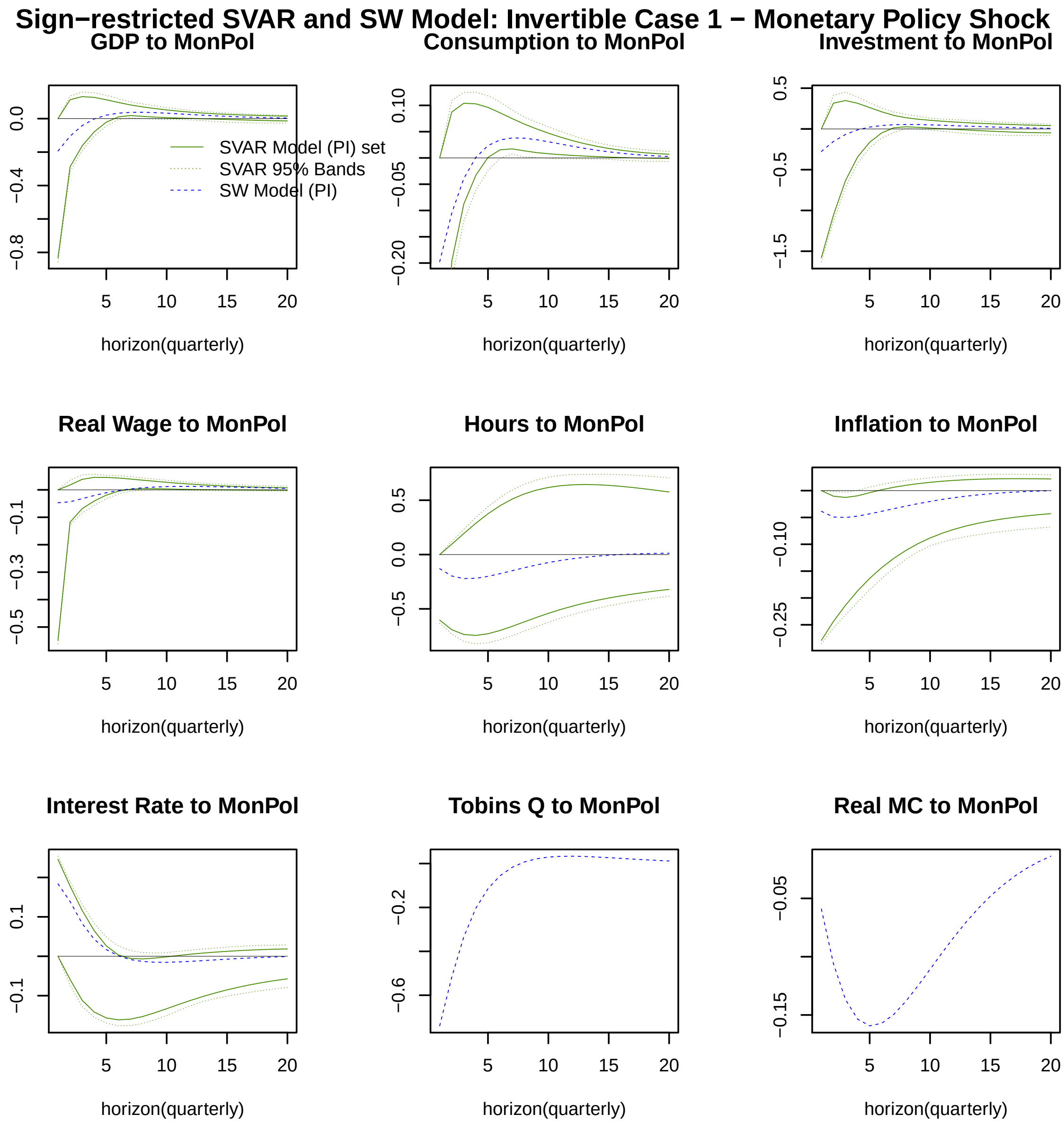


Figure 36: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using Sign-Restricted Robust Prior (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Monetary Policy Shock**

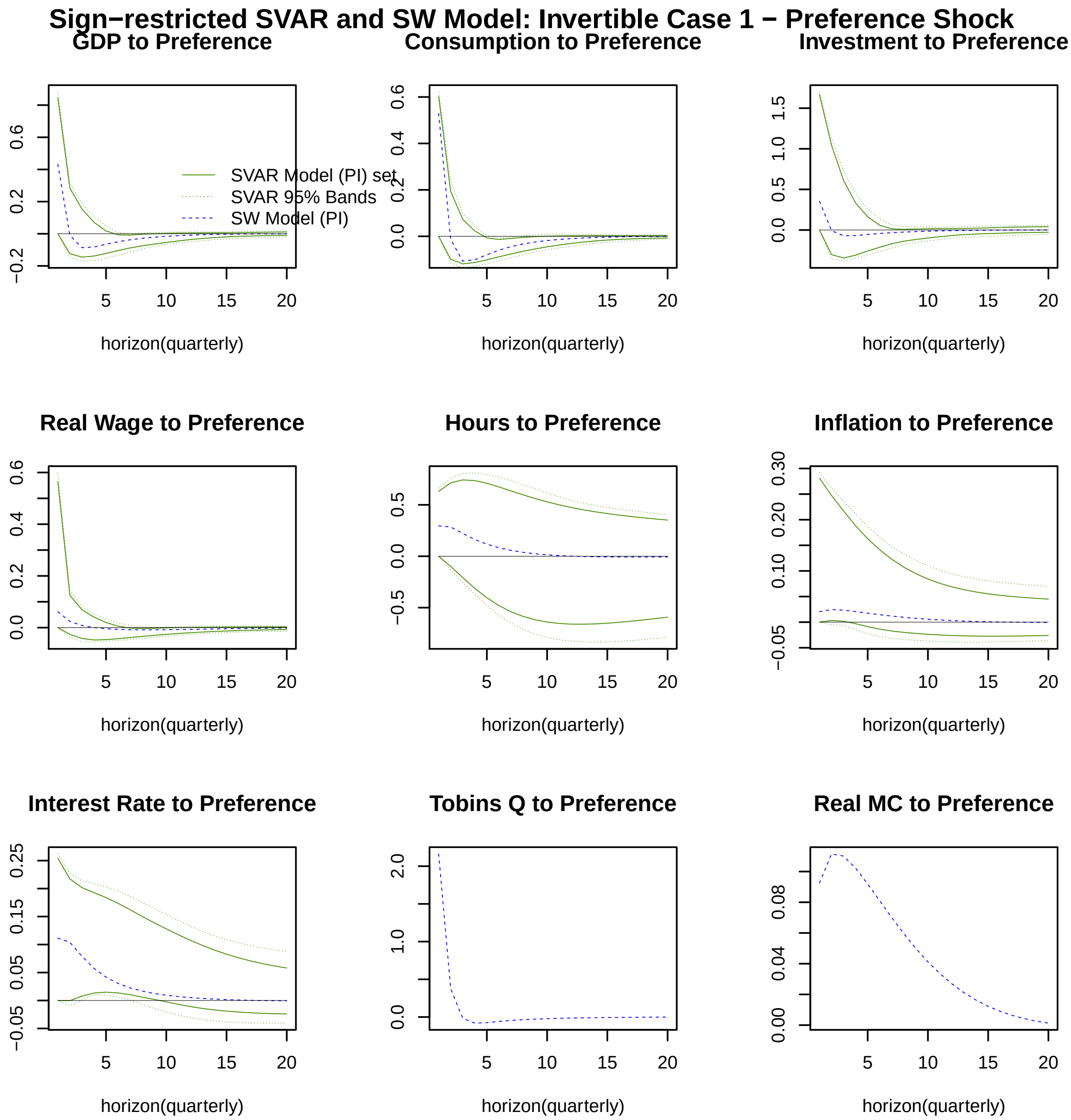


Figure 37: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using Sign-Restricted Robust Prior (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Preference Shock**

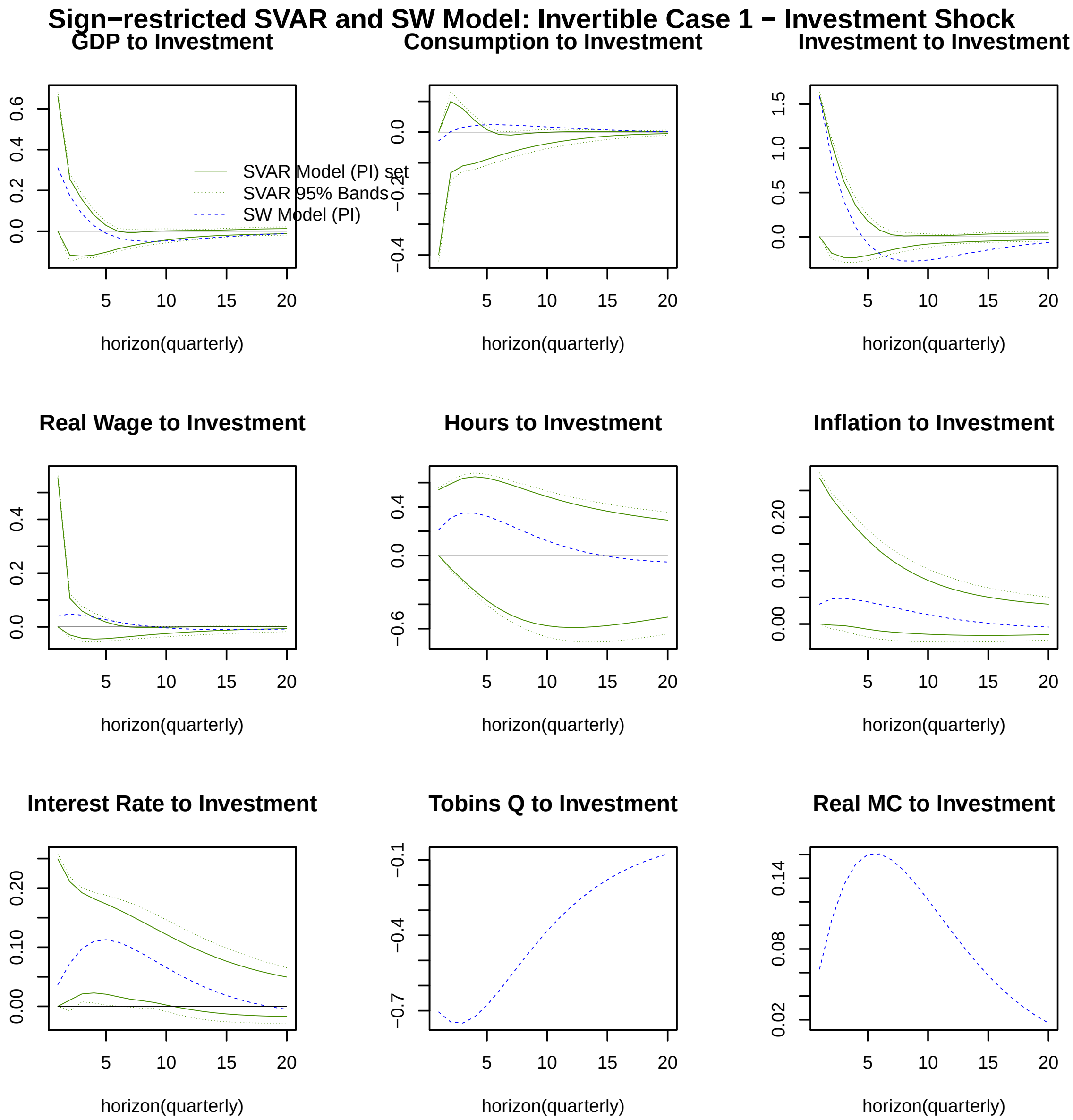


Figure 38: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using Sign-Restricted Robust Prior (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Investment Shock**

