Discussion Papers in Economics

IMPERFECT INFORMATION AND HIDDEN DYNAMICS

By

Paul Levine
(University of Surrey),

Joseph Pearlman
(City University),

Stephen Wright
(Birkbeck College)

&

Bo Yang
(Swansea University).

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School of Economics
University of Surrey
Guildford
Surrey GU2 7XH, UK
Telephone +44 (0)1483 689380
Facsimile +44 (0)1483 689548
Web https://www.surrey.ac.uk/school-economics
ISSN: 1749-5075

Imperfect Information and Hidden Dynamics*

Paul Levine† Joseph Pearlman‡ Stephen Wright§ Bo Yang¶

October 18, 2023

Abstract

In a DSGE rational expectations model, the agents' assumed information sets are crucial for both the dynamics of the solution and for whether a SVAR econometrician can infer impulse responses to structural shocks. We adopt a heterogeneous agent, incomplete markets general framework where agents have imperfect and idiosyncratic information sets. In the limiting empirically plausible case of extreme heterogeneity, we show that a unique finite state space solution exists taking the same form as a single agent problem. The solution induces higher-order dynamics, hidden both from the agents and the econometrician, that would be absent in a perfect information economy.

JEL Classification: C11; C18; C32; E32

Keywords: Imperfect Information; Hidden Dynamics; Heterogeneous agents; Invertibility-Fundamentalness; Recoverability; SVARs

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*Earlier (and very different) versions of this paper were presented at a number of seminars and at the 7th Annual Conference of the Centre for Economic Growth and Policy (CEGAP), Durham University held on 19-20 May, 2018; as a keynote lecture at the 20th Anniversary Conference of the CeNDEF, 18-19 October, 2018, University of Amsterdam; at the 5th Annual Workshop of the Birkbeck Centre for Applied Macroeconomics, 14 June, 2019; at the 2019 CEF Conference, 28-30 June, Ottawa; at the 2019 MMF Conference, 4-6 September, LSE; at the EARG Conference "Macroeconomics and Reality: Where Are We Now?", 24 November, 2020, University of Reading, the Scottish Economics Society Annual Conference, 26 April, 2021 in a session in honour of Andy Hughes Hallet and the 2023 CEF and MMF annual Conferences. We acknowledge comments by participants at all these events and from discussions with Cristiano Cantore, Miguel Leon-Ledesma, Luciano Rispoli, Hamidi Sahneh, Tom Sargent and Ron Smith.

†School of Economics, University of Surrey, Guildford, GU2 7XH, UK, and Department of Economics, City University London, London, EC1R 0JD, UK. E-mail: p.levine@surrey.ac.uk

‡Department of Economics, City University, London, EC1R 0JD, UK. E-mail: joseph.pearlman.1@city.ac.uk

§Department of Economics, Mathematics and Statistics, Birkbeck College, University of London, London, WC1E 7HX, UK. E-mail: s.wright@bbk.ac.uk

¶Department of Economics, Swansea University, Swansea, SA2 8PP, UK. E-mail: bo.yang@swansea.ac.uk

1 Introduction

In a DSGE rational expectations model, the nature of agents' information sets is crucial for both the dynamics of the solution and for whether an econometrician can infer structural shocks and impulse responses from such a DGP. In this paper we adopt a heterogeneous agent, incomplete markets framework where agents have incomplete and idiosyncratic information sets. Solving for a general rational expectations equilibrium in this class of models is far from straightforward. Techniques pioneered by Nimark (2008) and others typically involve hierarchies of expectations ("beauty contests"), which in general imply infinite-order state-space representations that can only be solved numerically, and, thus of necessity, sacrifice the simplicity and insights of a single agent economy. However, in two limiting cases, things simplify considerably.

As idiosyncratic variation tends to zero, everyone is the same, so straightforwardly the economy can be represented by the behaviour of a single agent, and the informational problem simply disappears. But the economy also simplifies as idiosyncratic variation becomes extreme. In such cases, any aggregate signals from the idiosyncratic economy are effectively swamped by idiosyncratic volatility but agents must rely on whatever purely aggregate signals are available. It should be stressed that in such an economy there need be no "noise", in the sense commonly used in the literature. Nothing need be measured with error, and idiosyncratic varation will typically affect the optimal behaviour of agents in the model, but the informational problem will arise from agents' inability to distinguish idiosyncratic shocks from aggregate shocks.

In this paper, we exploit the properties of this second limiting case. We show that the typical agent's signal extraction problem in such an economy will take the same form as a single agent signal extraction problem and that the resulting solution will have a finite state space. But, crucially, we also show that the aggregate solution that results from the solution will be affected by the nature of optimal responses to strictly idiosyncratic shocks, even when such such shocks aggregate to zero. There are clear gains from this approach in terms of simplicity, tractability and insights on how heterogeneity and imperfect information impact on aggregate dynamics. It also allows us to exploit well-established results on single agent signal extraction problems.

Exploiting this solution method allows us to address the question at the start of the paper: how the general nature of the agents' signal extraction problem under imperfect information impacts on the econometrician's problem of attempting to infer the nature of structural shocks and associated impulse responses from the data. A key feature is that, if agents cannot directly observe nor infer structural shocks and therefore make errors in their interpretation, this induces additional hidden dynamics in the aggregate economy that would simply be absent in a perfect information economy. We show this manifests itself as Blaschke factors in the DGP, the parameters of which cannot be derived even from an infinite data sample.

As always, with any simplifying assumption, there are also losses in descriptive accuracy. But there is quite a lot of evidence, which we discuss in Section 1.6 below, that strictly idiosyncratic variation is indeed much greater than aggregate variation, so we argue that, for empirically relevant volatilities, outcomes are likely to be closer to our case than to outcomes that can only arise under the (still very common) assumption that heterogeneous agents are simply endowed with perfect information.

Our results are general and not model-specific, but to motivate them we first examine an illustrative example. We then discuss how our analysis relates to the existing literature.

1.1 A Motivating Illustrative Example

We start by illustrating the key elements of our analysis with reference to the informational implications of a simplified log-linearized version of a heterogeneous agent RBC economy with idiosyncratic and aggregate uncertainties as in Krusell and Smith (1998). The model itself is taken from Graham and Wright (2010), hereafter GW:[1]

$$\begin{aligned} k^s_{i,t+1} &= \kappa_1 k^s_{i,t} + \kappa_2(a_t + \varepsilon_{i,t}) + (1-\kappa_1-\kappa_2)c_{i,t} && (1)\\ \mathbb{E}_{i,t}c_{i,t+1} &= c_{i,t} + \kappa_3 \mathbb{E}_{i,t} v_{t+1} && (2)\\ a_t &= \phi a_{t-1} + \varepsilon_{a,t} \text{ where } \varepsilon_{a,t} \sim \text{n.i.i.d}(0,\sigma_a^2) && (3)\\ v_t &= (1-\alpha)(a_t - k_t) && (4)\\ \text{Agents' Information Sets}: \quad m^A_t &= v_t\,; \quad m^A_{i,t} = a_t + \varepsilon_{i,t} \text{ where } \varepsilon_{i,t} \sim \text{n.i.i.d}(0,\sigma_i^2) && (5) \end{aligned}$$

where $k^s_{i,t}$ and $c_{i,t}$ are, respectively, the capital stock supplied by households[2] and their consumption on island i in period t; a_t is aggregate technology; $\varepsilon_{i,t}$ is an idiosyncratic technology shock that aggregates to zero; v_t is the rental rate on aggregate capital, $k_t = \int \mu(i)k_{i,t}di$ where $\mu(i)$ is the density of agent i; and κ_1, κ_2 and κ_3 are linearisation parameters.

The informational problem in this setting arises directly from heterogeneity in an incomplete markets economy. Agents are assumed to have information sets that derive only from the markets they trade in: thus they only observe the aggregate rental rate v_t[3] and their local (island-specific) wage. In Appendix E.5, we show that this is equivalent to the information assumption $m^A_{i,t} = a_t + \varepsilon_{i,t}$ in (5). Given this "market-consistent" information set, GW show that the decentralized market equilibrium cannot replicate,

[1] Appendix Section E.5 sets out the model as a special case of a standard RBC model with a fixed labour supply. The model of Rondina and Walker (2021) is a restricted case with 100% depreciation of capital, although its structure is formally equivalent to GW.

[2] Note that $k^s_{i,t}$ differs from capital stock rented by firms on island i since capital is free to flow from less to more productive islands.

[3] GW show that almost identical results arise if agents observe a common risk-free rate.

and differs in important ways from, the solution that would be achieved if all aggregate states were directly observable. Thus in this framework, PI, as assumed originally by Krusell and Smith (1998) (and still commonly assumed in much of the heterogeneous agent literature), is *not* market-consistent: it can only arise if the information is essentially provided as endowment.

Note that, in this model, there is no ‘noise’. Nothing is measured with error: the island-specific technology shock has real effects, which will always affect the optimal behaviour of agents in the model. But the information itself may be noisy which is the source of the informational problem.

An advantage of this simple model is that it is possible to derive a finite state space analytical solution for any value of the idiosyncratic variance σ_i,[4] which we can compare with our solution for the limiting case, to which we now turn.

1.2 Extreme Heterogeneity Leads to a Single-Agent Problem

In the limiting case, as varσ_i approaches infinity, the idiosyncratic wage provides essentially no information about aggregate technology, leaving agents with only the aggregate signal from the return on capital as an input to their filtering problem for the aggregate economy. Rational agents in such a heterogeneous economy will know that other agents face an identical problem; as a result, all agents will share (and know that they share) a common estimate of aggregate capital and aggregate technology. As a result, while the general solution analyzed by GW involves an infinite order hierarchy of expectations, in this special case, the hierarchy collapses, and hence the economy has a finite state-space representation.

Since all agents share a single common signal of the aggregate economy (the rental rate, v_t), this economy closely resembles, but is *not* the same as an economy with a notional single agent who only observes v_t. While there has been a substantial literature that assumes imperfect information in a single agent model, building on the foundations developed by Pearlman *et al.* (1986),[5] any such model is subject to the critique that it cannot explain *why* information is imperfect.[6]

But this limitation of the single agent model with imperfect information does not stop it being useful. If we solve the model for the limiting case of extreme heterogeneity (as $\sigma_i \to \infty$) we show that, as a general result (Theorem 2), the solution for the aggregate economy turns out to have the *same form* as for a parallel economy with a single agent

[4] See Appendix D.1.

[5] Theorem 1 shows how a general linear RE model can be converted into a form used by Pearlman *et al.* (1986).

[6] GW argue that the complete markets assumption required for the existence of a single agent in a heterogeneous economy must *imply* perfect information. And in the limiting homogeneous case ($\sigma_i \to 0$) by inspection the informational problem disappear.

with a censored imperfect information set who only observes v_t. But we also show that, crucially, the aggregate dynamics of this parallel economy are affected in important ways by the underlying heterogeneity.

Figure 1 illustrates the underlying mechanisms. It shows the responses to a positive technology shock in two cases of imperfect information (II): the first is the limiting case of extreme heterogeneity (which we denote II-HA(∞)); the second is the solution for a notional single agent with an artificially censored information set (which we denote II-SA). For comparison, it also shows responses in the case where perfect information (PI) is simply assumed. The key differences stem from the responses of aggregate consumption to agents' best estimate of the capital stock.

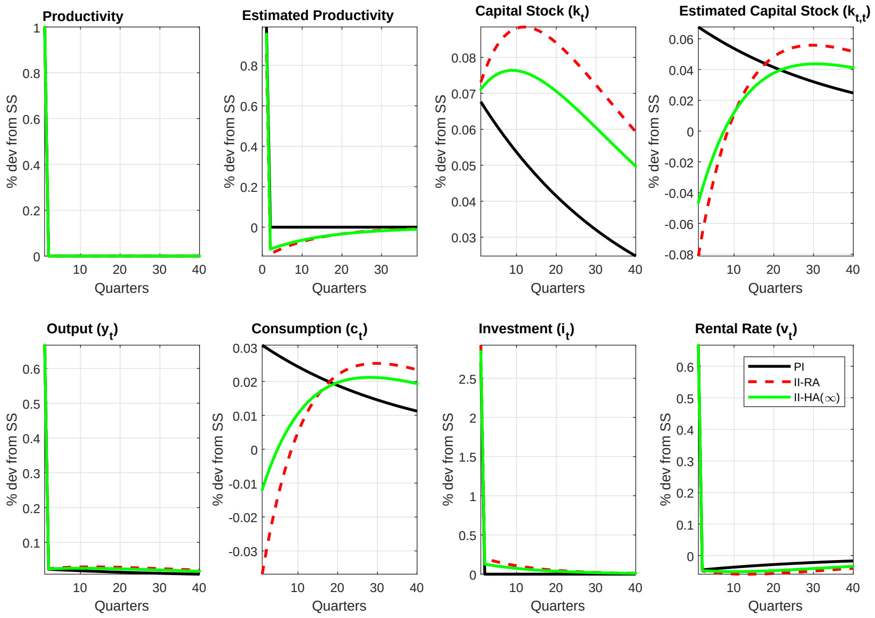

Figure 1: Simple RBC Model. Impulse Responses to a Temporary Technology Shock for PI and II-HA(Σ) as $\Sigma \to \infty$ Compared. Parameter Values: $r = 0.01$, $\alpha = 0.333$, $\delta = 0.025$, $\sigma = 2$

In the benchmark case in which heterogeneous agents are simply assumed to have PI (which we denote PI-HA), the approximate aggregation result of Krusell and Smith (1998) becomes exact, given the linearization, so the solution for the aggregate economy is identical to the solution for a single agent economy with PI (which we denote PI). Hence the productivity shock causes the familiar response of a temporary rise in consumption, with a modest degree of capital accumulation providing some element of consumption smoothing. Note that while the aggregate PI-HA solution is identical to the PI solution, heterogeneous agents also have optimal saddlepath responses to idiosyncratic states, which, for large values of σ_i will actually dominate individual behaviour; but these responses all cancel out at the aggregate level.

In stark contrast, in both the II-HA(∞) and II-SA cases, the positive productivity shock is initially misinterpreted as bad news, reflecting the offsetting effects of technology and capital on the return. In both cases, estimates of aggregate capital fall, triggering a fall in consumption. But the responses are not identical. The key difference is that, in the II-HA(∞) case, agents can observe their own capital. So bad news for aggregate capital must imply exactly offsetting good news for aggregate estimates of the idiosyncratic components of capital and technology (while idiosyncratic components must cancel in the aggregate, they do not cancel in aggregate expectations). But the optimal responses to estimates of idiosyncratic states in the II-HA(∞) case are small. Hence as Figure 1 shows, the consumption response is still negative. In contrast, in the II-SA case, there is only bad news, so consumption falls more sharply.

But while the II-SA case overstates the negative response, our Theorem 2 shows that the general II-HA(∞) case can be solved *as if* it were an II-SA case, and hence using the techniques of Pearlman *et al.* (1986). We show that the filtering problem for the aggregate economy that agents need to solve takes an identical form to the II-SA case; but a key matrix that feeds into the problem is shifted by the optimizing saddlepath responses to idiosyncratic states in the PI-HA case. So heterogeneity does nontrivially influence the dynamics of the aggregate economy, but in a way that can be captured exactly in a parallel single agent imperfect information economy, thereby allowing the application of a well-developed toolkit for solving the informational problem.

1.3 Extreme vs Intermediate Heterogeneity

An obvious question is how good an approximation our limiting case of II-HA$_\infty$ provides for less extreme degrees of heterogeneity. In the simple case of this example we can address this issue, since we can also derive an exact analytical solution for different degrees of heterogeneity, ie for the full range of values of $\Sigma = \frac{\sigma_i}{\sigma_a}$, which we denote II-HA$_\Sigma$.

Figure 2 illustrates the results. It shows the responses of aggregate consumption and other key aggregates to an iid aggregate technology shock for a wide range of values of Σ. For a quite wide range of empirically relevant values, the limiting case matches intermediate cases quite well, and via the same mechanism: the technology shock is interpreted as bad (or at least less good) news on the capital stock. Even when $\Sigma = 1$ (which would be very much at the low end of the empirically plausible range), the response of consumption to a technology shock is nontrivially damped, compared to the PI solution. Thus, while the limiting case is clearly restrictive, it also points clearly to the equally restrictive nature of solutions that simply rely on the assumption of PI, provided as an endowment. We also show in the next section that the implications for time series properties and hidden dynamics also apply in this more general case.

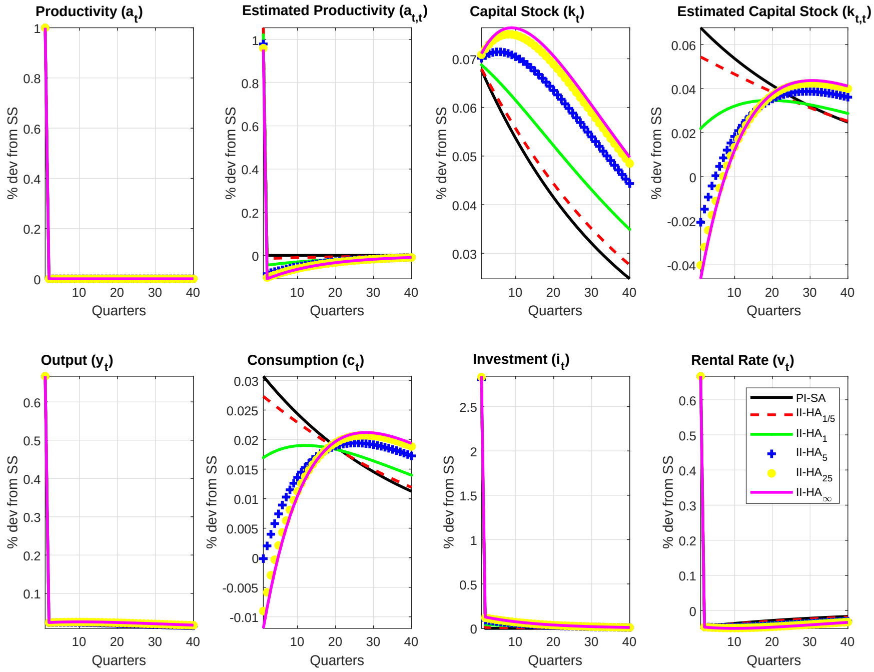

Figure 2: Simple RBC Model. Impulse Responses to a Temporary Technology Shock for PI, II-SA and II-HA(Σ) where $\Sigma \equiv \frac{\text{var}(\varepsilon_{i,t})}{\text{var}(a_t)}$. Parameter Values: $r = 0.01$, $\alpha = 0.333$, $\delta = 0.025$, $\sigma = 2$

1.4 Time Series Properties and Hidden Dynamics

Figures 1 and 2 also illustrate another crucial feature of all II cases: the initial errors in interpreting the productivity shock have prolonged impacts on capital accumulation, and thus induce additional dynamics in response to a productivity shock that are entirely absent under PI.

For an econometrician observing this model economy, this 'contamination' of aggregate dynamics by filtering errors has crucial implications. To illustrate, in the model in its simplest form, with no persistence in technology ($\phi = 0$), and for empirically plausible values of σ, using the general solution procedures set out in Section 2 and 3 it can be shown that for all possible cases, the single observable v_t in our simple example always has a fundamental ARMA(1,1) representation, of the general form

$$v_t \quad = \quad \left(\frac{1 - \psi_s L}{1 - \mu L} \right) e_{s,t} \tag{6}$$

where both the MA parameter ψ_s and the fundamental innovation $e_{s,t}$ differ across cases, for $s \in \{\text{II-HA}(\infty), \text{II-HA}(\Sigma), \text{II-SA,PI}\}$, with

$$
\begin{aligned}
e_{s,t} &= (1-\alpha) \underbrace{\left(\frac{1 - \frac{L}{\lambda_s}}{1 - \lambda_s L} \right)}_{\text{Blaschke Factor}} \varepsilon_{a,t} \qquad (7) \\
\lambda_{II-HA(\infty)} &= \frac{\kappa_1}{\kappa_1 + \kappa_2} > \lambda_{II-SA} = \beta \lambda_{II-HA(\infty)} \\
\psi_{II-HA(\infty)} &= \mu \lambda_{II-HA(\infty)} > \psi_{II-RA} = \beta^2 \psi_{II-HA(\infty)} \\
\lambda_{PI} &= \psi_{PI} = \frac{1}{\mu} \lambda_{II-HA(\infty)} > \lambda_{II-HA(\infty)}, \\
\lambda_{PI} &< 1 \quad \text{for empirically plausible values of } \sigma
\end{aligned}
$$

but where the AR parameter μ is common across all cases, and equal to the single stable eigenvalue in the PI case.[7] Details of the solution for the non-limiting case are given in Appendix B.3.10.

All the possible cases (including, for plausible parameters the PI case) have the common property that the fundamental innovation e_{st} that can be recovered from the history of v_t, is *not* a scaling of the true structural shock $\varepsilon_{a,t}$, but is driven by a Blaschke Factor that maps from the history of the structural shock to the observable fundamental innovation with the case-specific form (7); and the structural shock is always non-fundamental. To this extent, all versions of this simple economy share the common feature that, from the perspective of an econometrician observer at time t, there are hidden dynamics. But there is a key, and crucial, difference. In the PI case,[8] these dynamics are hidden from the econometrician, but they are, given the assumption of perfect information, visible to agents in the economy. In all the II cases they are also hidden to agents in the economy; and as Figures 1 and 2 illustrate, the errors agents make in estimating aggregate states result in higher order dynamics. This implies a key difference in the time series properties of v_t.

In the case of PI, the true DGP is a nonfundamental ARMA(1,1) process driven by the structural shock $\varepsilon_{a,t}$, i.e., of the same order as the fundamental ARMA. In Lippi and Reichlin's (1994) terms it is nonfundamental but "basic". In contrast, in all cases of II, the true DGP is a nonfundamental ARMA(2,2) (since in all cases $\lambda_s \neq \psi_s$) in the same shock: implying it is both nonfundamental and "nonbasic". This feature generalises: imperfect information economies will always have higher-order dynamics than full information economies.

This matters, because, while all the parameters of a basic representation can be re-

[7] See Appendices C.2–C.5 for details. We also show (in the Appendix Section E.7) that the properties given below hold for values of σ greater than around one half, hence in line with the majority of empirical estimates. The result generalizes easily to cases with $\phi > 0$.

[8] As noted above, at the aggregate level, PI is identical to PI-HA.

covered from the data (hence $\lambda_{PI} = \psi_{PI}$) this is not the case for nonbasic representations: hence for all of the II cases, nothing in the history of v_t would ever reveal the value of λ_s, and hence the true structural shock. In discussing such representations Lippi & Reichlin asserted that such representations were "*not likely to occur in models based on economic theory*". But our example provides a clear counter-argument to this assertion: in cases of imperfect information, nonbasic representations are actually very much to be expected; they arise directly from the agents'signal extraction problem. As a result the hidden dynamics of an imperfect information economy are much more deeply hidden than those of a perfect information economy.

1.5 Relating the example to our general results

The features of the simple example also illustrate the remainder of our general results.

Theorem 3 analyzes the general nature of the relationship between "A-invertibility" and "E-invertibility": whether, respectively, agents in the economy or an econometrician can observe, or infer structural shocks and states from what they observe. There is a crucial link between both properties and the "Poor Man's Invertibility Condition" (PMIC) of Fernandez-Villaverde *et al.* (2007).

For a given set of observables, Theorem 3 shows first that E-invertibility is impossible without A-invertibility - in itself perhaps an unsurprising result. But we also show that a necessary, but not sufficient condition for A-invertibility is that E-invertibility would hold if (hypothetically) agents were simply endowed with PI. In our example, for empirically plausible values of σ, the structural shock is non-fundamental even under PI, hence this condition is not satisfied, and as a result, both A- and E-invertibility fail at the first hurdle. However, for sufficiently low values of σ in our example, the productivity shock *would* be fundamental and hence E-invertible under PI.[9] But while E-invertibility under PI is necessary, it is not sufficient for A-invertibility under general conditions of II. Theorem 3 shows that applying a generalized version of the PMIC to the full state-space representation of the economy under II (which, it may be recalled, must be of higher dimension than under PI) implies additional conditions for A-invertibility. In the illustrative example, these are violated, for any value of σ, so both A- and E-invertibility fail.

The remainder of our results draw out key general implications of imperfect information, which can also be illustrated with reference to the example.

Theorem 4 shows that, in the absence of A- (and hence E-) invertibility, the solution for the aggregate economy can never replicate PI, and must always incorporate Blaschke factors of the same general form as in our example.

Theorem 5 then shows that, despite the higher dimension of the structural state-space

[9]Since $\partial\mu/\partial\sigma > 0$, for sufficiently low values of σ, $\lambda_{PI} > 1$, so the Blaschke factor disappears.

representation induced by imperfect information, there will always be a fundamental representation (the "innovations representation" of Fernandez-Villaverde *et al.*, 2007) of the same dimension as under PI, with the remainder of the structural dynamics captured by Blaschke factors. In our example, this implies the feature noted above that, in both cases of II, there is a fundamental ARMA(1,1) representation, i.e, of the same order as the structural ARMA under PI, despite the fact that, in both II cases, the true structural representations are ARMA(2,2). Crucially, however, the innovations in these fundamental representations are not equal to the true structural shock.

Theorem 6 relates our results to the property of "recoverability"(see Chahrour and Jurado, 2022): it shows that recoverability must also fail, at a practical level, when A-invertibility fails. To illustrate this property in our example, consider the position of an econometrician at some time $T >> t$. As $T - t \to \infty$, nonfundamental shocks at time t, like the productivity shock in our example, *would* in the limit be recoverable from the history v^T, since while Blaschke factors are not invertible working backwards in time, they *are* invertible working forward in time, which is a requirement for recoverability.[10] But, crucially, this will only have any practical applicability under PI, in which case the Blaschke parameter $\lambda_{PI} = \psi_{PI}$ can be estimated directly from the data (since the representation is basic) whereas for all cases of II $\lambda_s \neq \psi_s$,and hence λ_s cannot be estimated directly from the data. Thus in in practice recoverability faces an acute identification problem.

These results imply a clear health warning to anyone using estimated fundamental time series representations (which we refer to generically as VARs[11]) in an attempt to estimate structural shocks and impulse response functions. To do so without reference to the informational structure of the economy, and how this compares to the information set of the econometrician, may lead to nontrivial errors of inference. If the econometrician has an information set that is a weak subset of the agent's (in general, imperfect) information set, then fundamental innovations may be erroneously labelled as structural shocks, and impulse responses may differ nontrivially from true structural impulse responses.

But this raises an obvious question: can we assess how different structural shocks will be from observable innovations? The final Theorem 7 constructs a general measure of approximate fundamentalness that applies to both perfect and imperfect information

[10]Since from (6) and (7), the structural ARMA under PI implies

$$\varepsilon_{a,t} = \frac{1}{\alpha}\left(\frac{1-\mu L}{1-\lambda_{PI}^{-1}L}\right)v_t = \frac{\lambda_{PI}}{\alpha}\left(\frac{1-\mu L}{1-\lambda_{PI}F}\right)v_{t+1} \tag{8}$$

where $F = L^{-1}$ is the forward shift operator. Hence $\varepsilon_{a,t}$ is a convergent sum of current and future values of v_t. See Appendix H for other illustrative examples of recovering shocks from future values of observables.

[11]The true reduced form will typically be a VARMA, or VAR(∞), and may sometimes be estimated directly by state-space methods (e.g., Smets and Wouters, 2007) but will more commonly be a finite order approximation.

assumptions. In our illustrative example, this comes down to the correlation between the true structural innovation $\varepsilon_{a,t}$ and the fundamental innovation $e_{s,t}$. It is straightforward to show that, as λ_s, the MA parameter in the Blaschke polynomial becomes sufficiently close to unity, then $\text{corr}\left(\varepsilon_{a,t}, e_{s,t}\right)$ also tends to unity. On an empirically plausible calibration, $\lambda_{\text{HA}_\infty}$ is, on the one hand, sufficiently close to unity that $\varepsilon_{a,t}$ and $e_{s,t}$ would be expected to be quite strongly positively correlated; but on the other hand, sufficiently far from unity that impulse responses to true productivity shocks are distinctly more complex and prolonged than under PI.

1.6 Contributions to Existing Literature

There are four strands of literature related to our paper.

The first strand is a largely econometrics literature on the invertibility/fundamentalness problem which was first pointed out in the economics literature by Hansen and Sargent (1980). Two seminal papers are Lippi and Reichlin (1994) that introduces Blaschke matrices and Fernandez-Villaverde *et al.* (2007) that examines conditions for a solution of a rational expectations (henceforth RE) model to have a VAR representation. A comprehensive review is provided by Alessi *et al.* (2011) and much of this material is now found its way into two excellent macro-econometrics textbooks: Canova (2007) and Kilian and Lutkepohl (2017).

In the econometrics literature, a more recent approach bypasses the intervening step of a SVAR and uses external or internal instruments which are variables correlated with a particular shock of interest, but not with the other shocks. Instruments can then be used to directly estimate causal effects by direct instrumental-variables regressions using the method of local projections of Jorda (2005).

This invertibility/fundamentalness problem is often described in this first strand of literature as one of "missing information" when the econometrician does not have all the information that agents in the data generating process (henceforth DGP) have. This leads to a second literature that focuses on news shocks as an example of this extra information: see, for example, Leeper *et al.* (2013), Blanchard *et al.* (2013) and Forni *et al.* (2017).[12] In our paper, missing information of this form is not at the heart of the problem, but rather it is imperfect information on the part of both agents and the econometrician that takes centre stage; indeed the information sets can be the same for both without removing non-fundamentalness.

A third literature on imperfect information in single agent models was initiated by Minford and Peel (1983) and generalized by Pearlman *et al.* (1986) - henceforth PCL

[12]Recent surveys of these two strands of the literature and the relationship between VAR and DSGE models are provided by Sims (2012) and Giacomini (2013). However, in common with the literature, these surveys explore the issue without examining the main focus of our paper - the information assumptions of the agents in the underlying structural model.

- with further contributions by Pearlman (1992), Woodford (2003), Collard and Dellas (2010) and Baxter *et al.* (2011). A general theme of these papers is that II can act as an endogenous persistence mechanism in the business cycle. Ellison and Pearlman (2011) incorporated II into a statistical learning environment. Applications with estimation were made by Collard *et al.* (2009), Neri and Ropele (2012) and Levine *et al.* (2012). Leeper *et al.* (2013), Blanchard *et al.* (2013) mentioned above also study information issues in a single agent framework. Both these papers emphasize a main theme of our paper, namely, that macroeconomic variables in the DSGE DGP process can only convey information available to agents in the model. It follows that, if agents lack PI (non-A-invertibility in our terminology) and do not observe current structural shocks, then the macroeconomic time series cannot contain the information to recover the shocks in an estimated VAR.