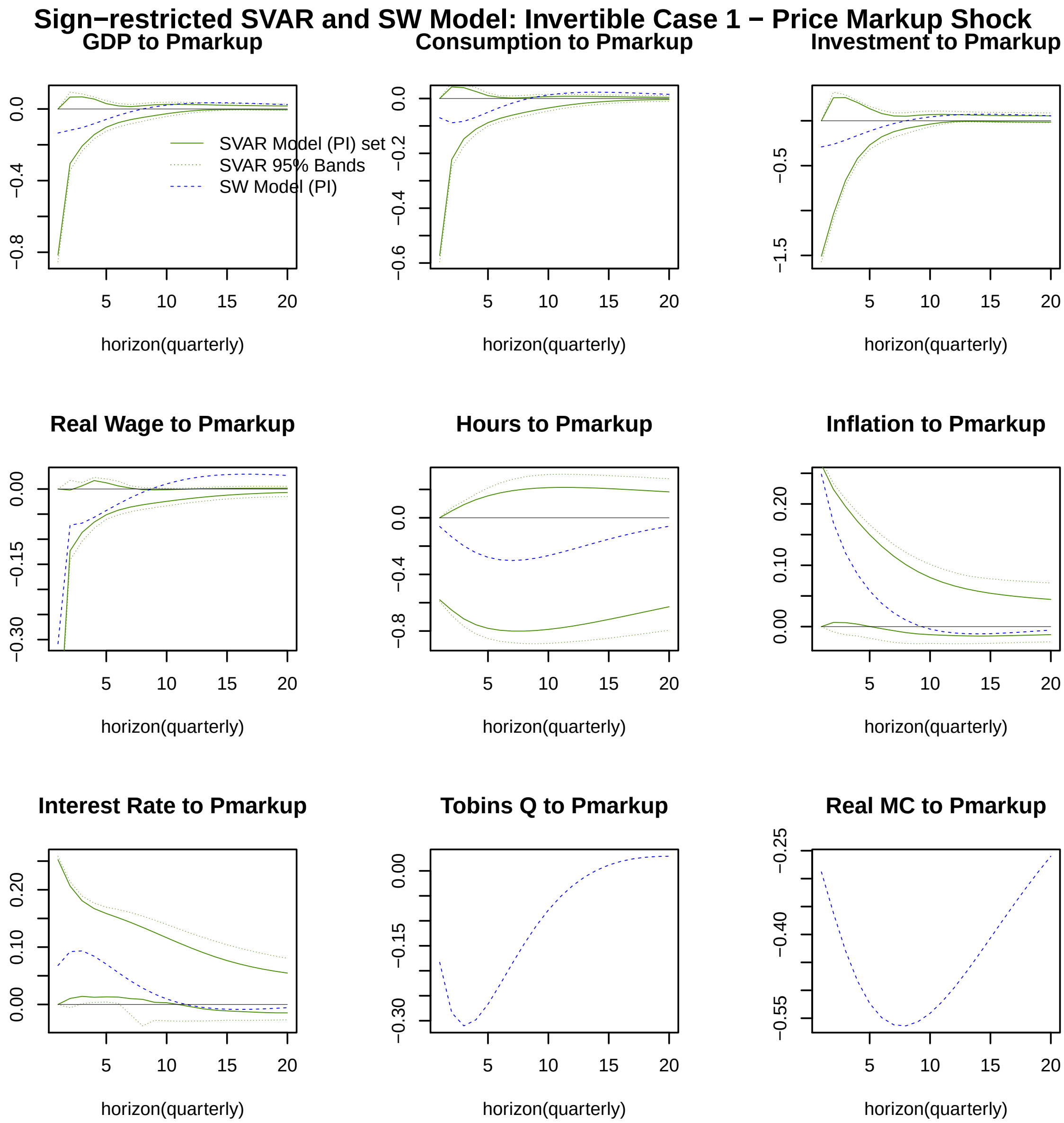


Figure 39: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using Sign-Restricted Robust Prior (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Price Markup Shock**

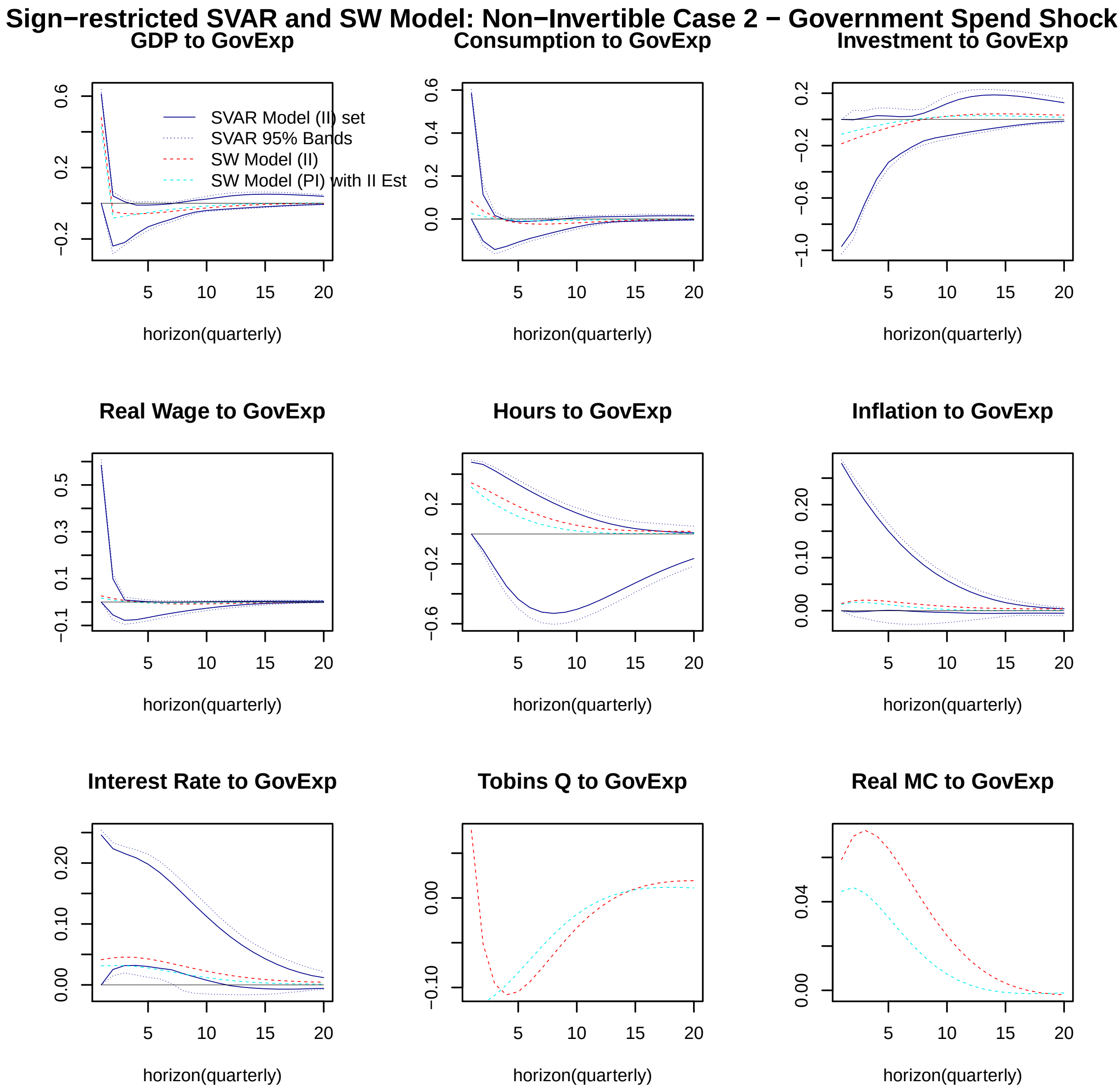


Figure 40: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using Sign-Restricted Robust Prior (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Government Spending Shock** ($\mathbb{F}_g^{II} = 0.0194$, $\rho_g = 0.90$, $\varepsilon_g = 0.43$)