A fourth literature is a class of heterogenous agent models that can be traced back to Townsend (1983) which distinguish local (idiosyncratic) information and (aggregate) information, e.g., Lucas (1975), Pearlman (1986), Woodford (2003), Pearlman and Sargent (2005), Nimark (2008), Angeletos and La'O (2009), Graham and Wright (2010), Nimark (2014), Adams (2021), Adams (2023), Ilut and Saijo (2021), Okuda *et al.* (2021), Rondina and Walker (2021), Huo and Pedroni (2020), Huo and Takayama (2021), Angeletos and Huo (2021) and Broer *et al.* (2021). Angeletos and Lian (2016) provide a recent comprehensive survey of what they refer to as the incomplete information literature.[13]

To elaborate on our paper's contribution to this fourth strand of literature, we follow Pearlman and Sargent (2005) who use the method of PCL to obtain a finite-space 'single-agent' RE solution that avoids higher-order beliefs (which is also a feature of many of the papers cited above). Our paper provides a general finite-space time-domain solution in a framework that encompasses all those on offer in these citations. Our time-domain contribution is closest to frequency domain solution of Rondina and Walker (2021) though more general in the sense that the HA framework allows for non-scalar states, but less general in the sense we assume the limiting case where idiosyncratic uncertainty far outweighs aggregate uncertainty. (However we do have a non-limiting case HA solution for our illustrative model above). Unlike that paper and Adams (2021) we do more than characterize the solution in that we provide existence and uniqueness results. Furthermore, unlike the other papers cited our solution is general and not model specific. We can therefore claim to have a HA (limiting case) solution that generalizes BK (for PI) and PCL (for II-RA) in a comparable general HA framework.

We also draw on empirical evidence on the relative magnitude of idiosyncratic vs ag-

[13]Here a comment on terminology is called for. Our use of perfect/imperfect Information (PI/II)is widely used in the second strand of literature when describing agents' information of the history of play driven by draws by Nature from the distributions of exogenous shocks. The complete/incomplete framework of the Angeletos and Lian (2016)'s survey (and other work by these authors) incorporates PI/II, but also refers to agent's beliefs regarding each other's payoffs. In our framework this informational friction (leading to "Global Games") is as yet absent.

gregate shocks, as a rationale for our limiting case. For instance, Ilut and Saijo (2021), in a general equilibrium heterogeneous firm framework, estimate the idiosyncratic component of the standard deviation of a total factor productivity (TFP) shock to be 50-100 times that of the aggregate component. Bloom *et al.* (2018) estimate, using macro- and industry-level data, the standard deviation of common and idiosyncratic technology shocks, and find evidence of substantial idiosyncratic uncertainty in causing business cycles and large increase in variance that characterises the crisis period.[14]

The final strand of literature proposes the concept of approximate invertibility (fundamentalness) for non-invertible (non-fundamental) RE linear solutions of DSGE models - see, for example, Beaudry *et al.* (2016) and Forni *et al.* (2019). We provide a generalization of the results of these papers to a DGP where agents have II. Related to this concept, Miranda-Agrippino and Ricco (2019) consider the case when a researcher only wants to partially identify the system, that is, to retrieve the dynamic effects of one or a subset of the structural shocks.

In summary, our paper makes several important contributions, addressing both methodological and substantive issues in model solution and in conducting applied time series and macroeconomics research related to DSGE models. Firstly, it provides a finite state-space solution to an important *general class* of heterogeneous agent RE problems first studied by of Townsend (1983). Secondly, it shows that, in this context, an a-theoretical VAR estimation of those variables may not generate the impulse response functions to the structural shocks of interest because the RE solution may incorporate Blaschke factors. Thirdly, it identifies and generalizes the conditions for invertibility of the RE solution of a SA/HA/PI/II DSGE model. Fourthly, it constructs the PI and II measures of approximate fundamentalness which can be used to assess the (non-) invertibility/fundamentalness of structural shocks for further model validation. Thus, our paper offers a unifying, general framework based on novel theoretical results to provide important insights into studies of heterogeneity, informational imperfections and time series properties in DSGE models.

1.7 Structure of Paper

In Section 2, we first set out our baseline framework for a single agent with II. Theorem 1 then shows that a general class of linear RE models can always be transformed into the

[14]See also David *et al.* (2016) who estimate the posterior variance of a firm-specific TFP process. Other important early contributions investigating the transmission of idiosyncratic uncertainty include Bloom (2009), Arellano *et al.* (2012) and Christiano *et al.* (2014). Using a panel of Compustat firms, Arellano *et al.* (2012) calibrate a model with credit frictions and heterogeneous firms, and show that exogenous increases in uninsurable idiosyncratic volatility help generate substantial volatility in business cycles. Similarly, Christiano *et al.* (2014) focus on the entrepreneurial idiosyncratic risk generating cross-sectional dispersion of firm-level productivity with tighter credit conditions that could lead to a recession and accounts for a large share of the macroeconomic fluctuations.

form that allows us to solve the informational problem using the techniques originally set out in PCL. In Section 3, we then show, in Theorem 2, that we can derive a representation of the aggregate economy with the same form from a limiting case of an incomplete markets, heterogeneous agent economy.

Section 4 shows how the econometrician's problem relates to the solution of the agents' problems presented in Sections 2 and 3. Section 5 examines measures of approximate fundamentalness when A-invertibility fails.

Section 6 provides a quantitative analysis illustrating Theorems 3–7 using a richer RBC model than the earlier analytical one. Section 7 provides concluding remarks.[15] Online appendices provide proofs of our key results as well as analysing a range of background issues.

2 The Single Agent's Problem

In this section, we first examine the general informational problem for the benchmark case of a single agent. We first show that a general class of linear rational expectations models can always be transformed into the form utilized by PCL to generalize the solution of Blanchard and Kahn (1980) under imperfect information (II-SA) rather than perfect information (PI). We then provide outline RE solutions in these two cases.

2.1 The Problem

We begin by writing a linearized RE model in the following general form

$$A_0 Y_{t+1,t} + A_1 Y_t = A_2 Y_{t-1} + \Psi \varepsilon_t \qquad m_t^E = L^E Y_t \qquad m_t^A = L^A Y_t \tag{9}$$

where matrix A_0 may be singular, Y_t is an $n \times 1$ vector of macroeconomic variables; and ε_t is a $k \times 1$ vector of Gaussian white noise structural shocks. We assume that the structural shocks are normalized such that their covariance matrix is given by the identity matrix i.e., $\varepsilon_t \sim N(0, I)$.

We define $Y_{t,s} \equiv \mathbb{E}\left[Y_t | I_s^A\right]$ where I_t^A is information available at time t to the single agent, given by $I_t^A = \{m_s^A : s \leq t\}$. We assume that this contains the history of a strict subset of the elements of Y_t, hence information is in general imperfect; but we do not at this stage seek to justify the restricted nature of the information set. Note that measurement errors can be accounted for by including them in the vector ε_t.

[15] Our II solution for simulation and Bayesian estimation alongside the invertibility checks are currently available in version 4.6.1 of Dynare. See Levine *et al.* (2020) and Appendix J for details.

2.2 Conversion to Generalized Blanchard-Kahn Form

The rest of this section is structured so that we first show how (9) can be transformed into the state-space form utilized by PCL, a generalization of the Blanchard-Kahn form (Theorem 1), and then describes the unique RE saddle-path stable solution to the problem under PI and II.

Anderson (2008) lists a selection of methods that can be used to solve (9) for the case when agents have PI. The most well-known of these are Klein (2000), Sims (2002) and Blanchard and Kahn (1980) - henceforth BK. Lubik *et al.* (2023) adopt the Klein-Sims approach to a general II environment with two kinds of agents with different information sets. However, we find that the generalized version of the BK form that was utilized by PCL is particularly suitable for comparing with the finite-space solutions of heterogeneous agent problems in Section 3.2. It is also important in Theorem 4 for revealing the spectrum of the II solution as non-minimal and incorporating a set of Blaschke factors.

In order to move seamlessly from (9) to results that are based on PCL, we introduce our first key result, which appears to be novel in the literature:

Theorem 1. *For any information set,* (9) *can always be converted into the following generalized BK form, as used by PCL*

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{10}$$

$$m_t^A = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \tag{11}$$

where z_t, x_t are vectors of backward and forward-looking variables, respectively.

Proof of Theorem 1. See Appendix B.1. □

The expressions involving $z_{t,t}$ and $x_{t,t}$ arise from rewriting the model in PCL form (10). This transformation (outlined in Appendix B.1) involves a novel iterative stage which replaces any forward-looking expectations with the appropriate model-consistent updating equations. This reduces the number of equations with forward-looking expectations, while increasing the number of backward-looking equations one-for-one. But at the same time it introduces a dependence of the additional backward-looking equations on both state estimates $z_{t,t}$ $\left(\equiv \mathbb{E}[z_t|I_t^A]\right)$ and estimates of forward-looking variables, $x_{t,t}$. The presence of the latter is the key feature that distinguishes our results on invertibility from those of Baxter *et al.* (2011) - henceforth BGW - the applicability of which is restricted to cases where all forward-looking variables are directly observable.

2.3 The Single Agent Solution Under Perfect Information (PI)

The PI solution is an important special case. Here we assume (without seeking to justify this assumption) that the single agent directly observes all elements of Y_t, hence of (z_t, x_t). Hence $z_{t,t} = z_t$, $x_{t,t} = x_t$, and using standard solution methods, there is a saddle path satisfying

$$x_t + Nz_t = 0 \quad \text{where} \quad \begin{bmatrix} N & I \end{bmatrix} (G + H) = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{12}$$

where Λ^U is a matrix with unstable eigenvalues. The saddlepath matrix N can be calculated by standard techniques. If the number of unstable eigenvalues of $(G + H)$ is the same as the dimension of x_t, then the system will be determinate.[16]

Given this determinacy condition, after substituting for x_t, a unique saddle-path stable RE solution exists for the states under PI of the following form

$$z_t = Az_{t-1} + B\varepsilon_t \tag{13}$$

where

$$A \equiv G_{11} + H_{11} - (G_{12} + H_{12})N \tag{14}$$

2.4 The Single Agent Solution Under Imperfect Information (II-SA)

Under II, the transformation of (9) into the form (10) and (11) in Theorem 1 allows us to apply the solution techniques originally derived in PCL. We briefly outline this solution method below.

We first define matrices G, in (10), and H, in (11), conformably with z_t and x_t, and define two more structural matrices F, J

$$G \equiv \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \qquad H \equiv \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{15}$$

$$F \equiv G_{11} - G_{12}G_{22}^{-1}G_{21} \qquad J \equiv M_1 - M_2G_{22}^{-1}G_{21} \tag{16}$$

where F and J capture intrinsic dynamics in the system, that are invariant to expectations formation. Both PCL and BGW show that the filtering problem is unaffected by these additional terms.[17]

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[16]Note that, in general, as Sims (2002) has pointed out, the dimension of x_t will not match the number of expectational variables in (9), as we see in the algorithm for the proof of Theorem 1 (see Appendix B.1).

[17]By substituting from the second block of equations in (10), we can write $z_t = Fz_{t-1} + \begin{bmatrix} B \\ 0 \end{bmatrix} \varepsilon_{t+1}$ plus additional terms involving expectations formed at time t; and $m_t^A = Jz_t +$ additional terms likewise. Since all expectational terms are known at time t, they do not affect the solution to the filtering problem.

Following PCL, we apply the Kalman filter updating given by

$$\begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} = \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} + K \left[m_t^A - \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} - \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \right]$$

The single agent's best estimate of (z_t, x_t) based on current information is a weighted average of their best estimate using last period's information and the new information m_t^A. Thus the best estimator of (z_t, x_t) at time $t-1$ is updated by the "Kalman gain" K of the error in the predicted value of the measurement. PCL show that K is solved endogenously as $K = \begin{bmatrix} P^A J' \\ -NP^A J' \end{bmatrix} [(M_1 - M_2 N) P^A J']^{-1}$, where P^A is defined below in (23), but this version of the Kalman gain is not directly incorporated into the solution for (z_t, x_t).

The unique saddle-path stable solution under II, as derived by Pearlman *et al.* (1986) for the pre-determined and non-predetermined variables z_t and x_t, can then be described by processes for the predictions $z_{t,t-1}$ and for the prediction errors $\tilde{z}_t \equiv z_t - z_{t,t-1}$:

$$\begin{aligned} \text{Predictions}: \quad & z_{t+1,t} = A\left(z_{t,t-1} + \mathcal{K}J\tilde{z}_t\right) && (17) \\ \text{Prediction Errors}: \quad & \tilde{z}_t = Q^A \tilde{z}_{t-1} + B\varepsilon_t && (18) \\ \text{Non-predetermined}: \quad & x_t = -N\left(z_{t,t-1} + \mathcal{K}J\tilde{z}_t\right) - G_{22}^{-1}G_{21}\left(I - \mathcal{K}J\right)\tilde{z}_t && (19) \\ \text{Measurement Equation}: \quad & m_t^A = E\left(z_{t,t-1} + \mathcal{K}J\tilde{z}_t\right) && (20) \end{aligned}$$

where

$$\mathcal{K} = P^A J' \left(JP^A J'\right)^{-1}; \quad Q^A = F\left[I - \mathcal{K}J\right] \tag{21}$$

F and J are as defined above in (16), $\mathcal{K}$ is an alternative Kalman gain matrix after stripping out the predictable variation in the state variables z_{t+1} arising from dependence on x_t. The matrix A, defined in (14), is the autoregressive matrix of the states z_t in the solution under PI. We have introduced another non-structural matrix E defined by

$$E \equiv M_1 + M_3 - (M_2 + M_4)N \tag{22}$$

which captures the impact of predictions and prediction errors for z_t on observable variables. B captures the direct (but unobservable) impact of the structural shocks ε_t and $P^A = \mathbb{E}[\tilde{z}_t \tilde{z}_t']$ is the solution of a Riccati equation given by

$$P^A = Q^A P^A Q^{A'} + BB' \tag{23}$$

To ensure stability of the solution P^A, we also need to satisfy the convergence condition, that Q^A has all eigenvalues in the unit circle. Since the matrix Q^A is also the autoregressive matrix of the prediction errors $\tilde{z}_t$ in (18), this is equivalent to requiring that prediction errors are stable. Since there is a unique solution of the Riccati equation under mild conditions that satisfies this condition, it follows that the solution (B.69)–(20) is also unique thereby extending this property of the PI BK solution to the II case.

We can thus see that the solution procedure above is a generalization of the BK solution for PI and that the determinacy of the system is independent of the information set.

We finally note that the II solution can be transformed into the PI solution when the agent's information set is (z_t, x_t). Choose just a subset of the information, $m_t = Jz_t$, such that JB is invertible. We then deduce from (23) that $P^A = BB'$ and hence $\tilde{z}_t = B\varepsilon_t$. Substituting into (B.69) yields $z_{t+1,t} = Az_{t,t-1} + AB\varepsilon_t = A(z_{t,t-1} + \tilde{z}_t) = Az_t$. Adding this to $\tilde{z}_{t+1} = B\varepsilon_{t+1}$ yields $z_{t+1} = Az_t + B\varepsilon_{t+1}$, the PI solution.

3 The Heterogeneous Agent (HA) Framework

We now move from a single agent (SA) to a heterogeneous agent (HA) framework.