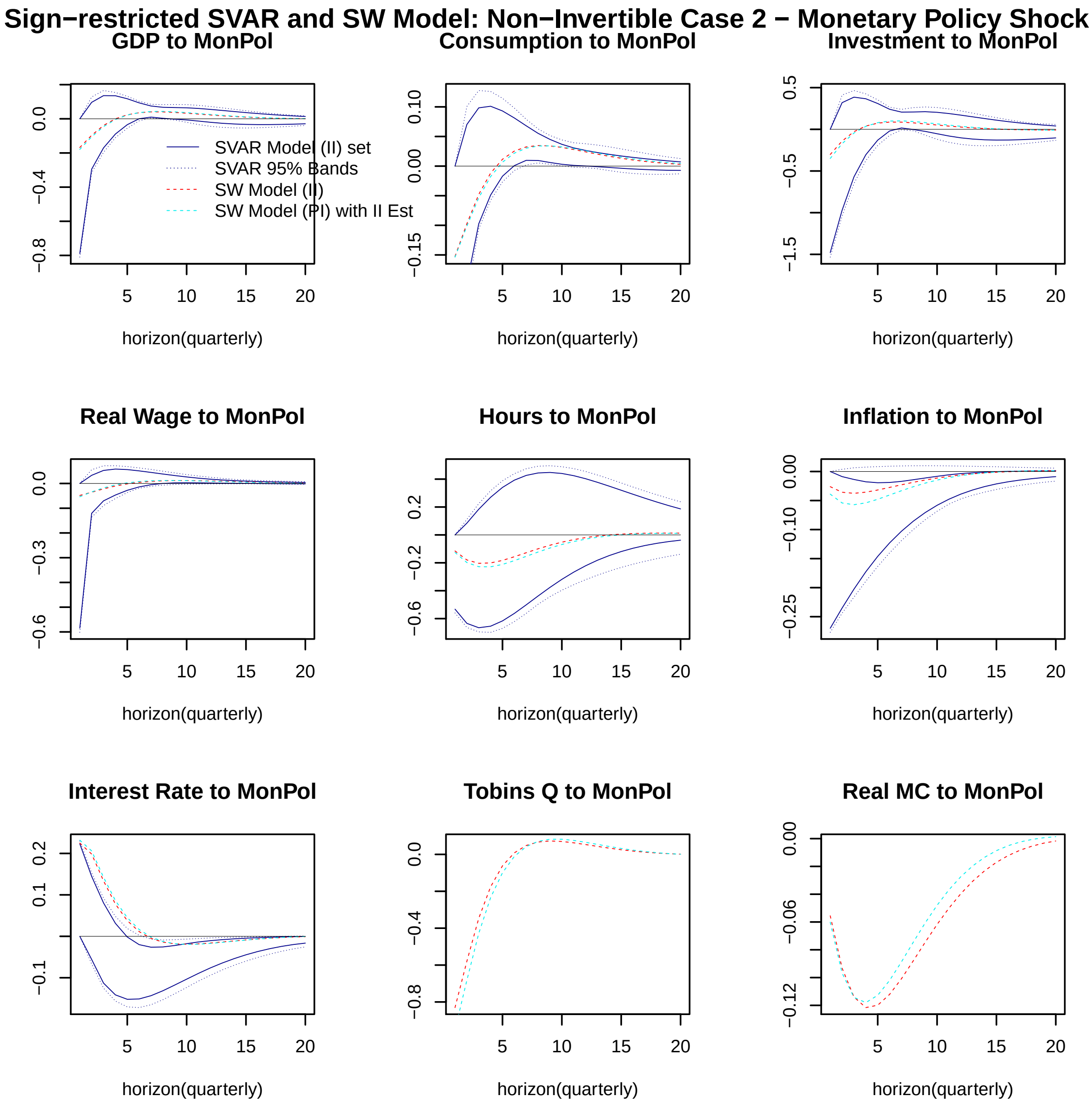


Figure 41: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using Sign-Restricted Robust Prior (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Monetary Policy Shock** ($\mathbb{F}_r^{II} = 0.0036$, $\rho_r = 0.25$, $\varepsilon_r = 0.25$)

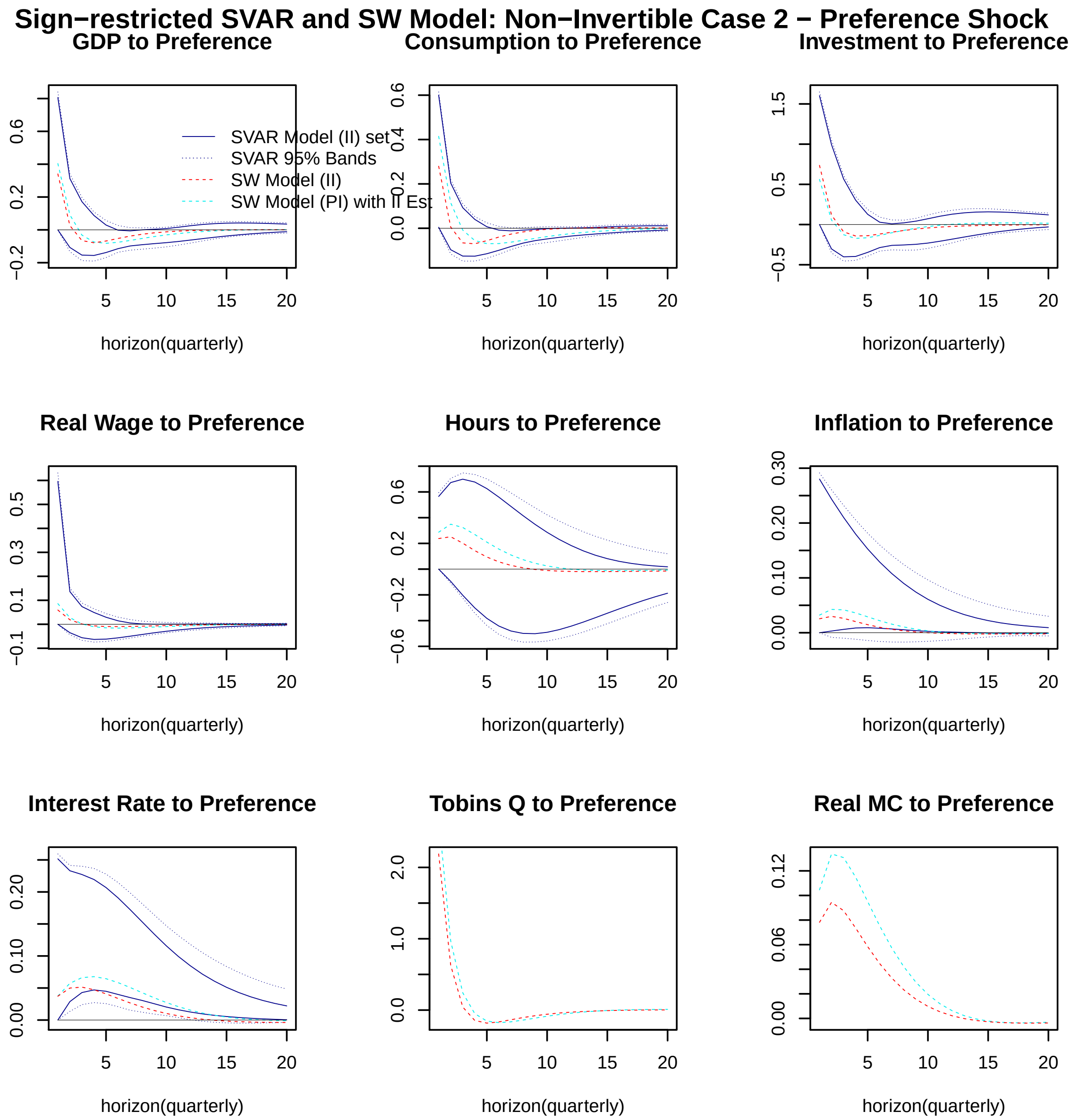


Figure 42: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using Sign-Restricted Robust Prior using Artificial Data (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Preference Shock** ($\mathbb{F}_b^{II} = 0.9526$, $\rho_b = 0.40$, $\varepsilon_b = 0.18$)

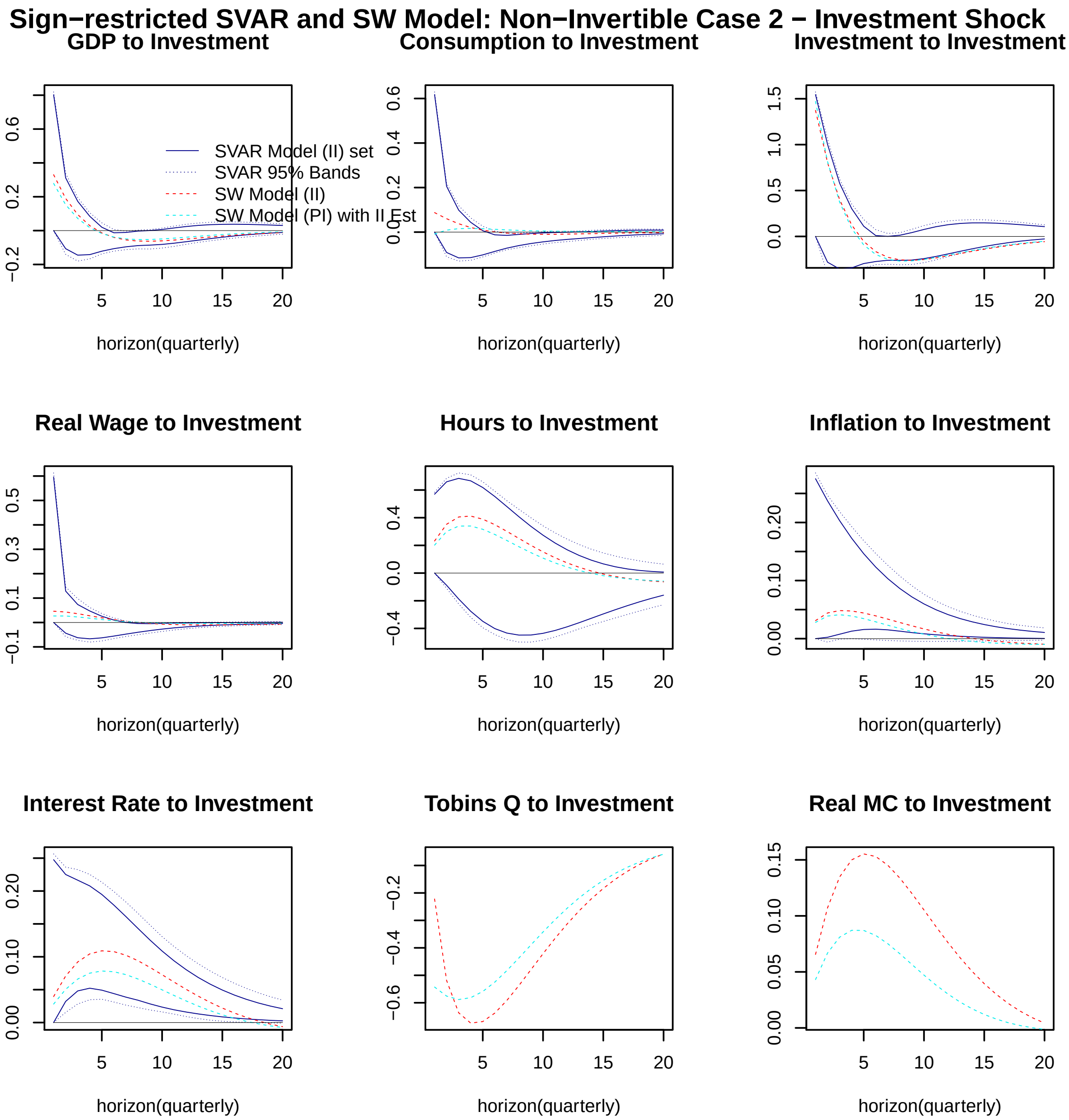


Figure 43: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using Sign-Restricted Robust Prior (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Investment Specific Shock** (($\mathbb{F}_i^{II} = 0.5085$, $\rho_i = 0.77$, $\varepsilon_i = 0.37$)

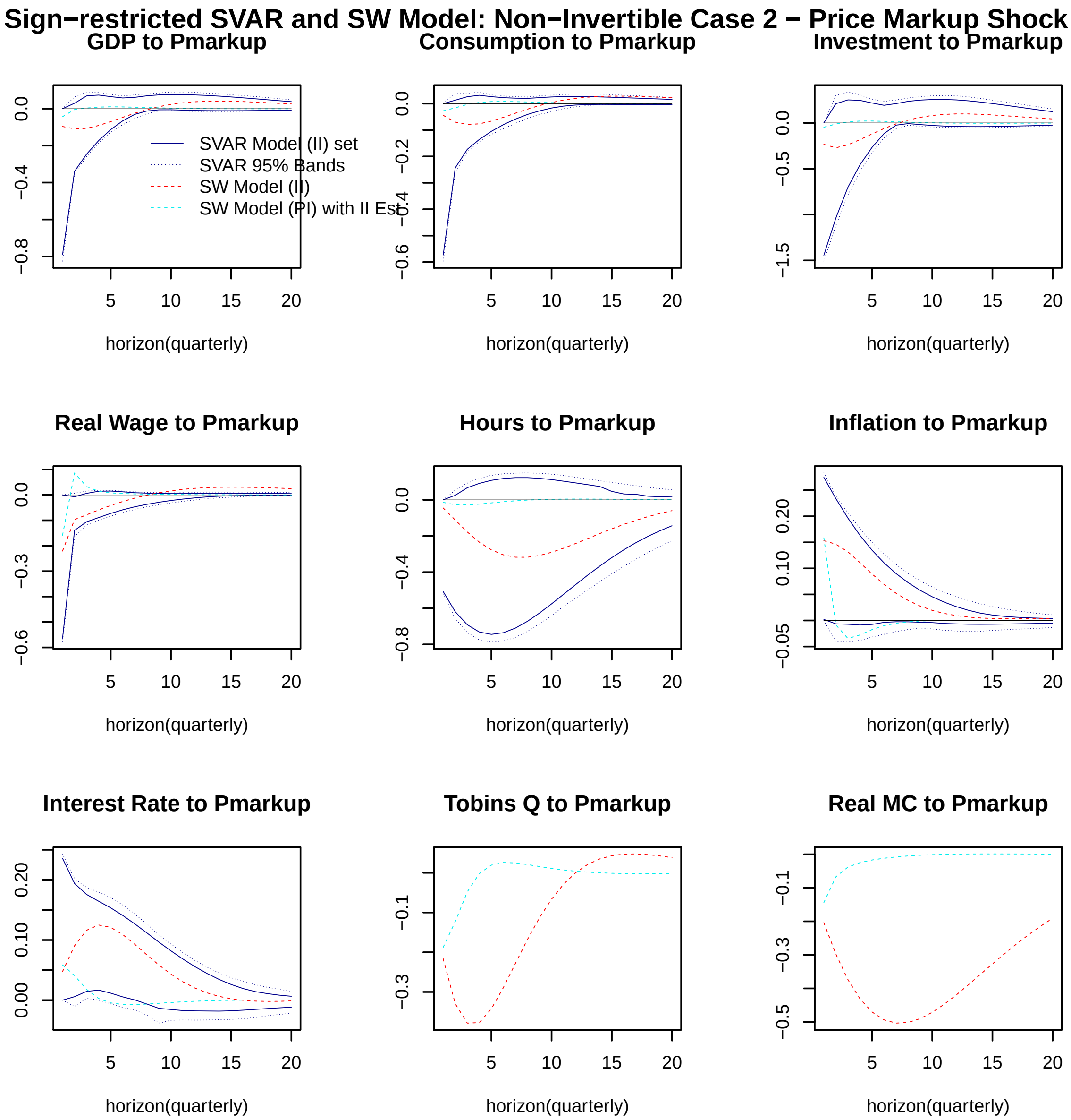


Figure 44: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using Sign-Restricted Robust Prior (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Price Markup Shock** ($\mathbb{F}_p^{II} = 0.6655$, $\rho_p = 0.89$, $\varepsilon_p = 0.09$)

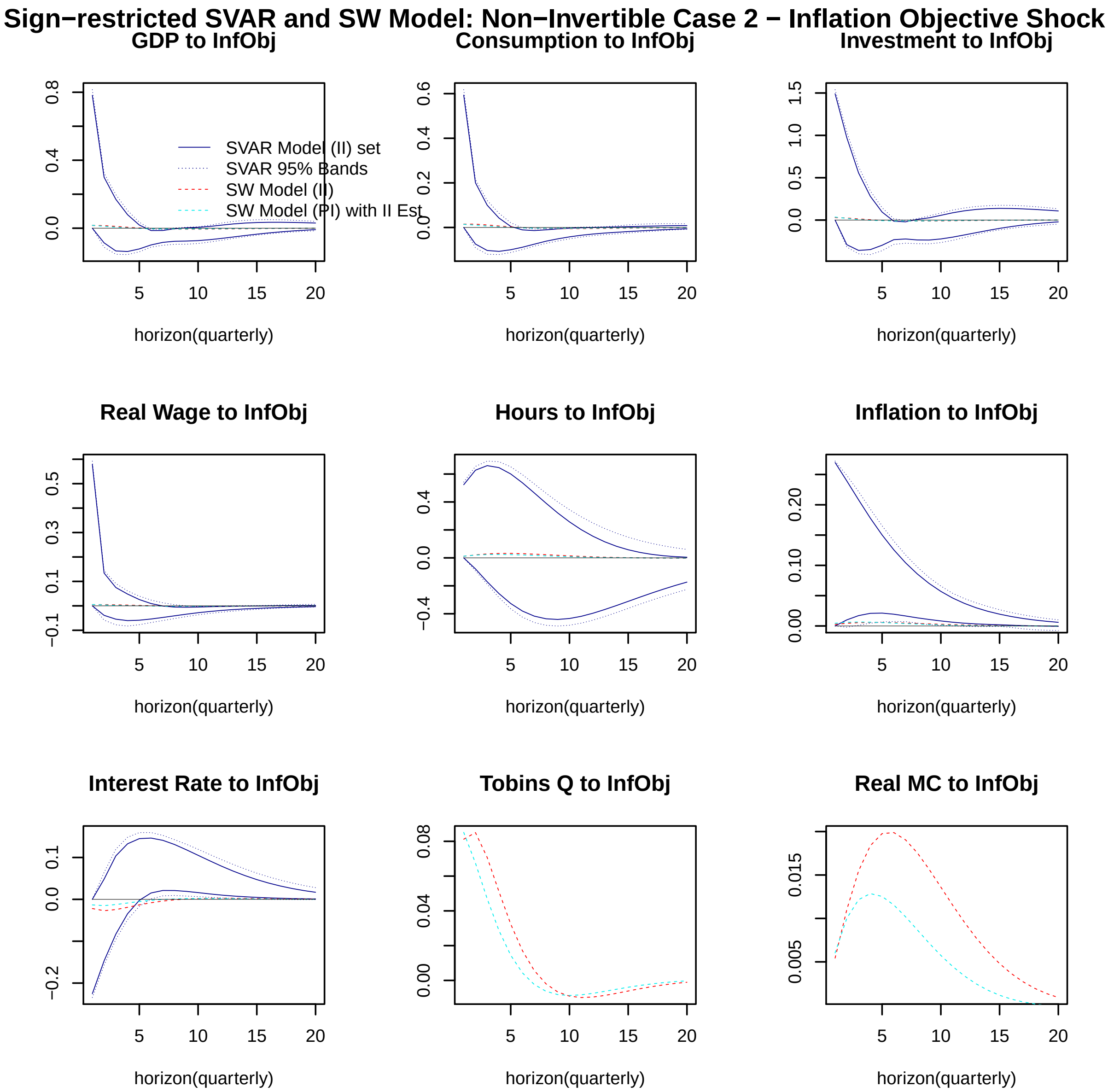


Figure 45: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using Sign-Restricted Robust Prior (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Inflation Objective Shock** ($\mathbb{F}_t^{II} = 0.9989$, $\rho_t = 0.60$, $\varepsilon_t = 0.08$)