3.1 General Framework

We start with the following generalized version of a linearized HA model from the perspective of agent i that encompasses all the papers discussed in the fourth strand of literature in Section 1.6[18]

$$\begin{bmatrix} \varpi_{t+1} \\ y_{i,t+1} \\ W\mathbb{E}_{i,t}x_{i,t+1} \end{bmatrix} = \begin{bmatrix} R & 0 & 0 \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \varpi_t \\ y_{i,t} \\ x_{i,t} \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & A_{21} \\ 0 & A_{31} \end{bmatrix} \begin{bmatrix} \epsilon_{t+1} \\ \epsilon_{i,t} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_1 & 0 & I_3 \\ H & I_2 & W-I & I_4 \end{bmatrix} \begin{bmatrix} \mathbb{E}_{i,t}z_{t+1} \\ \mathbb{E}_{i,t}z_t \\ \mathbb{E}_{i,t}x_{t+1} \\ \mathbb{E}_{i,t}x_t \end{bmatrix} \tag{24}$$

[18]Note that I_1, I_2 are general, not identity matrices. We have normalized the equation that includes forward expectations so that, under PI, the coefficient on expectations of x_{t+1} is the identity matrix. More generally, one would include $\mathbb{E}_{i,t}x_t$, but in the solution we would find that this is a linear function, via the saddlepath relationship, of $\mathbb{E}_{i,t}z_t$, to reduce the amount of notation we therefore omit it. Also note that $\begin{bmatrix} R & 0 \\ A_{21} & A_{22} \end{bmatrix}$ corresponds to G_{11} in (10), and A_{33} to G_{22}, along with other correspondences.

where $z_t \equiv [\varpi_t' y_t']'$ are predetermined variables, ϖ_t shock processes, $y_t = \int \mu(i) y_i di$ aggregates and $\mu(i)$ the agent i density. Measurements of agent i are given by

$$m_t^A = M_1 z_t + M_2 x_t \qquad m_{i,t}^A = \varpi_t + \varepsilon_{i,t} \qquad \text{var}(\varepsilon_{i,t}) = \Sigma \tag{25}$$

where m_t^A is a common information set, $\mathbb{E}_{i,t}$ are expectations over the diverse information set $\mathbb{I}_{i,t} = \{m_k^A, m_{ik}^A : \ k \leq t\}$. x_t are non-predetermined variables. This framework encompasses the less general frameworks of Rondina and Walker (2021), Huo and Pedroni (2020), Huo and Takayama (2021) and Angeletos and Huo (2021) by allowing for predetermined endogenous variables $y_{i,t}$ and y_t[19]

When there are no idiosyncratic shocks, then all agents are identical, so that $\mathbb{E}_{it} = \mathbb{E}_t$, then the system can be written in the following Blanchard-Kahn form of Theorem 1[20]

$$\begin{bmatrix} \varpi_{t+1} \\ y_{t+1} \\ \mathbb{E}_t x_{t+1} \end{bmatrix} = \begin{bmatrix} R & 0 & 0 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} \\ \hat{A}_{31} & \hat{A}_{32} & \hat{A}_{33} \end{bmatrix} \begin{bmatrix} \varpi_t \\ y_t \\ x_t \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \epsilon_{t+1} \tag{26}$$

where matrices $\hat{A}_{ij}$ are obtained by straightforward substitution. The information set of econometricians, m_t^E, is m_t^A as in (25). In what follows we assume the Blanchard-Kahn determinacy eigenvalue condition holds for (26).

A critical if uncontroversial assumption is that current aggregate shocks affect observations of aggregates with their input/output relationship being full rank. If not, then there is no chance of VAR estimation being able to relate residuals to structural shocks.

To reduce notation, we also assume that the exogenous variables follow a VAR(1) process with autoregressive matrix R.

We assume for convenience that expectations of future variables do not affect agents' decisions on $y_{i,t}$; this can be justified within an optimizing framework because if an agent makes a decision on a variable that depends on expectations of future values of aggregate variables, then this would be coupled with expectations of that variable's future value. So the variable would be an element of the vector $x_{i,t}$.

Note that, crucially, we make the assumption that agents observe their own actions perfectly, so that $\mathbb{E}_{i,t}[Y_{i,t}] = Y_{i,t}$; in that respect, the derivation of the state-space representation via Theorem 1 is more straightforward. However, account has to be taken of agent i's best estimate of current and future values of aggregate variables.

[19] Angeletos and Huo (2021) assume a different information set where typically there are fewer observations of aggregates than there are aggregate shocks, so their work is not applicable to VAR comparison.. Also, our solution procedure is entirely in the time domain (as in Sections 2.3 and 2.4) as opposed to the frequency domain used in these studies. It therefore provides a seamless progression from the familiar Blanchard-Kahn PI solution, conducted in the time-domain, to II-HA case.

[20] Note in a HA framework we must put $M_3 = M_4 = 0$ in (11) in our choice of aggregate information set. The exact representation of the $\hat{A}_{ij}$ appears in the proof of Theorem 2 below.

There are, however, some modifications, and in addition, some simplifications that we use to ease the burden of notation in the proof of the main theorem of this section. Firstly, we make the usual assumption for heterogeneous agents in this literature, that any aggregate shocks ϖ_t are both perceived and acted upon with the addition of an idiosyncratic component $\varepsilon_{i,t}$; agent i therefore only observes the composite $\varpi_t + \varepsilon_{i,t}$. But, crucially, as noted in the introduction, $\varepsilon_{i,t}$ is not simply 'noise': it also directly affects the agent's state variable $y_{i,t}$. In addition, we make the usual assumption in this literature that $\varepsilon_{i,t}$ is a vector white noise process.

While we write the system (26) in an unrestricted form, our results below will focus on a limiting case where we allow $\text{var}(\varepsilon_{i,t}) \to \infty$, so there is no useful information about z_t provided by $m_{i,t}$. Then $\mathbb{E}_{i,t} z_{t+1} = \mathbb{E}_t z_{t+1}$. We denote this case by II-HA(∞) which is the focus of the next section. On the basis of available evidence, cited in Section 1.6, we argue that this limiting case is of empirical interest.

As in the single agent case, we also exploit properties of the heterogeneous case where all agents have PI (which we denote PI-HA) where this information is simply assumed to exist as an endowment so agents (somehow) observe all current realizations of the shock processes ε_t and $\varepsilon_{i,t}$. In this solution, the saddlepath matrix N for any individual agent i will include optimal responses to purely idiosyncratic components, but, as in the example analysed in the Introduction, it is straightforward to show that, given the linearity of the setting, these responses cancel out in aggregate, so that the PI-HA solution for the aggregate economy is identical to the PI- SA solution derived above. However, a key feature of our next result is that the saddlepath responses to purely idiosyncratic responses in the PI-HA case still play a key role in determining the nature of the filtering problem when information is imperfect.

Lemma 1. *The solution for agent i under PI-HA is given by*

$$y_{i,t+1} - y_{t+1} = (A_{21} - A_{23}N_{\varepsilon_i})\varepsilon_{it} + (A_{22} - A_{23}N_{y_i})(y_{it} - y_t) \qquad x_{it} - x_t = -N_{y_i}(y_{it} - y_t) - N_{\varepsilon_i}\varepsilon_{it} \tag{27}$$

where y_t and x_t are the solutions to the aggregate economy under PI as set out in subsection 2.3 to be consistent with (24)*, provided that $A_{22} - A_{23}N_{y_i}$ is a stable matrix.*

Proof of Lemma 1. See Appendix B.2. □

3.2 The Limiting Case of a Heterogeneous Agent Model

Before stating our main theorem, we introduce the following notation that resets matrices J and E:

$$J = M_1 - M_2 A_{33}^{-1} A_{32} \qquad E = M_1 - M_2 N \tag{28}$$

Note that if $M_2 = 0$, then $E = J$. Writing $J = [J_1 \quad J_2]$ conformably with z_t so that $Jz_t = J_1\varpi_t + J_2 y_t$, we define $S = J_1^{-1}J_2$.

Theorem 2. *Assume a general HA framework as in* (24)*. Assume that the number of observables equals the number of shocks* ($m = k$)*. Then, as the diagonal elements and determinant of* Σ*, the idiosyncratic shock covariance matrix, tend to infinity, the limiting aggregate solution, II-HA*(∞) *will have two possibilities: (a) it is equivalent to the PI case or (b) it will be different from PI but still with a finite set of states. A unique saddle-path solution exists with an identical structure to the single agent II-SA solution of* (9)*, as in (B.69) to (20), but replacing* F *in* (16) *with*

$$F(\infty) = \begin{bmatrix} R & 0 \\ A_{21} - A_{23}N_{\varepsilon_i} & A_{22} - A_{23}N_{y_i} \end{bmatrix} \tag{29}$$

where N_{ε_i} and N_{y_i} are saddlepath responses to $\varepsilon_{i,t}$ and $y_{i,t}$ in the PI-HA case as in Lemma 1.

Proof of Theorem 2. See Appendix B.3. □

Thus we have shown that the aggregate solution in such an economy can be derived from (but in important respects differs from) the informational problem of a notional single agent with the same aggregate information set, and will therefore, in contrast to intermediate cases, have a finite state-space solution.

It is important to note that Theorem 2 does *not* say that the II-HA(∞) case is in general identical to the II-SA case with the same aggregate observables. Instead it says that the solution of the agents' signalling extraction problem for the aggregate economy always takes the same *form* as the solution of a notional II-SA problem, but, crucially, with amendments to the underlying structure of this notional economy. The original definition of F in (16) for the II-SA economy is entirely independent of the saddlepath matrix N; whereas (29) shows that, in the notional $II - \infty\infty$ problem solved in Theorem 2, it is shifted by saddlepath responses to idiosyncratic shocks and states in the PI-HA case.[21] Thus, the nature of the idiosyncratic economy impacts both on the solution to the signal extraction problem but also, as a result, on aggregate dynamics.

4 The Econometrician's Problem

We now show how the econometrician's problem relates to the solution of the agents' problem presented in Section 2 and hence (from Theorem 2) the limiting case of Section

[21] Although the general PI-HA solution has not been specified, the proof of this theorem in Appendix B shows that N_{ε_i} and N_{y_i} are unrelated to the filtering problem, and thus must be identical to the solution in the PI-HA case. The additional HA saddlepath is precisely analogous to that in Rondina and Walker (2021) â€“ see their condition (A.20)

3.

4.1 Informational Assumptions

In our central case, we assume that, under imperfect information, the econometrician always has the same information set for the aggregate economy as the aggregate information set available to the agents under II, thus $m_t^E = m_t^A$. Having derived three key results below (Theorems 3–4) under this assumption, in Corollary 5.2 of Section 4.7, we consider the implications of the econometrician's information set being a strict *subset* of that of the agents.[22] In Section 4.8, we then consider the case that, at least over the course of time, the econometrician has, at some T, *more* information than agents at time $t << T$.

4.2 A-invertibility: When II Replicates PI

It is evident that, for the general case, in both II-SA and II-HA(∞) cases, imperfect information introduces non-trivial additional dynamics into the responses to structural shocks - a contrast which is crucial to much of our later analysis. However, there is a special case of the general problem under II, which asymptotically replicates PI, and hence where $P^A = BB'$.

Definition 1. ***A-invertibility****: An information set is A-invertible if agents can infer the true values of the structural shocks ε_t (and hence, in the II-HA case, $\varepsilon_{i,t}$) from the history of their observables, or equivalently, $P^A = BB'$ is a stable fixed point of the agents' Ricatti equation, (23). Hence Q^A must be a stable matrix evaluated at this fixed point.*

4.3 E-invertibility: The ABCD (and E) of VARs

Corresponding to A-invertibility we now define the corresponding concept from the viewpoint of the econometrician:

Definition 2. ***E-invertibility****: An aggregate information set is E-invertible if an econometrician can infer the true values of the shocks ε_t from the history of the econometrician's observables, $\left\{m_s^E : s \leq t\right\}$.*

To see how the two concepts of A- and E-invertibility relate to each other, consider an econometrician's state-space representations of the aggregate economy of the type that arise naturally from our solution method in Section 2, of the general form

$$s_t = \tilde{A}s_{t-1} + \tilde{B}\varepsilon_t \qquad m_t^E = \tilde{E}s_t \tag{30}$$

[22] This includes the popular example non-invertibility arising from a "missing information problem" where agents but not the econometrician observe "news shocks".

where the tildes over each of the matrices distinguish this state-space representation from the particular form (without tildes) under perfect information. It is straightforward to show that both the PI and II-SA (and hence, from Theorem 2, II-HA(∞)) representations of the previous two sections are in the ABE form of (30).

For the PI case, given the informational assumptions set out above, we have, straightforwardly, $s_t = z_t, \tilde{A} = A, \tilde{B} = B, \tilde{E} = E$. As noted above, given our linearity assumption, this is also the solution for the aggregate economy in the PI-HA case.

For the II-SA case, we have

$$s_t = \begin{bmatrix} z_{t,t-1} \\ \tilde{z}_t \end{bmatrix} \tag{31}$$

$$\tilde{A} \equiv \begin{bmatrix} A & A\mathcal{K}J \\ 0 & Q^A \end{bmatrix} \tag{32}$$

$$\tilde{B} \equiv \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{33}$$

$$\tilde{E} \equiv \begin{bmatrix} E & E\mathcal{K}J \end{bmatrix} \tag{34}$$

where A, $\mathcal{K}$, J, Q^A and E are as defined after (B.69) to (20).

Given Theorem 2, there is also an equivalent representation of the aggregate economy in the II-HA(∞) case.

This "ABE" representation form is the form usually found in the statistics literature. In contrast, the following "ABCD" form is often but not exclusively used in the economics literature, e.g., Fernandez-Villaverde *et al.* (2007)

$$s_t = \tilde{A}s_{t-1} + \tilde{B}\varepsilon_t \qquad m_t^E = \tilde{C}s_{t-1} + \tilde{D}\varepsilon_t \tag{35}$$

It is straightforward to show that any ABE form implies an ABCD form, with $\tilde{C} = \tilde{E}\tilde{A}$ and $\tilde{D} = \tilde{E}\tilde{B}$. Appendix A.1 shows that (less obviously) the reverse also applies; it also shows that all of the state-space models that are used in the statistics, control theory and econometrics literature can be rewritten in terms of one another.

The condition for the system (30) to be E-invertible, which we exploit below in Theorem 3, is then a generalization of the PMIC of Fernandez-Villaverde *et al.* (2007),[23] which is obtained by some algebraic manipulation of (30):

Lemma 2. ***PMIC****: For a general "ABE" system of the form in (30), necessary and sufficient conditions for E-invertibility are: (a) A 'square system' with $m = k$; (b) $\tilde{E}\tilde{B}$ (now a square matrix) is non-singular; (c) $\tilde{A}(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E})$ has stable eigenvalues.*

[23]This result appears to date back at least to the work of Brockett and Mesarovic (1965).

Proof. See Appendix A.2.[24] □

A final observation is that invertibility does not require the ABE representation to be in minimal (i.e., controllable and observable) form; we mention this since the ABE representation of the II solution below might not be minimal.[25]

The advantages of using the ABE state-space form in what follows are (i) the Riccati equation is simpler than for any of the other formulations, (ii) the solution under II is much simpler to express and, most usefully, (iii) the representation of the model using the innovations process (see Appendix A.5) has the same structure as the original model.