F.5 Comparison of SVAR Identified using FEV Bounds and Estimated SW Model

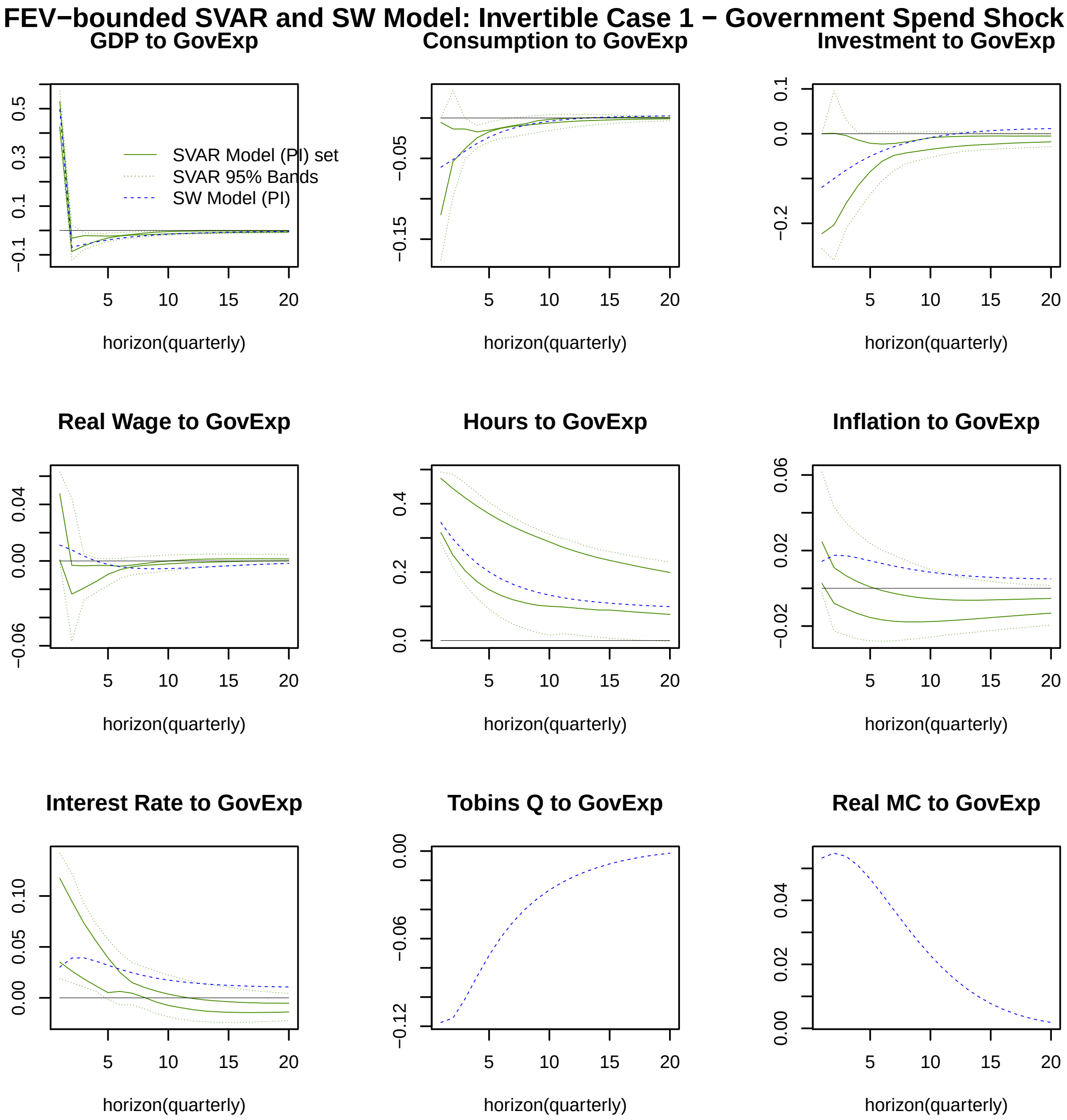


Figure 46: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using FEV Bounds (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Government Spending Shock**

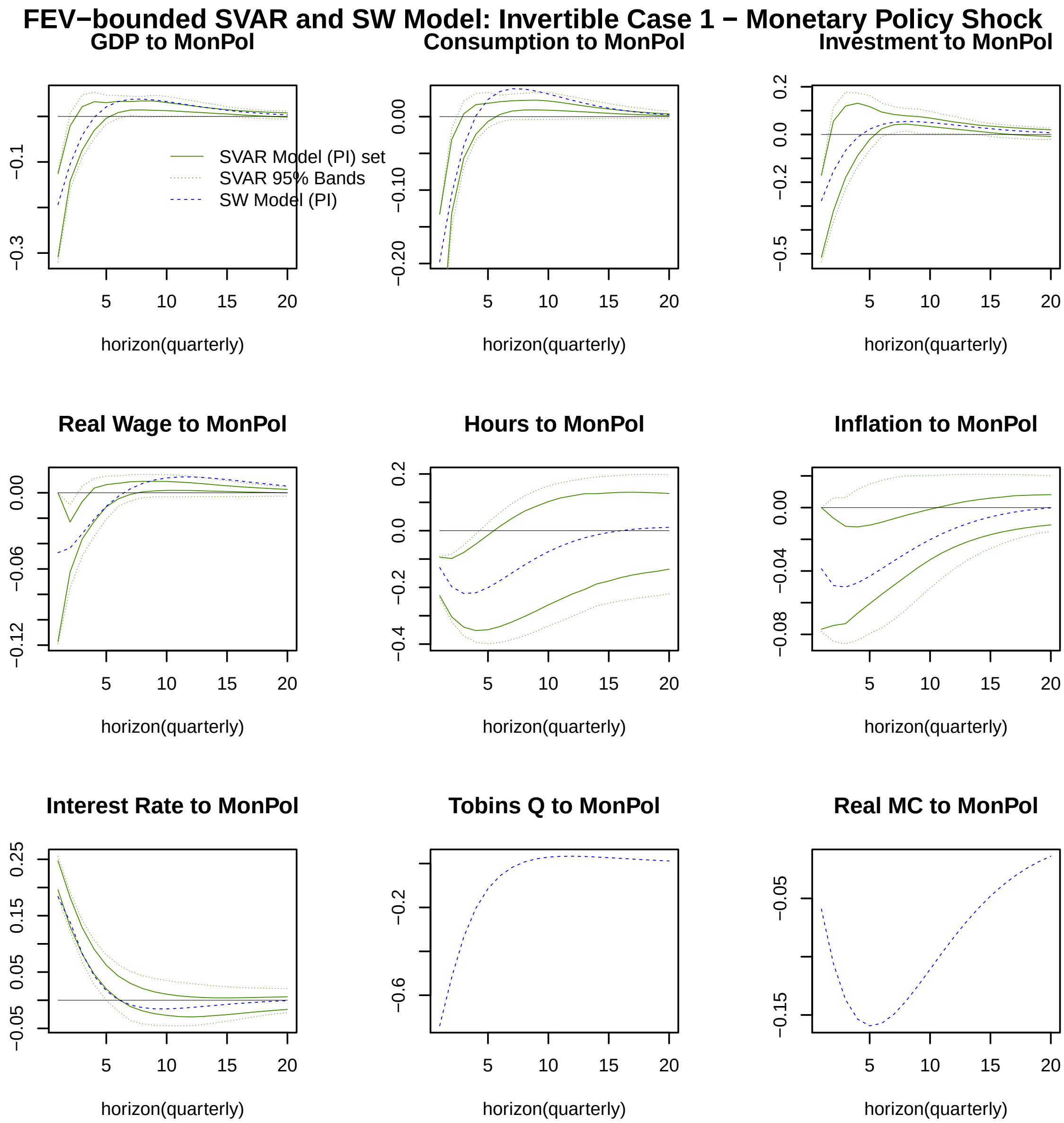


Figure 47: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using FEV Bounds (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Monetary Policy Shock**

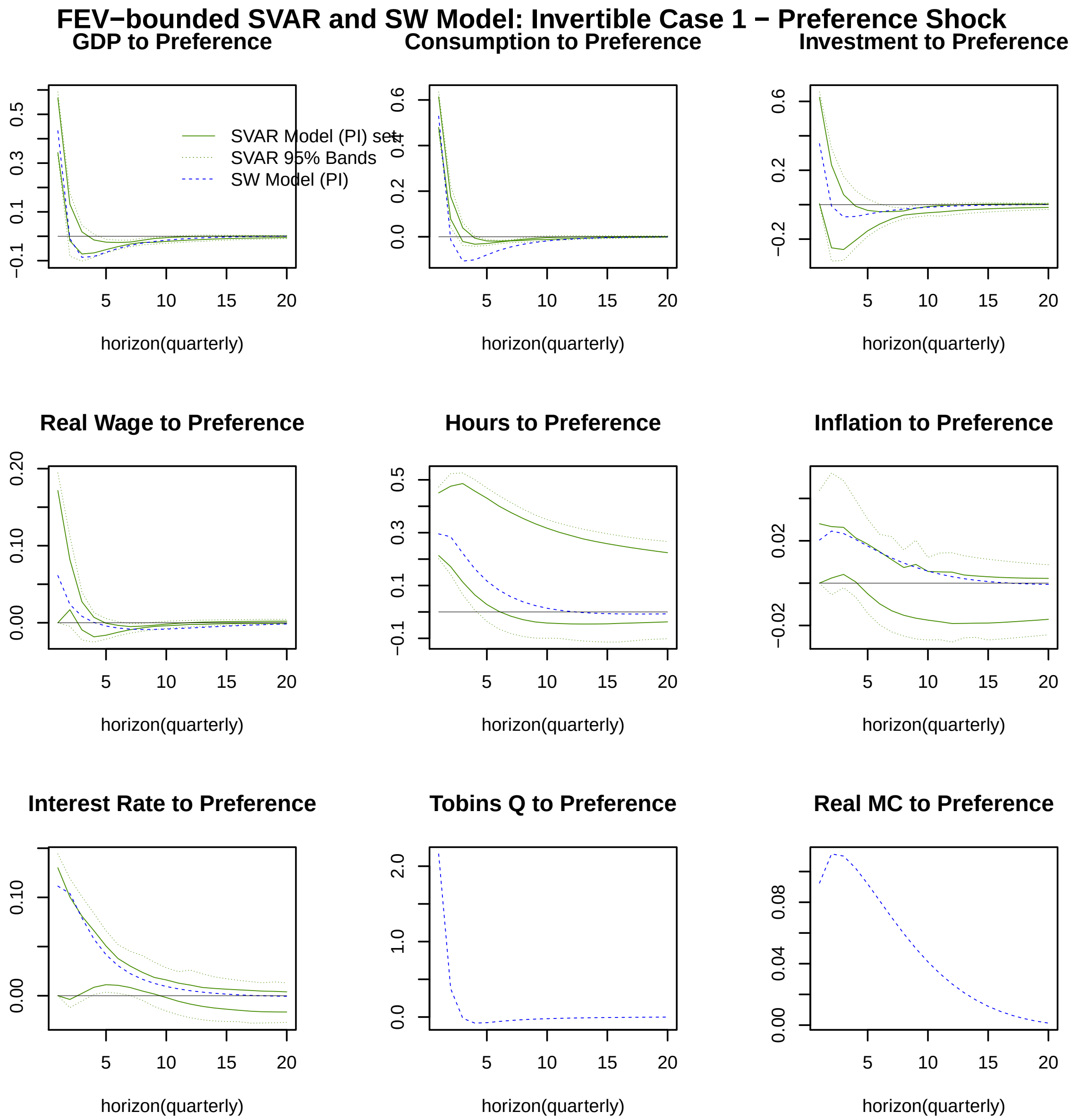


Figure 48: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using FEV Bounds (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Preference Shock**

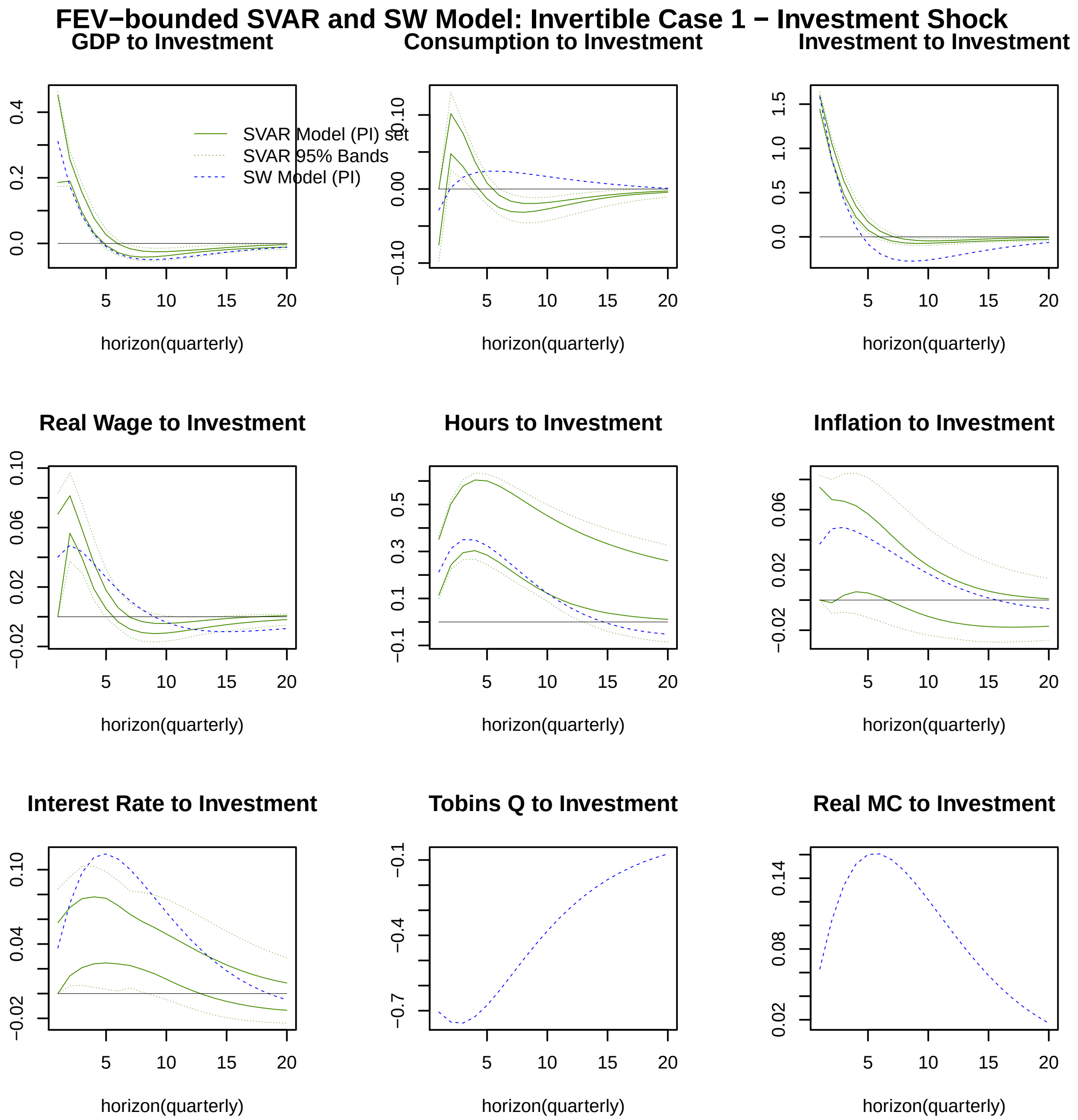


Figure 49: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using FEV Bounds (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Investment Shock**

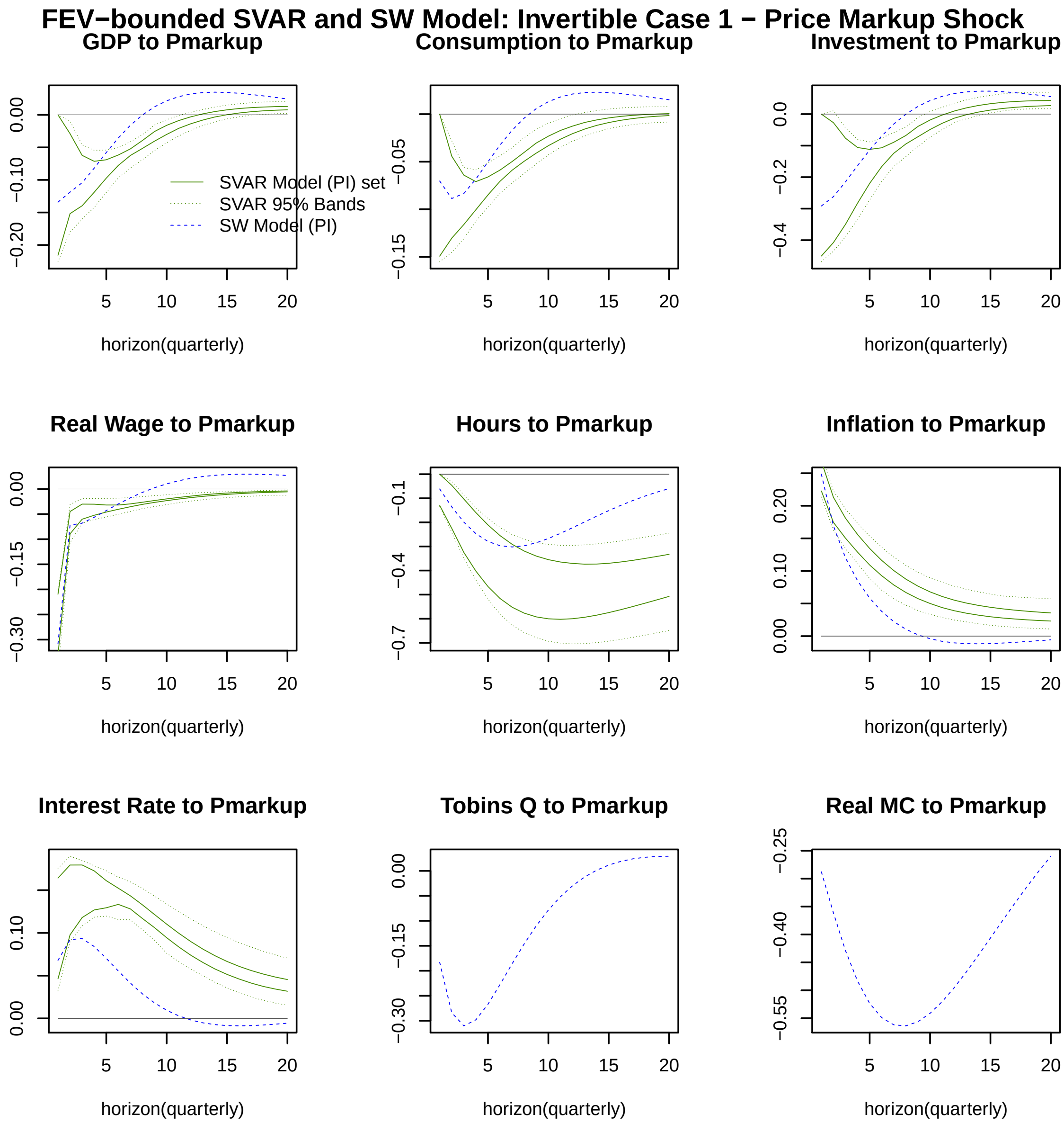


Figure 50: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using FEV Bounds (Solid) and Estimated SW Models (Dashed): Invertible Case 1 – Price Markup Shock**

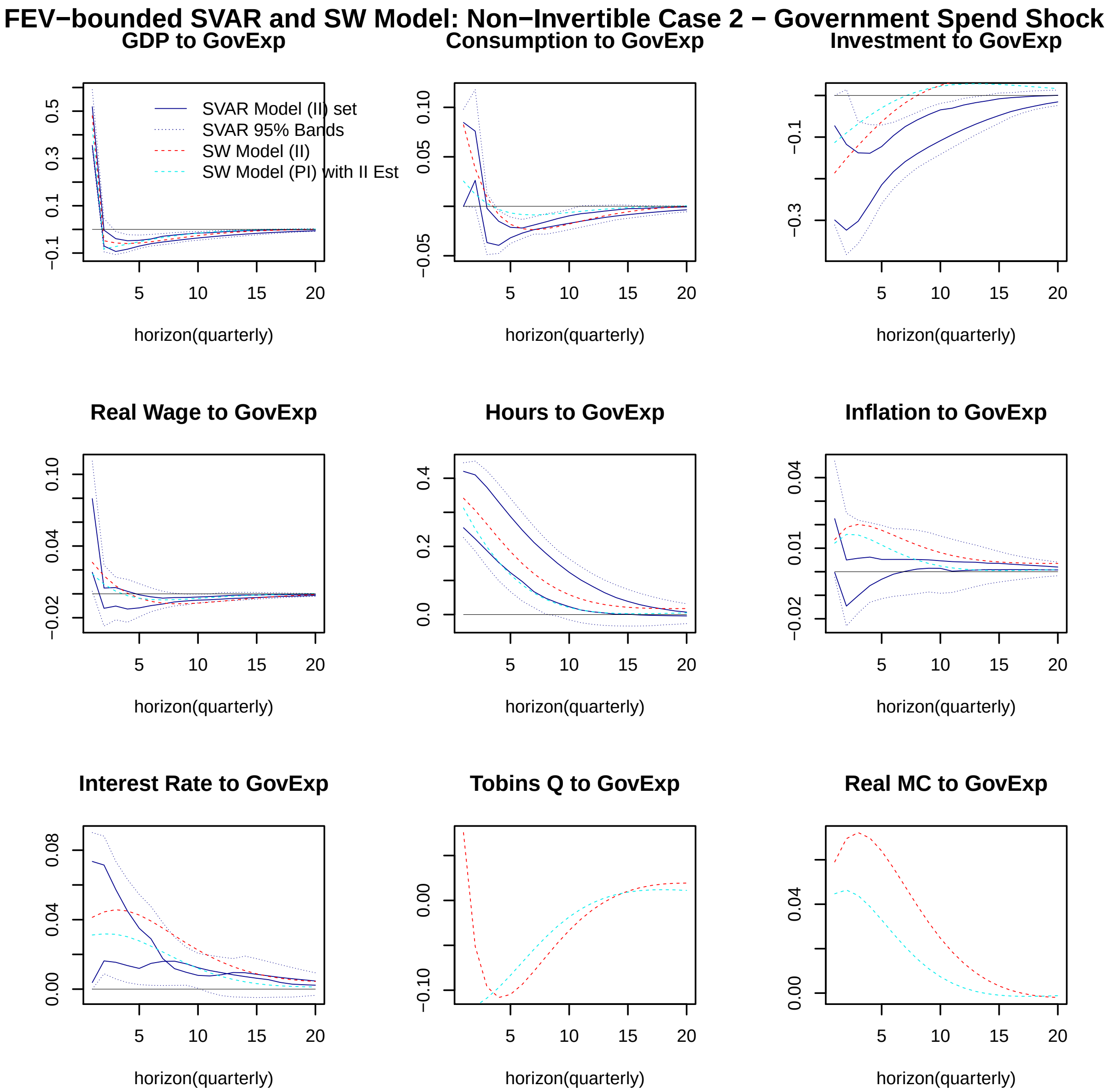


Figure 51: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using FEV Bounds (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Government Spending Shock** ($\mathbb{F}_g^{II} = 0.0194$, $\rho_g = 0.90$, $\varepsilon_g = 0.43$)