4.4 E-invertibility: When Agents Have PI

The conditions for E-invertibility under PI are straightforward, and merely mimic the PMIC requirements of the previous section, but with $\tilde{A} = A, \tilde{B} = B, \tilde{E} = E, s_t = z_t$. Hence we immediately have:

Lemma 3. *If agents have PI, the conditions for E-invertibility (as in Definition 2) are: the square matrix EB is of full rank and $A(I - B(EB)^{-1}E)$ is a stable matrix.*

It is straightforward to show that this is identical to the original PMIC, derived from the ABCD representation, in Fernandez-Villaverde *et al.* (2007). Since, as noted above, the aggregate solution under PI in a heterogeneous agent economy (PI-HA) is the same as in a standard single agent economy (PI), the same condition implies in both PI cases.

4.5 E-invertibility: When Agents Have II

Now consider the equilibrium where agents *do not* have PI and the more general case of E-invertibility under II (which, from Theorem 2, subsumes both II-SA and II-HA(∞) cases). The result is straightforward, but powerful:

Theorem 3. *Assume that the number of observables equals the number of shocks $(m = k)$. Assume further that the PMIC conditions in Lemma 3 hold (so the RE solution would be E-invertible under PI) but agents* ***do not have PI*** *. Then E-invertibility under II holds if and only if A-invertibility holds, and this requires that the square matrix JB is of full rank, and $Q_A = F(I - B(JB)^{-1}J)$ is a stable matrix.*

[24] A slightly weaker condition than invertibility is fundamentalness which allows some eigenvalues to be on the unit circle. However, we use the two terms interchangeably and, in fact, if we restrict our models to have only stationary variables, then the two concepts are equivalent.

[25] To show this, suppose that $(\tilde{A}, \tilde{B})$ is not controllable; then there exists an eigenvalue-eigenvector pair (λ, x) such that $x'\tilde{A} = \lambda x', x'\tilde{B} = 0$. It is then trivial to show that $x'\tilde{A}(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}) = \lambda x'$. But we have assumed that $\tilde{A}$ is a stable matrix, so an uncontrollable mode cannot be the source of non-invertibility. The same conclusion can be drawn for non-observability, for which there exists an eigenvalue-eigenvector pair (μ, y) such that $\tilde{A}y = \mu y, \tilde{E}y = 0$.

Proof. See Appendix B.4. $\square$

The following corollary follows from Theorem 3 involving a key matrix U_2 in the proof of Theorem 2:

Corollary 3.1. *For the II-HA(∞) case, after making the substitutions for J and F from Theorem 2, the conditions for E-invertibility in Theorem 3 reduce to the condition that $U_2 \equiv (A_{21} - A_{23}N_{\varepsilon_i})S - (A_{22} - A_{23}N_{y_i})$ is a stable matrix which then in turn becomes the condition for PI, II-SA and II-HA(∞) solutions to be equivalent in the case where agents do not have PI.*

4.6 If A-invertibility Fails

The conditions for A- (and hence E-) invertibility to be satisfied are stringent. The following result forms the basis for our remaining analysis of cases where these conditions are not satisfied.

Theorem 4. *Under the assumptions of Theorem 3, if A- (and hence E-) invertibility fail, then the solution for aggregate variables can never be identical to the PI case, and will incorporate Blaschke factors in the impulse response functions.*

Proof. See Appendix B.6. $\square$

The first element of this theorem is unsurprising: if agents cannot correctly identify the true structural shocks then their responses are bound to differ from those under PI. But the key feature that the aggregate solution that results from these responses must incorporate Blaschke factors is crucial for what follows.

Note also that, given the equivalence of II-SA and II-HA(∞) representations established in Theorem 2, this result, applies in both cases, as do the later results that follow from it.

4.7 The Innovations Process for II When A-invertibility Fails

In the absence of A- and hence E-invertibility, there is still an "innovations representation" (see Fernandez-Villaverde *et al.*, 2007) under mild conditions.[26] The counterpart to the innovations representation is, in population, a finite order fundamental[27] VARMA (or VAR(∞)) in the observables, m_t^E, with innovations e_t. This can either be directly estimated via its state-space representation (using Dynare, for example), or, more commonly, it may be approximated by a finite-order VAR(p) approximation. When the

[26]See Appendix A.5 for standard results for the innovations representation and Lemma 6 in particular.

[27]We deliberately use the term fundamental here, rather than invertible, to reflect the fact that estimated VARs may contain stationary transformations of unit root processes.

conditions stated in Theorem 3 do not hold, the VARMA or VAR approximation will generate a series of reduced-form residuals that are a linear transformation of innovations $e_t \equiv m_t^E - \mathbb{E}_{t-1} m_t^E$ but *not* of the structural shocks ε_t.

We now examine the properties of the innovations representation under general conditions when a failure of A-invertibility leads to a failure of E-invertibility.

Theorem 5. *Consider the case where there is a failure of A-invertibility under II, and hence (from Theorem 3) of E-invertibility. The innovations representation of the RE saddle-path solution is of the same dimension as under PI and is given by*

$$\xi_{t+1} = A\xi_t + ZE'(EZE'^{-1} e_{t+1} \qquad e_t \sim N(0, EZE') \tag{36}$$

where ξ_t is a vector process of precisely half the dimension of the state-space process that generates the impulse response functions (henceforth IRFs) of the structural shocks of the form (B.69)–(20) *and* $e_t \equiv m_t^E - \mathbb{E}_{t-1} m_t^E$

$$Z = AZA' - AZE'(EZE')^{-1}EZA' + P^A J'(JP^A J')^{-1} JP^A \tag{37}$$

Proof. See Appendix B.7. □

Notably, this result tells us that, even though the dynamics of the RE saddle-path solution under II are considerably more complex and add more inertia than under PI (and hence have a state-space representation of twice the dimension), the innovations process e_t is generated by equations that are of the *same* dimension as under PI.[28]

The implication of this result is of major significance for empirical work:

Corollary 5.1. *Since the spectrum of* (36) *must be identical to that of* (B.69)–(20)*, it follows that in the absence of A-invertibility, the latter is a non-minimal spectral factorization. It therefore incorporates a set of Blaschke factors whose presence cannot be detected by an estimated a-theoretical representation. Hence the statistical properties of data as generated by the model under II and represented by a fundamental VARMA or VAR approximation cannot, in general, generate the true IRFs.*

In empirical work, a common approach (in the tradition of, for example, Christiano *et al.*, 2005) is to compare impulse responses by applying a structural identification scheme to the estimated VAR(p) with the impulse responses implied by their structural DSGE model. In contrast, Kehoe (2006) advocates the approach of Sims (1989) and Cogley and Nason (1995) which compares impulse responses of a finite order, finite sample structural VAR estimated on the data with a VAR with the same structure, run on artificially generated data from the model.

[28]This result is a generalization of BGW, Corollary 1, p302, but without relying on their assumption that all forward-looking variables are observable.

However, for both approaches in the absence of A-invertibility (and therefore E-invertibility), the reduced-form residuals in the data VAR are not a linear transformation of the structural shocks ε_t (even with correct choices of identification matrix), but are instead a finite-order, finite-sample estimate of the innovations, e_t. Then the innovations e_t are not a linear transformation of ε_t and it follows that comparisons of IRFs may be seriously misleading.

Up to now we have deliberately avoided the "missing information" problem highlighted in the original fundamentalness literature by assuming that the econometrician and the agents have the same II set. Now consider the following corollary where agents have *more* information about the variables of the model (such as news shocks), although this does not imply that agents have PI.

Corollary 5.2. *If the econometrician's information set is a subset of that of the agents and the system is not A-invertible, then the innovations process as estimated by the econometrician will again be of the same dimension as under PI, and thus will be of lower dimension than the true system in* (B.69)–(20).

The implication therefore is that, with any failure of A-invertibility, then provided the econometrician is no better informed than the agents, one should be wary of using an unrestricted VAR to generate the IRFs of the structural shocks.

4.8 Are the Structural Shocks Recoverable When E-invertibility Fails?

As we have seen, in the absence of A-invertibility, the best the econometrician can do, given the history of the observations, is to estimate the innovations representation (see below) of the true model. However, a recent literature, initiated by Chahrour and Jurado (2022), has raised the possibility that non-invertible structural shocks may be *recoverable*, in a finite sample of length T, from the full sample history $\left\{m_i^E : i = 1, ..., T\right\}$ for $t \in (\tau, T - \tau)$ for τ sufficiently large. Analogously to invertibility, recoverability is an asymptotic concept: the shock ε_t is recoverable if it can be written as a convergent sum of both past *and* future observables, in which case the impact of both initial and terminal conditions on any observation in the interior of the sample becomes vanishingly small as $T \to \infty$.

In Lemma 4 in Appendix A.4, we show that the innovations process for a non-invertible VARMA model is represented by one of the minimal spectral factorizations of the spectrum of the observables. All other spectral factorizations can then be generated via other symmetric solutions of the associated Riccati matrix, and one of these will be equivalent to the original VARMA model to within a simple linear transformation *provided* that the latter is also a minimal spectral factorization. However, we have shown that, when

A-invertibilty fails for our II setup, the true DGP implies a *non*-minimal spectral factorization[29] due to the presence of Blaschke factors that map the true structural shocks, ε_t, to e_t, the innovations to the observables. Thus we have the following further result:

Theorem 6. *If the model has the BK-type representation of* (10) *and* (11)*, with x_t of non-zero dimension (i.e., has saddlepath dynamics), and is not A-invertible, then the true DGP is a non-minimal spectral factorization of the spectrum of the agents' information set. Hence the parameters required to render the structural shocks recoverable cannot be identified from an a-theoretical time series representation of the observables (or VAR approximation thereof)*

Proof. The result follows immediately from the previous paragraph and Appendix A.4. □

The VAR assumes a minimal spectral factorization of the data, and this is why it cannot be a true representation of the model even after applying the Forni *et al.* (2017) transformation. Thus in the absence of A-invertibility, and where there are saddle-path dynamics, when converting the innovations process representation of the former into any non-invertible representation, such alternative representations will always retain the dimension of the innovations process. Since the latter, as we have seen, is of dimension lower than that of the state-space describing the effect of individual shocks under II, it follows that the two representations can never be equivalent. Hence the non-A-invertible structural shocks are not recoverable from any stochastically minimal representation, whether fundamental or non-fundamental.[30] Thus recoverability cannot, in general, provide an alternative means of using VARs for deriving IRFs of structural shocks under II in the absence of E-invertibility.[31]

Does this mean that recoverability has no applicability at all to such models? On the contrary, Theorem 5 and Corollary 5.1 showed that the true model has a non-minimal stochastic representation, incorporating a set of Blaschke factors. From an a-theoretical perspective, while any such factors may exist in principle, they can be of arbitrary form.

[29]Or equivalently, in Lippi and Reichlin (1994)'s terminology, the implied non-fundamental VARMA representation is also non-*basic* (i.e., is of higher order).

[30]Note that Forni *et al.* (2017) have an example where recoverability does hold but their very simple model (See Appendix I) incorporates recursive expectations that automatically render the system backward-looking, and therefore is not of the BK-type referred to in Theorem 6.

[31]Exceptions to this general result arise only in tightly restricted cases. Thus we showed in the discussion of our illustrative example that in the special case that agents are endowed with PI, the structural shock *is* recoverable even when E-invertibility fails. But this arises only as a result of the combination of PI, a single structural shock, and a resulting non-fundamental ARMA representation of the observable that is, in Lippi and Reichlin (1994)'s terms, "basic": i.e., of the same order as the fundamental representation - hence the econometrician simply has to flip the single MA root, and solve backwards. It might appear that this result would generalize to any solution that generates a basic non-fundamental representation (for which PI for agents is a necessary, but not sufficient condition). However, this is not the case, since with multiple MA roots the econometrician will not know a priori which roots to flip.

In the context of a structural model with II, these Blaschke factors are not arbitrary, since they can be related back to the underlying structure of the model. However in the non-basic higher order non-fundamental representation, their estimation is subject to the identification problem (highlighted in the illustrative example) of parameters that generate Blaschke factors, so the VAR (a-theoretical) econometrician is unable to recover structural shocks even using data from $-\infty$ to $+\infty$.

4.9 Can the Econometrician Bypass Non-invertibility?

Our central results have been derived under the assumption that, at time t, the econometrician either has the same information set as the agent or a strict subset thereof. Both assumptions are commonly made in the literature. However, it is worth considering the possibility that, at least after the passage of time, the econometrician may in some cases have a bigger t-dated information set. The illustrative example of Section 1.1 is a potential case in point. It is straightforward to show that the agents' information problem arises because they do not have any information on the aggregate wage or aggregate output: if they did, the system would be A-invertible.[32] Yet, at least over the passage of time, econometricians will acquire estimates of aggregate output and wages at time t, albeit possibly measured with error. While this takes us outside the framework of our key results, it is straightforward to show that failures of A-invertibility still have important implications for time series properties.

In the light of our results, we now return to the econometrics literature briefly reviewed in Section 1.6 that bypasses the intervening step of a SVAR using external or internal instruments and the method of local projections of Jorda (2005). This method does not require invertibility.

Plagborg-Moller and Wolf (2021), building on Stock and Watson (2018), show that the addition of an instrumental variable whether external or internal, to the econometrician's information set may enable estimation of at least a scaling of the true IRF even when structural shocks are non-invertible. Their Corollary 1 shows that this is equivalent to a Cholesky ordering of the VAR provided that the instrumental variable is the first variable of the VAR.

To focus our analysis, we take a very simple example of a reduced form observation vector m_t whose impulse response is given by a Blaschke factor that multiplies the structural shock ε_t. We then assume that the instrumental variable, x_t, is a noisy observation

[32] See BGW Section 6, which shows that, if the wage or output is observable, the rental rate of capital becomes informationally redundant.

of the structural shock,[33] so the system is given by

$$x_t = sv_t + \varepsilon_t \qquad m_t = \frac{b-L}{1-bL}\varepsilon_t \qquad \text{var}(\varepsilon_t) = \text{var}(v_t) = 1 \tag{38}$$

where v_t is iid and independent of ε_t at all leads and lags. Our illustrative example again provides motivation: m_t can be interpreted as a scaling of the observable fundamental innovation $\left(\frac{1-\gamma^{-1}L}{1-\gamma L}\right)\alpha\varepsilon_{a,t} \equiv \left(\frac{1-\mu L}{1-\psi L}\right)v_t$ in (6) and (7), the time series representation of the single observable, v_t, the rental rate on capital. In population, at least, observing the history of m_t is equivalent to observing the history of v_t. Conditional upon this information set alone, m_t is clearly white noise. However, an estimated truncated VAR(∞) in x_t and m_t with x_t ordered first in a Cholesky decomposition will yield the following innovations representation for m_t

$$m_t = \frac{b-L}{1-bL}\frac{1}{\sqrt{1+s^2}}e_{1t} + \frac{s}{\sqrt{1+s^2}}e_{2t} \tag{39}$$

where e_{1t}, e_{2t} are orthogonal white noise processes with unit variance.[34] Since e_{1t} is simply a scaling of x_t, by substitution this can be rewritten as

$$m_t = \frac{b-L}{1-bL}\frac{1}{1+s^2}x_t + \frac{s}{\sqrt{1+s^2}}e_{2t} = \frac{b-L}{1-bL}\frac{1}{1+s^2}(sv_t + \varepsilon_t) + \frac{s}{\sqrt{1+s^2}}e_{2t} \tag{40}$$

where $\sqrt{1+s^2}e_{1t}$ is the prediction error for x_t (given by $x_t - \mathbb{E}_{t-1}x_t$) and $\frac{be_{1t}+se_{2t}}{\sqrt{1+s^2}}$ is the prediction error for m_t. Thus, as pointed out by Plagborg-Moller and Wolf (2021), the structural VAR produces a scaling of the true impulse response to the shock, with attentuation bias driven by s. This is another no free lunch result as in Stock and Watson (2018). Indeed, by inspection of (40), only in the limit as s goes to zero is the IRF correctly estimated, and the system becomes invertible, and, by substitution back into the fundamental representation, the IRF for the underlying observable v_t can also be derived.