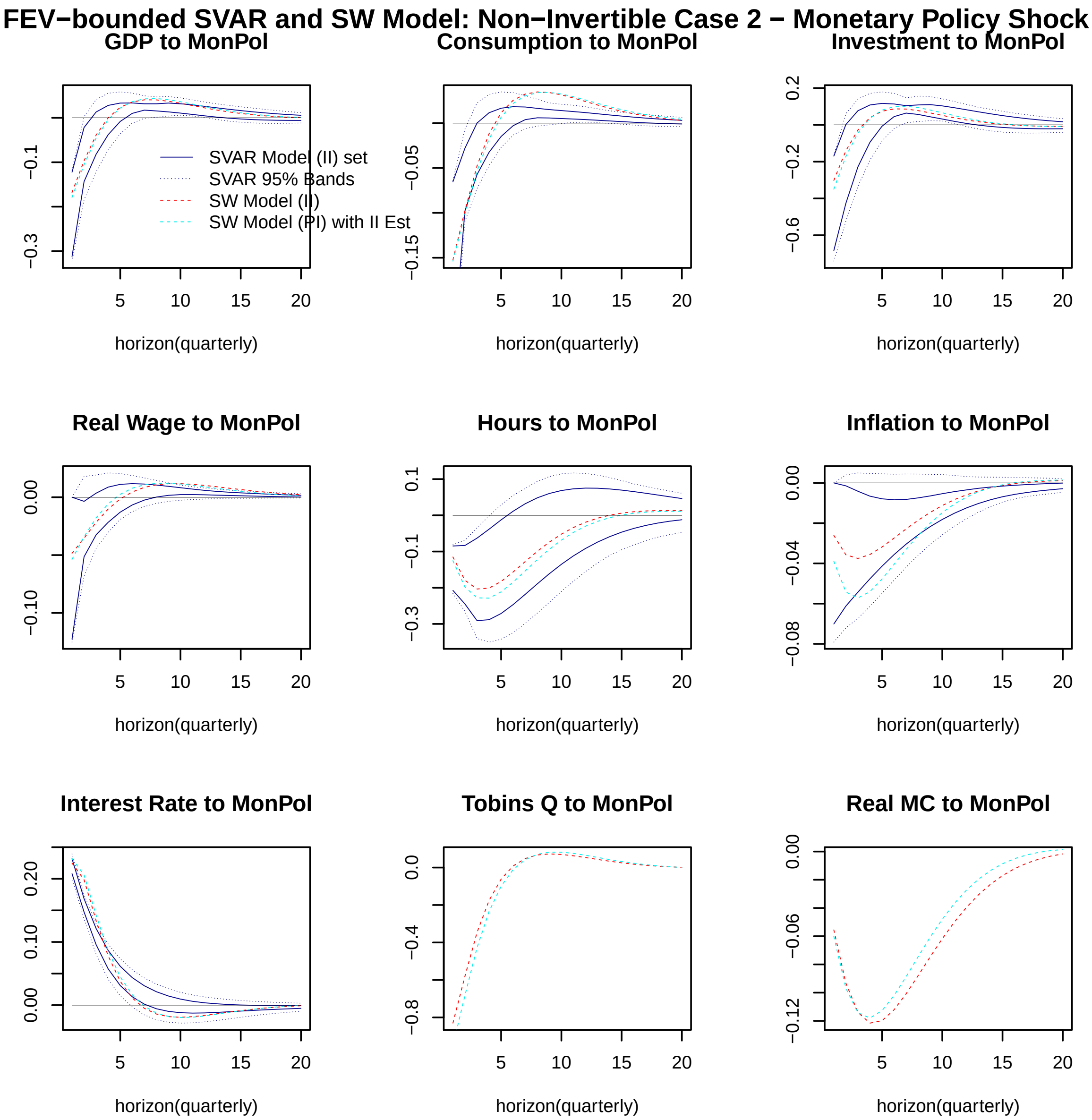


Figure 52: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using FEV Bounds (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Monetary Policy Shock** ($\mathbb{F}_r^{II} = 0.0036$, $\rho_r = 0.25$, $\varepsilon_r = 0.25$)

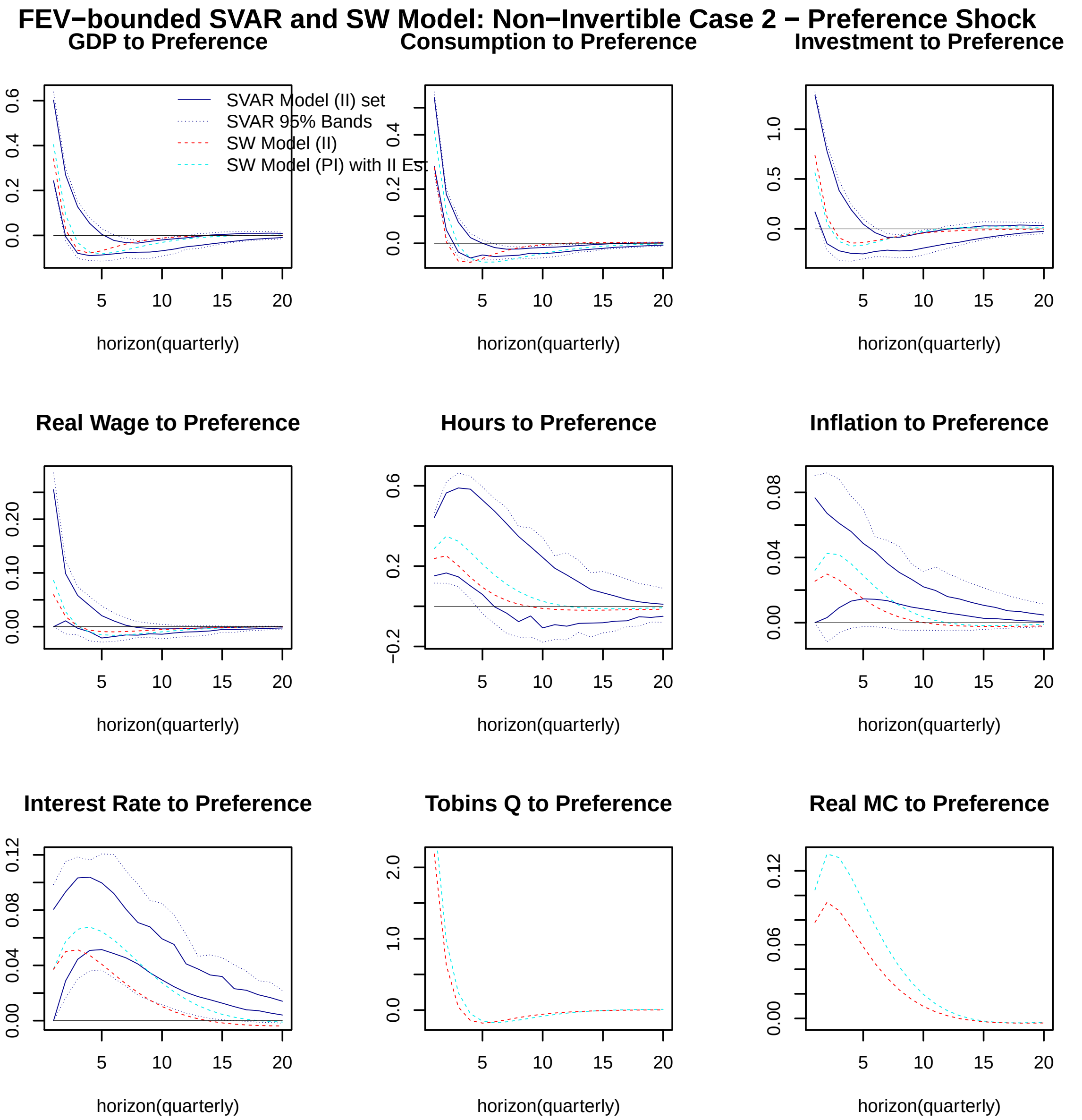


Figure 53: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using FEV Bounds (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Preference Shock** ($\mathbb{F}_b^{II} = 0.9526$, $\rho_b = 0.40$, $\varepsilon_b = 0.18$)

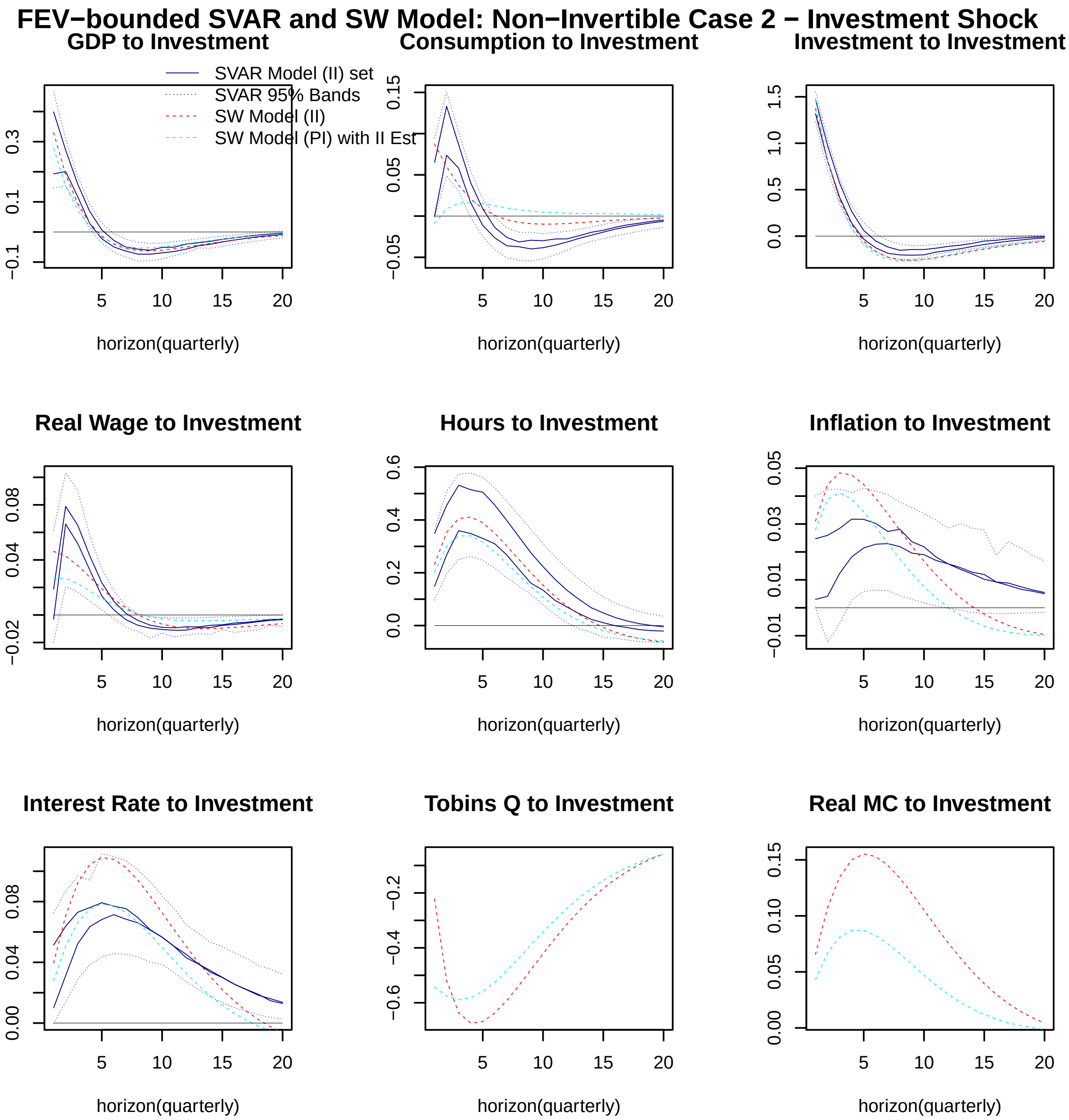


Figure 54: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using FEV Bounds (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Investment Specific Shock** (($\mathbb{F}_i^{II} = 0.5085$, $\rho_i = 0.77$, $\varepsilon_i = 0.37$)

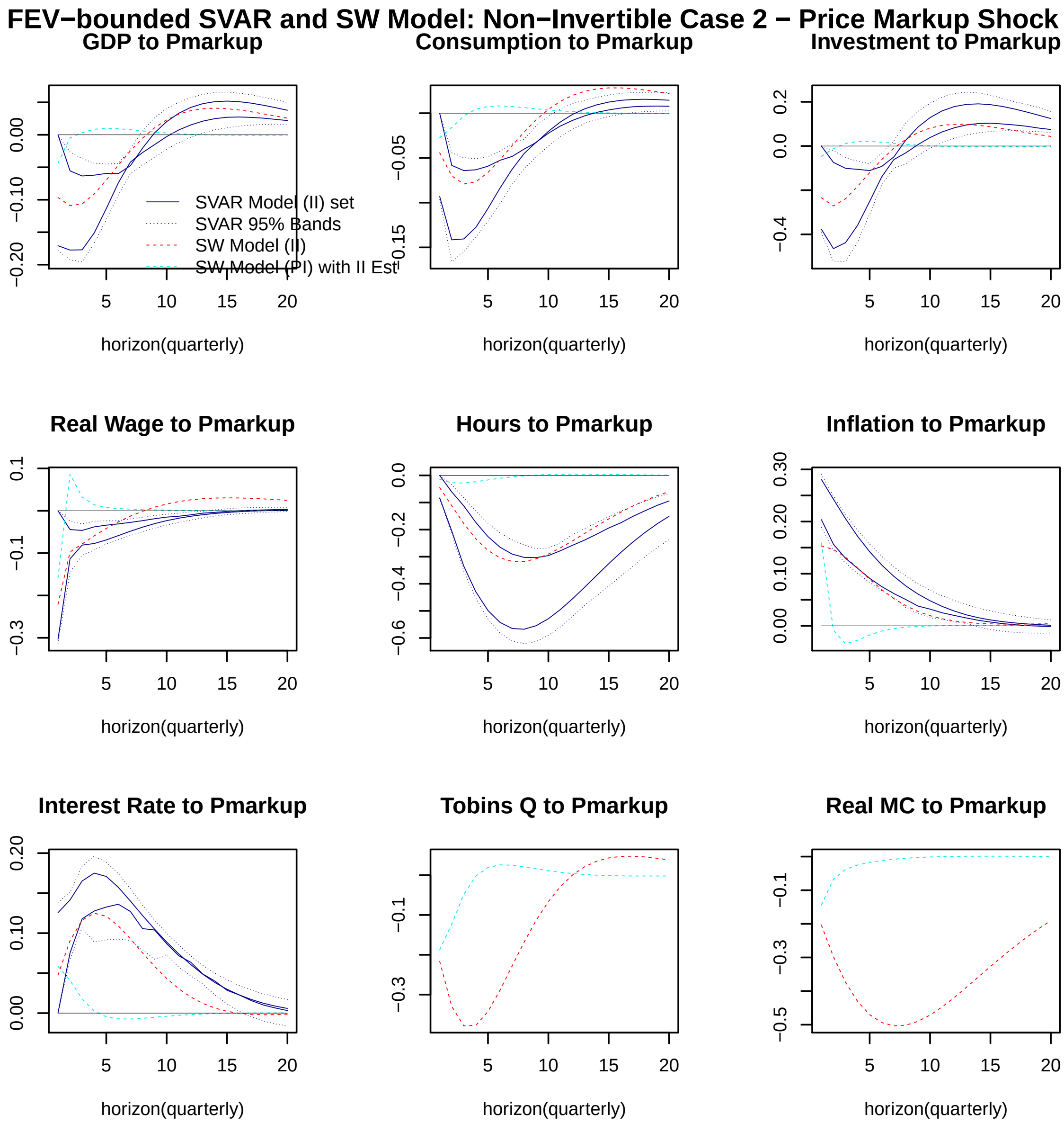


Figure 55: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using FEV Bounds (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Price Markup Shock** ($\mathbb{F}_p^{II} = 0.6655$, $\rho_p = 0.89$, $\varepsilon_p = 0.09$)

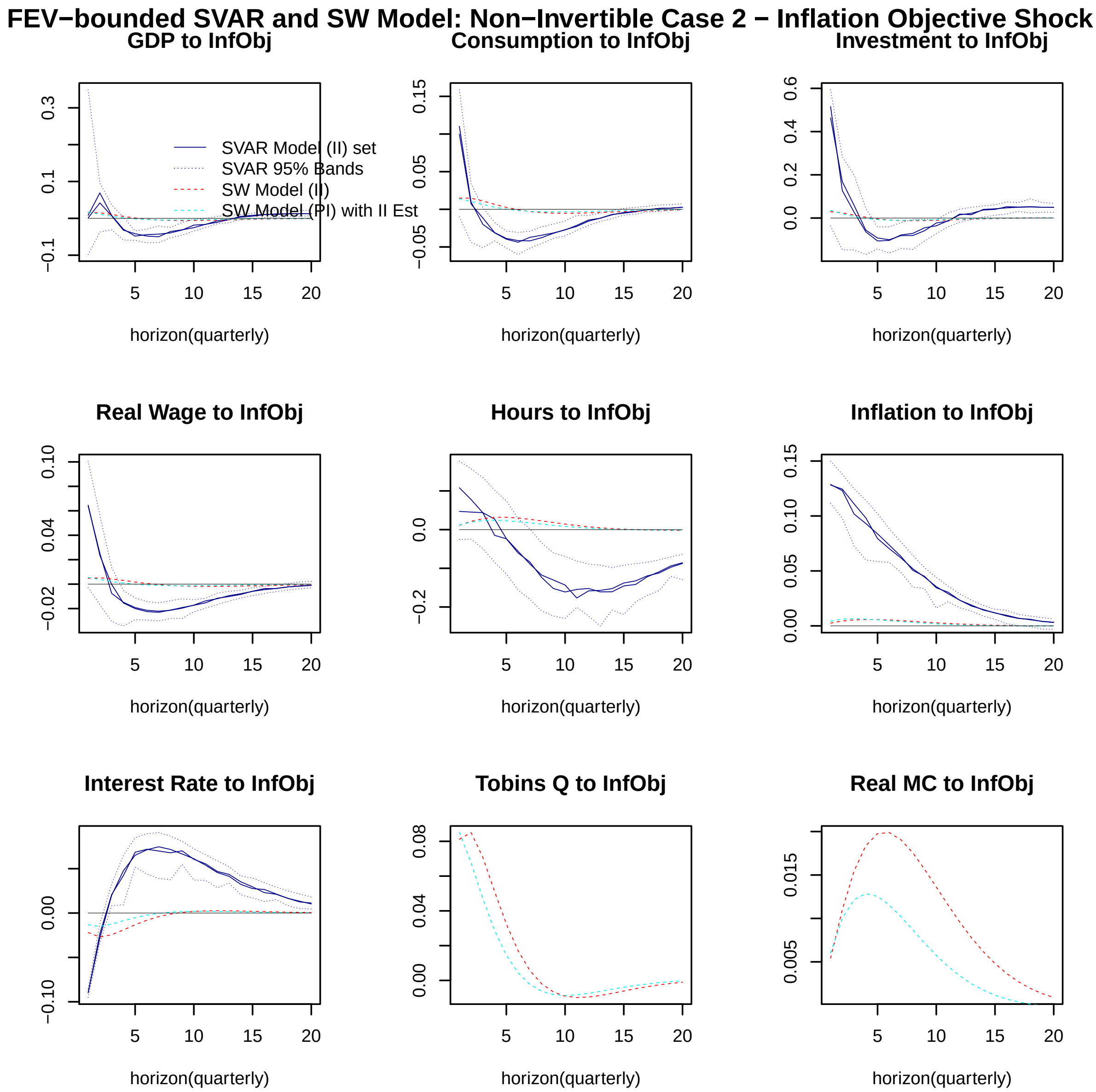


Figure 56: **Impulse Response Identified Set: Estimated SVAR(1) Model Identified using FEV Bounds (Solid) and Estimated SW Models (Dashed): Non-invertible Case 2 – Inflation Objective Shock** ($\mathbb{F}_t^{II} = 0.9989$, $\rho_t = 0.60$, $\varepsilon_t = 0.08$)