But even in the extreme limiting case of $s = 0$, in which the econometrician's superior information set allows them to bypass non-invertibility entirely, a key feature of our results is still central: the *nature* of this IRF is driven by the failure of A-invertibility. As noted in our earlier discussion, while under the assumption of PI, v_t is a (non-fundamental) ARMA(1,1), the failure of A-invertibility, and the resulting Blaschke factor, means that the true DGP for v_t is a non-fundamental (and, in Lippi and Reichlin (1994)'s terms, "non-basic") ARMA(2,2). So even in this most favourable case, the agent's informational

[33]This is effectively Plagborg-Moller and Wolf (2021)'s Assumption 4, which requires that the observable error from a linear projection of the instrumental variable on lagged observables is equal to the structural shock plus iid noise.

[34]For derivation, see Appendix A.3.

problem fundamentally changes the time series properties of the economy.[35]

This example has allowed the econometrician to at least mitigate the non-invertibility problem by adding more information, in this case, x_t. This is also the approach of factor-augmented VAR models which extract common factors from large cross section of time series data.[36] However, in the context of our paper, this then begs the question why agents are not able to observe the additional information provided by (38) as well. What are the consequences of agents having this additional source of information?

Trivially, if agents can observe the same information as the econometrician at time t, then all our results still go through, since this is the baseline case for all our results. However, while the additional information is indeed likely to become common knowledge, in most cases, this will only occur with at least some lag. If A-invertibility fails in the absence of the additional information, while the additional lagged information must at least somewhat reduce agents' filtering errors, it will not change the key feature of our results, namely, that the solution will include Blaschke factors, in both II-SA and II-HA(∞) cases.[37]

5 Approximate Invertibility-Fundamentalness

This section examines, for possibly non-square systems, measures of approximate fundamentalness when A-invertibility fails.[38]

Two methods are notable in this regard: the first measure from Beaudry *et al.* (2016) recommends using the difference in variances between the innovations process and the structural shocks, motivated by the PI case (A.17) which can be written as

$$e_t = m_t - Ez_{t,t-1} = E(z_t - z_{t,t-1}) = EA(z_{t-1} - z_{t-1.t-1}) + EB\varepsilon_t \tag{41}$$

Under invertibility, $z_{t-1} - z_{t-1.t-1}$ has a value of 0, so that regressing the innovations process e_t on this latter term yields (in the scalar case) a perfect lack of fit $R^2 = 0$. For

[35] It must be an open question how the econometrician would interpret this IRF in this case. If estimation is predicated on the assumption of PI, then the additional dynamics would not be easy to interpret in a structural framework.

[36] A general form of the factor model adds a measurement

$$x_t = \Delta(L)f_t + v_t, \quad \Gamma(L)f_t = \eta_t, \quad A(L)v_t = u_t$$

where the vector f_t contains unobserved common factors, v_t is a vector of idiosyncratic components and $(\eta_t,\, u_t)$ are white noise vectors such that $\mathbb{E}(u_t\eta_t') = 0$. Then principle component estimation is used to obtain estimates of the factor loadings $\hat{f}_t$. Chapter 16 of Kilian and Lutkepohl (2017) provides a very comprehensive treatment of this "data-rich environment" approach and the relevant recent literature.

[37] GW show that, in an extended version of our illustrative example, the inclusion of lagged noisy information on GDP mitigates, but does not eliminate, the agents' informational problem.

[38] See also Canova and Ferroni (2022) for a treatment of (what we call) E-invertibility and the interpretation of SVAR where the number of structural shocks exceeds the number of observables.

the univariate case, in general, we have $R^2 = 1 - \text{var}(\varepsilon_t)/\text{var}(e_t)$. In the multivariate case, $\text{cov}(e_t) = EP^E E'$, so that the departure of this from $\text{cov}(EB\varepsilon_t)$ yields a measure of how similar the innovations process is to the structural shocks.

However, in the empirical literature using VARs, it is common to focus on just one shock such as in the examination of the hours-technology question in Gali (1999). To address fundamentalness on a shock-by-shock basis, one requires the Cholesky decomposition of $EP^E E' = \tilde{V}\tilde{V}'$, or else a decomposition that depends for example on long run effects of each shock, i.e., an SVAR decomposition. The corresponding R_i^2 for each shock is then given by

$$R_i^2 = 1 - u_{ii} \qquad U = \tilde{V}^{-1} EBB'E'(\tilde{V}')^{-1} = u_{ij} \tag{42}$$

The further is R_i^2 from 0, the worse is the fit.

5.1 A Multivariate Measure with Perfect Information

An obvious multivariate version of this is $R = I - \tilde{V}^{-1}EBB'E'(\tilde{V}')^{-1}$, and the maximum eigenvalue of R would then be a measure of the overall fit of the innovations to the fundamentals. In addition, one can check whether any fundamentals can be perfectly identified by examining the eigenvalues of the difference between the variances of the innovations and and the fundamentals

$$\mathbb{B}^{PI} = EP^E E' - EBB'E' \tag{43}$$

Any zero eigenvalues coupled with the corresponding eigenvector will provide a means of decomposing the covariance matrix of the innovations $EP^E E'$.

Turning to our second measure, Forni *et al.* (2019) suggest that one can use VARs as well for 'short systems', where the number of observables is smaller than the number of shocks.[39] Utilizing the underlying VARMA model, they suggest regressing the structural shocks against the innovations process, i.e., for the structural shock i, choose the least-squares vector m_i by minimizing the sum of squares of $\varepsilon_{i,t} - m_i' e_t$. Clearly, the theoretical value of this is

$$\hat{m}_i = \text{cov}(e_t)^{-1}\text{cov}(e_t, \varepsilon_{i,t}) = (EP^E E'^{-1}(EB)_i \tag{44}$$

where $(EB)_i$ denotes the ith column of EB. A measure of goodness of fit is then

$$\mathbb{F}_i^{PI} = \text{cov}(\varepsilon_{i,t}) - \text{cov}(\varepsilon_{i,t}, e_t)\text{cov}(e_t)^{-1}\text{cov}(e_t, \varepsilon_{i,t}) = 1 - (EB)_i'(EP^E E'^{-1}(EB)_i \tag{45}$$

Thus one can as usual define a linear transformation of Me_t (where M is made up of

[39]Miranda-Agrippino and Ricco (2019) propose a related concept of "partial invertibility" when only a subset of structural shocks is of interest and needs to be recovered for impulse response functions. Approximate fundamentalness can then be viewed as a generalization to a continuous measure of the degree of invertibility-fundamentalness.

the rows m_i') as representing the structural shocks, but only take serious note of those shocks where the goodness of fit is close to 0. Once again, one can use the multivariate measure of goodness of fit[40]

$$\mathbb{F}^{PI} = I - B'E'(EP^E E')^{-1}EB \tag{46}$$

where the diagonal terms then correspond to the terms $\mathbb{F}_i$ of (45). In (46), we note that $EP^E E' = \text{cov}(e_t)$ from the steady state of (B.27), and $(EB)_i = \text{cov}(e_t, \varepsilon_{i,t})$.

If the number of measurements is equal to the number of shocks, and if $\mathbb{F}_i = 0$ for all i, then since $\mathbb{F}^{PI}$ is by definition a positive definite matrix, it must be identically equal to 0. Of course, it may be the case that none of the $\mathbb{F}_i$ are zero, but that a linear combination of the structural shocks are exactly equal to a linear combination of the residuals. In addition, we might specify a particular value of the R^2 (e.g., $R_s^2 = 0.9$) fit of residuals to fundamentals such that we are happy to approximate the fundamental by the best fit of residuals.[41]

The maximum eigenvalue of $\mathbb{F}^{PI}$ then provides a measure of overall non-fundamentalness. It must of course be emphasized that none of these measures can be obtained directly from the data. The papers cited above all provide details of how simulations on the underlying VARMA models can indicate how to make appropriate inferences on the structural shocks using just the data and a VAR estimation.

5.2 A Multivariate Measure with Imperfect Information

Collard and Dellas (2010) provide examples where there are large differences in the IRFs under II and PI, and indeed Theorem 3 appears to indicate that this may be a major issue. In addition, Levine *et al.* (2012), for an estimated DSGE model, find that such differences are quite large as well.

As we have seen for the PI case above, it is quite straightforward to obtain goodness of fit measures for the individual shocks from the multivariate measures, so for convenience, we only list the latter. Firstly, the Beaudry *et al.* (2016) measure, which can be abbreviated to the difference between the variances of the innovations and the fundamentals, is given by

$$\mathbb{B}^{II} = EZE' - EBB'E' \tag{47}$$

where from the Theorem 5 Z satisfies

$$Z = AZA' - AZE'(EZE')^{-1}EZA' + P^A J'(JP^A J')^{-1}JP^A \tag{48}$$

[40]If the theoretical model is estimated with constraints on B and with direct estimates of the shock variances $\sigma_1^2, \sigma_2^2, ...$, then the last term in (46) must be pre- and post-multiplied by the matrix $S = diag(1/\sigma_1, 1/\sigma_2, ...)$.

[41]A perfect fit in the Forni *et al.* (2019) case is $\mathbb{F}_i = 0,\ R_i^2 = 1$.

Likewise, the multivariate Forni *et al.* (2019) measure can, after some effort, be written[42]

$$\mathbb{F}^{II} = I - B'J'(JP^AJ')^{-1}JP^AE'(EZE')^{-1}EP^AJ'(JP^AJ')^{-1}JB \qquad (49)$$

Analogously to the PI case, $EZE' = \text{cov}(e_t)$, with $EP^AJ'(JP^AJ')^{-1}JB = \text{cov}(e_t, \varepsilon_t)$. The latter follows firstly because, from (20) and the innovations representation, we can write $e_t = E(z_{t,t-1} - \bar{s}_{1t}) + EP^AJ'(JP^AJ')^{-1}J\tilde{z}_t$. The first term is clearly independent of ε_t, while the covariance of the second term with ε_t is obtained by calculating $\mathbb{E}[\tilde{z}_{t+1}\varepsilon'_{t+1}]$ in (B.69).

Analogously to the PI case, $EZE' = \text{cov}(e_t)$, with $EP^AJ'(JP^AJ')^{-1}JB = \text{cov}(e_t, \varepsilon_t)$. The latter follows firstly because, from (20) and the innovations representation we can write $e_t = E(z_{t,t-1} - \bar{s}_{1t}) + EP^AJ'(JP^AJ')^{-1}J\tilde{z}_t$. The first term is clearly independent of ε_t, while the covariance of the second term with ε_t is obtained by calculating $\mathbb{E}[\tilde{z}_{t+1}\varepsilon'_{t+1}]$ in (B.69).

For the remainder of this section, we only discuss the Forni measure, because (1) the Beaudry measure is only suitable for the square case when numbers of measurements and shocks are the same; (2) given our main results on the role of Blaschke factors under II, if $y = (L-a)/(1-aL)\varepsilon$, then it is easy to show that the $\mathbb{B}$ measure for y is 0, whereas the $\mathbb{F}$ measure is $1 - a^2$.

We can bring together (46) and (49) in the following final theorem of the paper.

Theorem 7. *Consider the more general case with the number of structural shocks possibly greater than the number of measurements. (a) All zero eigenvalues of $\mathbb{F}^{PI}$ or $\mathbb{F}^{II}$, for the PI or II cases respectively, correspond to a perfect fit between a linear combination of fundamentals and a best regression fit of residuals; (b) The number of eigenvalues of $\mathbb{F}^{PI}$ or $\mathbb{F}^{II}$ that are less than $1 - R_s^2$, where R_s^2 is the chosen threshold for R^2, correspond to the number of linear combinations of fundamentals that can be obtained approximately from the residuals.*

Proof. See Appendix B.10. □

In addition, diagonal terms $\mathbb{F}_i^{II}$ correspond to a measure of goodness of fit of the innovations residuals to the ith structural shock and provide information for each shock individually. Note however that these measures correspond to the case when all observables are of current variables. While it is not difficult to perform the appropriate calculations in the case when some variables are current and others are lagged, it is not straightforward to write down a mathematical expression in such a case. Nevertheless,

[42]The same comment applies as in the footnote to (46). This follows because P^A depends on $Bcov(\varepsilon_t)B'$, so is invariant to whether the variances of the shocks are normalized to 1 or not.

we can apply the ideas above when *all* variables are lagged. In particular, the theoretical value of $\mathbb{F}^{II,lagged}$ can now be defined as

$$\mathbb{F}^{II,lagged} = \text{cov}(\varepsilon_t) - \text{cov}(\varepsilon_t, e_{t-1})\text{cov}(e_{t-1})^{-1}\text{cov}(e_{t-1}, \varepsilon_t) \tag{50}$$

where $\text{cov}(e_{t-1})$ is of course equal to $\text{cov}(e_t) = EZE'$, so the only change is to $\text{cov}(e_{t-1}, \varepsilon_t)$, which after a little effort can be derived as

$$\begin{aligned} \text{cov}(e_{t-1}, \varepsilon_t) = & EAP^A J'^A J'^{-1} JB - EAZE'(EZE'^{-1} EP^A J'^A J'^{-1} JB \\ & + EP^A J'^A J'^{-1} JFB - EP^A J'^A J'^{-1} JFP^A J'^A J'^{-1} JB \end{aligned} \tag{51}$$

Then the fit $\mathbb{F}_i^{II,lagged}$ to the ith shock is just given by the ith main diagonal term of $\mathbb{F}^{II,lagged}$.

In the next section, we compare numerically these PI and II multivariate measures of the fit of the innovations to the fundamentals for a DSGE model.

6 Numerical Application to a Richer RBC Model

This section further illustrates our theoretical results using numerical solutions of a more general RBC model than that used in Section C. The model has investment adjustment costs, variable hours and two shock processes. This first feature introduces more forward-looking behaviour into the model and two more non-predetermined variables, investment and the cost of capital. These provide an extra source of divergence between the PI and II solutions[43] and therefore an additional source of non-invertibility as well. See Appendix E for full details.

We implement the invertibility conditions of Theorem 3 and the multivariate measure of goodness of fit set out in Section 5. For the latter, our focus is on (46) and (49), the corresponding measures of correlation between e_t and ε_t, for the PI and II cases, respectively, where $\text{cov}(e_t) = EP^E E'$ and $\text{cov}(e_t) = EZE'$ are the covariance matrices of the innovation processes for the two cases, and $\text{cov}(\varepsilon_t)$ of the structural shocks in the model. As noted, the maximum eigenvalue provides a measure of overall non-fundamentalness. In addition, any eigenvalues close to zero provide information as to which structural shocks can be satisfactorily recovered even though the RE solution as a whole is not invertible.

The model is solved and simulated through Theorem 1 and the conversion procedure set out in Appendix B.1. Table 1 below summarizes a complete set of combinations of two observables for this model, i.e., $c = \frac{8!}{(8-2)!2!} = 28$, based on the rank and stability conditions of Theorem 3. Table 1 also checks the difference between PI and II in terms of identifying the fundamental (structural) shocks from the perspective of VARs via the

[43] In these computations, II refers to the II-SA equilibrium.

eigenvalues of $\mathbb{F}^{PI}$ and $\mathbb{F}^{II}$, assuming that the RBC Model is the true DGP. We consider two parameter settings for the risk parameter in the a Cobb-Douglas households utility function: $\sigma = 0.3, 2$.

Information Set $c = \frac{8!}{(8-2)!2!} = 28$	E-invertibility under PI?	A-invertibility	Notes	Eigenvalues of $\mathbb{F}^{PI}$ and $\mathbb{F}^{II}$	Diagonal Values of $\mathbb{F}^{PI}$ and $\mathbb{F}^{II}$
RBC Case 1: $\sigma = 2$ and $\alpha = 0.67$					
$(C_t, I_t), (H_t, R_t)$ $(I_t, R_t), (I_t, W_t)$ $(I_t, R_{K,t}), (I_t, H_t), (I_t, V_t)$	YES	YES	E, EB, J,JB are of full rank $A(I-B(EB)^{-1}E)$ is stable $F(I-B(JB)^{-1}J)$ is stable	$eig(\mathbb{F}^{PI}) \equiv eig(\mathbb{F}^{II}) = [0,0]$	$\mathbb{F}_i^{PI} = \mathbb{F}_i^{II} = [0,0]$
$(Y_t, C_t), (C_t, H_t), (Y_t, R_t)$ $(Y_t, H_t), (C_t, W_t), (C_t, R_t)$ $(Y_t, I_t), (H_t, W_t), (W_t, R_t)$ $(Y_t, W_t), (C_t, V_t), (R_t, V_t), (W_t, V_t)$	YES	NO	E, EB are of full rank $A(I-B(EB)^{-1}E)$ is stable J,JB are of full rank $F(I-B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0,0]$ $eig(\mathbb{F}^{II}) > 0$	$\mathbb{F}_i^{PI} = [0,0]$ $\mathbb{F}_i^{II} = [0.011, 0.781]$
$(Y_t, R_{K,t}), (C_t, R_{K,t}), (H_t, V_t)$ $(H_t, R_{K,t}), (W_t, R_{K,t}), (R_{K,t}, V_t)$	NO	NO	E, EB are of full rank $A(I-B(EB)^{-1}E)$ is **not stable** J,JB are of full rank $F(I-B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) > 0$ $eig(\mathbb{F}^{II}) > 0$	$\mathbb{F}_i^{PI} = [0.000, 0.086]$ $\mathbb{F}_i^{II} = [0.014, 0.336]$
$(R_t, R_{K,t})$	YES	NO	E, EB are of full rank $A(I-B(EB)^{-1}E)$ is stable J,JB are **rank deficient**	$eig(\mathbb{F}^{PI}) = [0,0]$ $eig(\mathbb{F}^{II}) > 0$	$\mathbb{F}_i^{PI} = [0,0]$ $\mathbb{F}_i^{II} = [0.096, 0.983]$
(Y_t, V_t)	NO	NO	EB, JB are **rank deficient**	$eig(\mathbb{F}^{PI}) > 0$ $eig(\mathbb{F}^{II}) > 0$	$\mathbb{F}_i^{PI} = [0.032, 0.968]$ $\mathbb{F}_i^{II} = [0.049, 0.953]$
RBC Case 2: $\sigma = 0.3$ and $\alpha = 0.67$					
$(C_t, I_t), (C_t, R_t), (C_t, R_{K,t})$ $(I_t, R_t), (I_t, H_t), (I_t, R_{K,t})$ $(H_t, R_t), (W_t, R_t), (I_t, W_t)$	YES	YES	E, EB, J,JB are of full rank $A(I-B(EB)^{-1}E)$ is stable $F(I-B(JB)^{-1}J)$ is stable	$eig(\mathbb{F}^{PI}) \equiv eig(\mathbb{F}^{II}) = [0,0]$	$\mathbb{F}_i^{PI} = \mathbb{F}_i^{II} = [0,0]$
$(Y_t, C_t), (C_t, H_t), (C_t, V_t)$ $(Y_t, H_t), (C_t, W_t), (I_t, V_t)$ $(Y_t, I_t), (H_t, W_t), (H_t, V_t)$ $(Y_t, W_t), (Y_t, R_t), (R_t, V_t), (W_t, V_t)$	YES	NO	E, EB are of full rank $A(I-B(EB)^{-1}E)$ is stable J,JB are of full rank $F(I-B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0,0]$ $eig(\mathbb{F}^{II}) > 0$	$\mathbb{F}_i^{PI} = [0,0]$ $\mathbb{F}_i^{II} = [0.014, 0.748]$
$(Y_t, R_{K,t}), (R_{K,t}, V_t)$	NO	NO	E, EB are of full rank $A(I-B(EB)^{-1}E)$ is **not stable** J,JB are of full rank $F(I-B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) > 0$ $eig(\mathbb{F}^{II}) > 0$	$\mathbb{F}_i^{PI} = [0.001, 0.089]$ $\mathbb{F}_i^{II} = [0.014, 0.320]$
$(H_t, R_{K,t}), (W_t, R_{K,t})$	NO	NO	E, EB are of full rank $A(I-B(EB)^{-1}E)$ is **not stable** J,JB are of full rank $F(I-B(JB)^{-1}J)$ is stable	$eig(\mathbb{F}^{PI}) > 0$ $eig(\mathbb{F}^{II}) > 0$	$\mathbb{F}_i^{PI} = [0.004, 0.686]$ $\mathbb{F}_i^{II} = [0.004, 0.686]$
$(R_t, R_{K,t})$	YES	NO	E, EB are of full rank $A(I-B(EB)^{-1}E)$ is stable J,JB are **rank deficient**	$eig(\mathbb{F}^{PI}) = [0,0]$ $eig(\mathbb{F}^{II}) > 0$	$\mathbb{F}_i^{PI} = [0,0]$ $\mathbb{F}_i^{II} = [0.039, 0.995]$
(Y_t, V_t)	NO	NO	EB, JB are **rank deficient**	$eig(\mathbb{F}^{PI}) > 0$ $eig(\mathbb{F}^{II}) > 0$	$\mathbb{F}_i^{PI} = [0.039, 0.961]$ $\mathbb{F}_i^{II} = [0.048, 0.953]$

Notes: We check Conditions in Lemma 3 and Theorem 2 for the full RBC model with investment adjustment costs and variable hours. We consider two cases for $(\sigma, \alpha) = (2, 0.67)$ and $(\sigma, \alpha) = (0.3, 0.67)$. Note that diagonal values of $\mathbb{F}^{PI}$ and $\mathbb{F}^{II}$ differ for different choices of information sets in each category; the values reported are for the first entry and are only indicative.

Table 1: Exact and Approximate Invertibility Checks for Full RBC Model

With two shock processes, A_t and G_t (normalized such that $\text{cov}(\varepsilon_t) = I$), for the case $\sigma = 2$, the following 7 combinations of two observables (from a set of 8 possible observables: $(Y_t, H_t, C_t, I_t, W_t, R_t, R_{K,t}, V_t)$) result in A-invertibility: $m_t^E = m_t^A = (C_t, I_t)$, (H_t, R_t), (I_t, R_t), (I_t, W_t), $(I_t, R_{K,t})$, (I_t, H_t) and (I_t, V_t). Since $m_t^E = m_t^A$, these combinations also imply E-invertibility. On the other hand, for the remaining 21 combinations, A-invertibility fails. Only 7 combinations, $(Y_t, R_{K,t})$, $(C_t, R_{K,t})$, (H_t, V_t), $(H_t, R_{K,t})$, $(W_t, R_{K,t})$, $(V_t, R_{K,t})$ and (Y_t, V_t), fail the PMIC under the assumption of PI and would not be picked up by a standard RE solution of the DGP (the model) that imposes PI as an endowment. For the case $\sigma = 0.3$, the set of A-invertible combinations is increased for the analytical model and 4 combinations, $(Y_t, R_{K,t})$, $(R_{K,t}, V_t)$, $(H_t, R_{K,t})$, $(W_t, R_{K,t})$ and (Y_t, V_t), would fail the PMIC in the absence of informational considerations (i.e., under

PI as an endowment).

Recall Theorem 3 that establishes an extra condition, given that models PI are E-invertible, that the matrices J and JB are of full rank, *and* $F(I - B(JB)^{-1}J)$ is a stable matrix (has all eigenvalues inside the unit circle), for the model to be A-invertible too. In Table 1, for both values of σ, we find only two cases, $(R_t, R_{K,t})$ and (Y_t, V_t), when this rank condition is not satisfied.

The last column of Table 1 reports the diagonal values of the $\mathbb{F}^{PI}$ and $\mathbb{F}^{II}$ matrices for the first entry in the first column. Any value close to zero reported in the diagonal matrices indicates an exact fit of the innovations to the structural shocks in the models. Then these shocks can be described as approximately fundamental. The reported values are indicative of those for all the combinations in each of the cells in the first column. We find that, in all cases the technology shock, A_t, is in fact approximately fundamental, but the government spending shock G_t is not. This is illustrated in Figures 3 and 4 for the case of $\sigma = 2$ with observables (Y_t, C_t). By contrast Figures 5 and 6 examine the case with observables (Y_t, V_t) which results in the technology shock a poor approximation to being fundamental.[44]

7 Concluding Comments

This paper brings together in a unifying framework for studying the invertibility (fundamentalness) of possible SVAR representations of DSGE RE solutions to heterogeneous agent models with imperfect dispersed or common information. Imperfect information (dispersed or otherwise) introduces significant changes into the dynamics of an economy compared to the still-common assumption of perfect information. The hidden dynamics in the title of this paper not only imply different impulse responses to structural shocks; they imply that, due to the presence of Blaschke Factors, these dynamics and the associated shocks are inherently unobservable, even in long samples of data. Our general results illustrated by a simple analytically tractable illustrative model shows how the hidden dynamics significantly effects the PMIC, recoverability and measures of approximate fundamentalness.

There are a number of possible avenues for future research. First, as Angeletos and Lian (2016) have pointed out, the solution of dynamic heterogeneous agent models with time-varying shock processes and dispersed information in a HA setting remains a major challenge. We have provided a general solution only for the limiting case where idiosyncratic shocks dominate aggregate shocks. We show both existence and uniqueness of the solution subject to the standard saddlepath stability conditions. One direction for research which we are pursuing is to investigate in the time domain how a variation

[44] Appendix F carries out a further illustrative exercise on a RBC model with fiscal policy and a tax news shock.

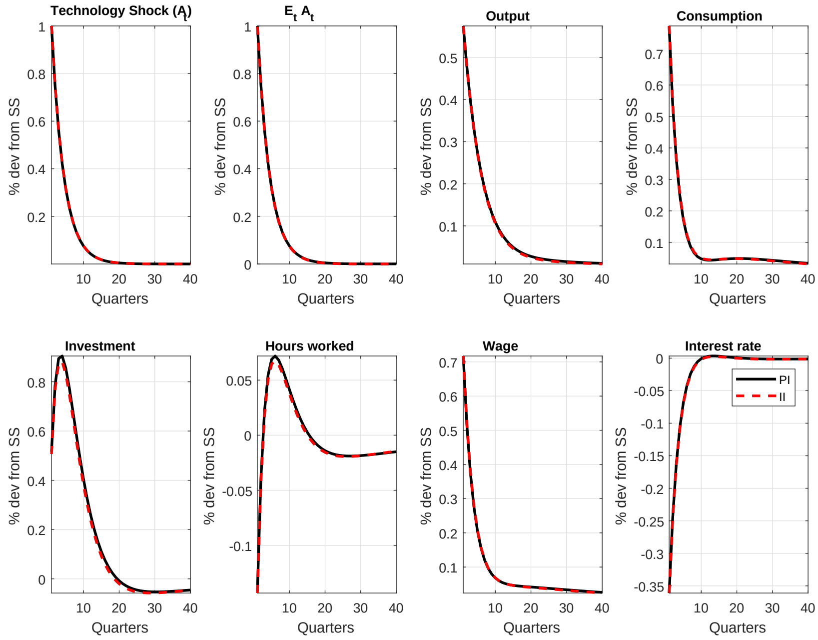

Figure 3: Full RBC Model with $\sigma = 2$: Impulse Responses to a Technology Shock, A_t. Observables Y_t, C_t. An Example of an Approximately Fundamental Shock

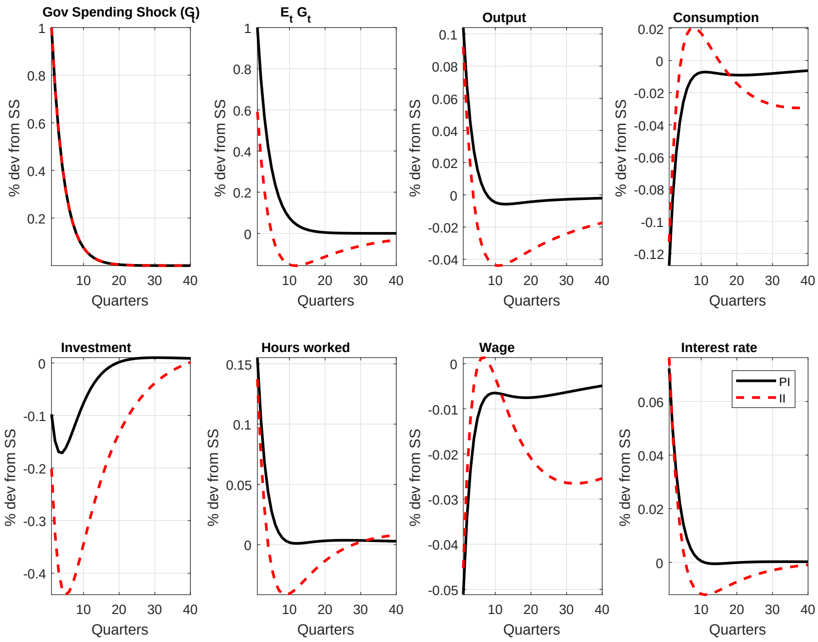

Figure 4: Full RBC Model with $\sigma = 2$: Impulse Responses to a Government Spending Shock, G_t. Observables Y_t, C_t. An Example of a Non-fundamental Shock

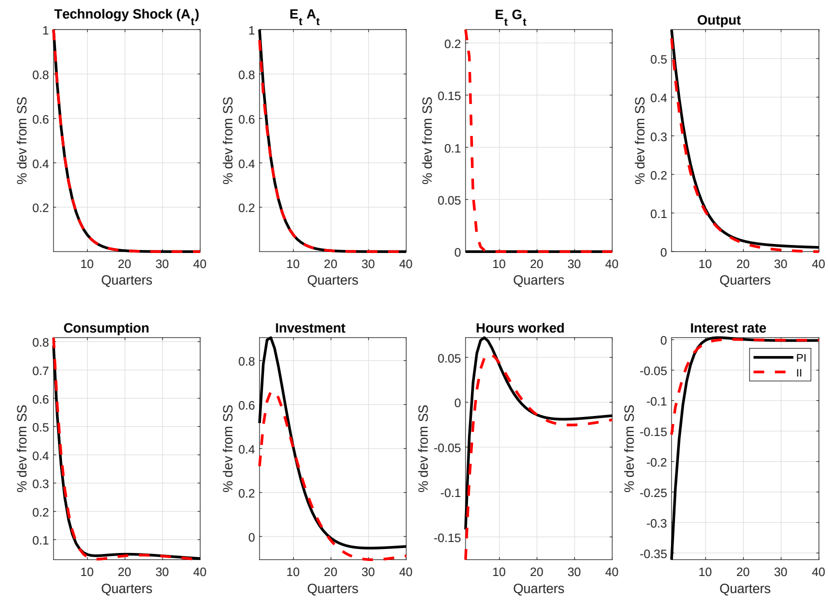

Figure 5: Full RBC Model with $\sigma = 2$: Impulse Responses to a Technology Shock, A_t. Observables Y_t, V_t. An Example of an Non-Fundamental Shock

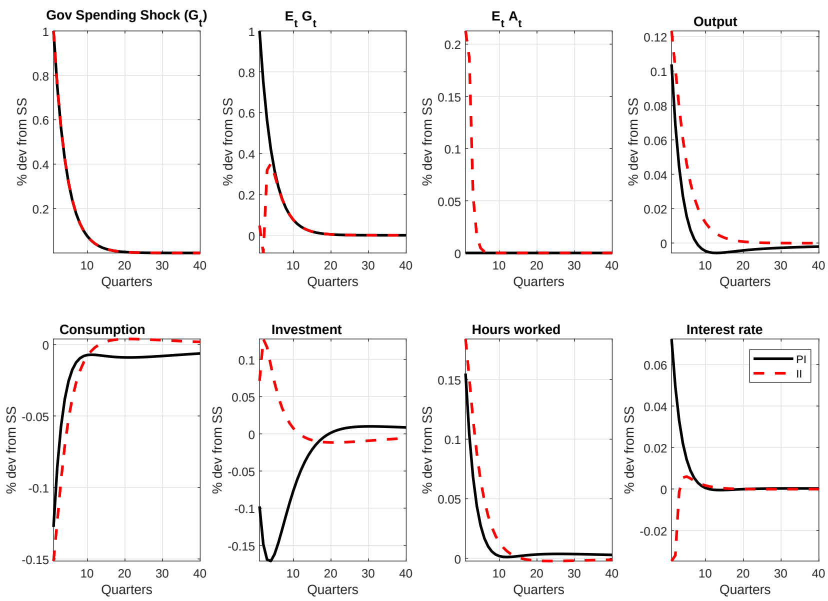

Figure 6: Full RBC Model with $\sigma = 2$: Impulse Responses to a Government Spending Shock, G_t. Observables Y_t, V_t. An Example of a Non-fundamental Shock

of this in Theorem 2 can be implemented that will generalize to the time domain, the finite-space results in the frequency domain in Rondina and Walker (2021) using the Wiener-Kolmogorov prediction formulae. Such a solution would also generalize the II-HA(Σ) case for our motivating example in Section 1.3 and results in Huo and Takayama (2021) and Angeletos and Huo (2021).[45] Our analysis of II is also restrictive in another sense that all agents have the same aggregate data in their information sets (although Corollary 5.2 allows it to differ from that of the econometrician). It would be of interest to relax this assumption to allow for agents with different II observables m_t^A as studied in Lubik *et al.* (2023).

Second, as is usual in the related literature, a Gaussian framework is adopted throughout our paper. Gouriéroux *et al.* (2020)[46] relax this assumption in their examination of both identification and fundamentalness issues. Although technically challenging a generalization of our results in this direction focusing on the information assumptions in the DGP would be of interest.

Finally as discussed in Section 4.9, much of the recent applied macroeconometrics has moved away from SVARS towards a direct measurements of shocks and their irfs. That section indicates, in an example, that the agents' informational problem still fundamentally changes the nature of the econometrician's problem to correctly estimate IRFs. However we provide no *general* results comparable to those related to SVARs in the rest of the paper. This remains a major challenge for future research.

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[45] See the discussion in Section 1.6.

[46] See also Funovitis (2020) for a critical comment.

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Online Appendix

A Background Results

First, we justify our form of state-space representation used in the paper and prove the Poor Man's Invertibility Condition (PMIC). Next, we set out several fairly standard results on the solution to Riccati equation, spectral analysis and recoverability and the econometrician's innovations representation that are essential to understand the theorems in the paper, and we here we cover these briefly.

A.1 Equivalence of Various State-Space Models

We show that all of the state-space models that are used in the statistics, control theory and econometrics literature can be represented by that used in the main text.

The usual model used in the statistics literature, Model 1, includes measurement error η_{1t}

$$s_{t+1} = A_1 s_t + B_1 \varepsilon_{1,t+1} \qquad m_t = C_1 s_t + D_1 \eta_{1t} \tag{A.1}$$

In the control theory literature, with possible correlation between ε_{2t} and measurement error η_{2t}, Model 2 is given by

$$w_{t+1} = A_2 w_t + B_2 \varepsilon_{2t} \qquad m_t = C_2 w_t + D_2 \eta_{2t} \tag{A.2}$$

In Fernandez-Villaverde *et al.* (2007) and much of the econometrics literature, Model 3 is given by

$$x_{t+1} = A_3 x_t + B_3 \varepsilon_{3,t+1} \qquad (\text{i.e., } x_t = A_3 x_{t-1} + B_3 \varepsilon_{3,t}) \qquad m_t = C_3 x_{t-1} + D_3 \varepsilon_{3t} \tag{A.3}$$

For Model 1, add η_{1t} to the state-space, so that it can be rewritten as

$$\begin{bmatrix} \eta_{1,t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} \eta_{1,t} \\ v_t \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & B_1 \end{bmatrix} \begin{bmatrix} \eta_{1,t+1} \\ \varepsilon_{1,t+1} \end{bmatrix} \qquad m_t = \begin{bmatrix} D_1 & C_1 \end{bmatrix} \begin{bmatrix} \eta_{1,t} \\ v_t \end{bmatrix} \tag{A.4}$$

For Model 2, if $D_2 = 0$, then the statistical properties of w_t are identical whether we date the shock as ε_{2t} or $\varepsilon_{2,t+1}$; thus Model 2 is equivalent to the main text model when $D_2 = 0$. Otherwise, include ε_{2t} and η_{2t} into the state-space

$$\begin{bmatrix} \varepsilon_{2,t+1} \\ \eta_{2,t+1} \\ w_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_2 & 0 & A_2 \end{bmatrix} \begin{bmatrix} \varepsilon_{2,t} \\ \eta_{2,t} \\ w_t \end{bmatrix} + \begin{bmatrix} 0 & I \\ I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_{2,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix} \qquad m_t = \begin{bmatrix} 0 & D_2 & C_2 \end{bmatrix} \begin{bmatrix} \varepsilon_{2,t} \\ \eta_{2,t} \\ w_t \end{bmatrix} \tag{A.5}$$

Model 3 can be written in the form of the main text model by appending both ε_{3t} and x_{t-1} to the state-space

$$\begin{bmatrix} \varepsilon_{3,t+1} \\ x_t \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{3,t} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ B_3 \end{bmatrix} \varepsilon_{3,t+1} \qquad m_t = \begin{bmatrix} D_3 & C_3 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{3,t} \\ x_{t-1} \\ w_t \end{bmatrix} \tag{A.6}$$

A.2 Proof of the PMIC

From (35) we have $\varepsilon_t = \tilde{D}^{-1}(m_t^E - \tilde{C}Ls_t)$ where L is the lag operator. Hence from (35) we have

$$(I - \tilde{A}L)s_t = \tilde{B}\varepsilon_t = \tilde{B}\tilde{D}^{-1}(m_t^E - \tilde{C}Ls_t)$$

from which we obtain $s_t = [I - (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})L]^{-1}\tilde{B}\tilde{D}^{-1}m_t^E$ and hence

$$\varepsilon_t = \tilde{D}^{-1}(m_t^E - \tilde{C}s_{t-1}) = \tilde{D}^{-1}(m_t^E - \tilde{C}[I - (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})L]^{-1}\tilde{B}\tilde{D}^{-1}m_{t-1}^E) \tag{A.7}$$

Expanding $(I - X)^{-1} = I + X + X^2 + \cdots$ we then have

$$\varepsilon_t = \tilde{D}^{-1}\left(m_t^E - \tilde{C}\sum_{j=1}^{\infty}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^j\tilde{B}\tilde{D}^{-1}m_{t-j}^E\right) \tag{A.8}$$

A necessary and sufficient condition for the summation to converge is that $\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}$ has stable eigenvalues (eigenvalues within the unit circle in the complex plane).

The PMIC transforms into ABE notation as follows: we note that the following term in (A.8) can be written in two equivalent ways

$$\tilde{C}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^j = \tilde{E}\tilde{A}(\tilde{A} - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}\tilde{A})^j = \tilde{E}(\tilde{A}(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}))^j\tilde{A} \tag{A.9}$$

so that the PMIC requirements are that $\tilde{E}\tilde{B}$ is invertible and that $\tilde{A}(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E})$ has stable eigenvalues.

A.3 The Spectrum of a Stochastic Process, Blaschke Factors

The spectrum of a stochastic process is a representation of all its second moments - auto, cross and auto-cross covariances, so that a VAR with sufficient lags will pick up all of these moments to a high degree of accuracy.

The spectrum (or spectral density) $\Phi_y(L)$ of a stochastic process y_t of dimension r is defined to be $\Phi_y(L) = \sum_{k=-\infty}^{\infty} \mathrm{cov}(y_t, y_{t-k})L^k$, and this is a rational function of L if y_t can be expressed as a state-space system with finite dimension. It is a standard result that the spectrum of the ABE system above is given by $\tilde{E}(I-\tilde{A}L)^{-1}\tilde{B}\tilde{B}'(I-\tilde{A}'L)^{-1}\tilde{E}'$.

Definition 3. *A rational spectral density $\Phi_y(L)$ admits a spectral factorization of the form $\Phi_y(L) = Z(L)Z'(L^{-1})$. A* minimal spectral factorization *(Baggio and Ferrante, 2016) is one where the McMillan degree of $Z(L)$ is a minimum.*[47]

Of importance for our main results below is the Blaschke factor $b(L) = (1-aL)/(L-a)$, which has the easily verifiable property that $b(L)b(L^{-1}) = 1$. This implies that if $y_{1t} = \varepsilon_t$ is a scalar white noise process, with spectrum given by $\Phi_{y_1}(L) = var(\varepsilon_t)$, then $y_{2t} = b(L)\varepsilon_t$ has the same spectrum. The second-moment properties of y_{1t} and y_{2t} are therefore identical; however, although there is a minimal realization of y_{2t} in ABCD form ($x_t = \frac{1}{a}x_{t-1} + (a-\frac{1}{a})\varepsilon_t,\ y_t = x_{t-1} - a\varepsilon_t$), it is not a minimal *spectral* factorization of the process, which is given by the fundamental representation $y_{2t} = \eta_t$, where $\text{var}(\eta_t) = \text{var}(\varepsilon_t)$. Crucially the IRFs of y_{1t} and y_{2t} in response to a shock to ε_t are completely different, with the latter being non-zero at all lags.

More generally, for the scalar case, suppose $Z(L) = n(L)/d(L)$. Now use a Blaschke factor to define $Z_1(L) = (1-aL)/(L-a)Z(L)$, so that $Z_1(L)Z_1(L^{-1}) = Z(L)Z(L^{-1})$. This changes $n(L)$ to $n(L)(1-aL)$ and $d(L)$ to $d(L)(L-a)$. The degree of the latter is obviously greater than that of $d(L)$, so that $Z_1(L)$ is a non-minimal spectral factorization. To reiterate the point raised earlier, if $y_t = Z_1(L)\varepsilon_t$ represents the true response to the structural shock, then a VAR econometrician will estimate a very good approximation to $Z(L)$ but would have no way of inferring the correct impulse response.

We can now draw on these general results to prove (A.12) in Section 4.9 of the main text. The joint spectrum of (x_t, m_t) is given by

$$\begin{bmatrix} \sigma & 1 \\ 0 & \frac{b-L}{1-bL} \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ 1 & \frac{b-L^{-1}}{1-bL^{-1}} \end{bmatrix} = \begin{bmatrix} 1+\sigma^2 & \frac{b-L^{-1}}{1-bL^{-1}} \\ \frac{b-L}{1-bL} & 1 \end{bmatrix} \tag{A.10}$$

Factorizing this in invertible form yields

$$\begin{bmatrix} \sqrt{1+\sigma^2} & 0 \\ \frac{b-L}{1-bL}\frac{1}{\sqrt{1+\sigma^2}} & \frac{\sigma}{\sqrt{1+\sigma^2}} \end{bmatrix} \begin{bmatrix} \sqrt{1+\sigma^2} & \frac{b-L^{-1}}{1-bL^{-1}}\frac{1}{\sqrt{1+\sigma^2}} \\ 0 & \frac{\sigma}{\sqrt{1+\sigma^2}} \end{bmatrix} \tag{A.11}$$

It follows that

$$x_t = \sqrt{1+\sigma^2}e_{1t} \qquad m_t = \frac{b-L}{1-bL}\frac{1}{\sqrt{1+\sigma^2}}e_{1t} + \frac{\sigma}{\sqrt{1+\sigma^2}}e_{2t} \tag{A.12}$$

where e_{1t}, e_{2t} are orthogonal white noise processes with unit variance. The expression for

[47] The Smith-McMillan representation (Youla, 1961) of a rational matrix function $Z(L)$ is given by $Z(L) = \Gamma(L)diag(\frac{n_1(L)}{d_1(L)}, ..., \frac{n_r(L)}{d_r(L)})\Theta(L)$, where $\Gamma(L), \Theta(L)$ have determinants equal to a constant, $d_k(L)$ divides $d_{k+1}(L)$ and $n_k(L)$ divides $n_{k-1}(L)$. The McMillan degree of $Z(L)$ is the highest power of L in $d_1(L)d_2(L)...d_r(L)$.

m_t can be rewritten as

$$m_t = \frac{b}{\sqrt{1+\sigma^2}} e_{1t} - \frac{(1-b^2)L}{1-bL} \frac{1}{\sqrt{1+\sigma^2}} e_{1t} + \frac{\sigma}{\sqrt{1+\sigma^2}} e_{2t} \tag{A.13}$$

It follows that $\sqrt{1+\sigma^2} e_{1t}$ is the prediction error for x_t and $\frac{be_{1t}+\sigma e_{2t}}{\sqrt{1+\sigma^2}}$ is the prediction error for m_t.

A.4 Recoverability and Agents' Information Sets

Recoverability, reviewed more didactically in Appendix H, makes the assumption that a vector process can be represented as a finite order VARMA: whether by direct estimation, or as an approximation, based on a finite order VAR.[48] A fundamental VARMA representation is a minimal spectral factorization; but there is a finite set of alternative non-fundamental representations of the same order that have an identical autocovariance (Lippi and Reichlin, 1994: each of these is also a *minimal* spectral factorization of the same process.

Thus a VAR econometrician who is well enough informed can reconstruct an alternative minimal spectral factorization that can approximate a true minimal spectral factorization, and the shocks to any such representation are recoverable. However, the VAR econometrician *cannot* reconstruct a *non*-minimal spectral factorization; we show below that this arises under II , in the absence of A-invertibility. Key to this is the following lesser-known result due to Lindquist and Picci (2015) in their Corollary 16.5.7 and Lemma 16.5.8:

Lemma 4. *Let* (30) *be a minimal representation of the spectral factor of a stationary stochastic process. There is a one-to-one correspondence between symmetric solutions of the Riccati equation* $P = \tilde{A}P\tilde{A}' - \tilde{A}P\tilde{E}'(\tilde{E}P\tilde{E}')^{-1}\tilde{E}P\tilde{A}' + \tilde{B}\tilde{B}'$ *and minimal spectral factors that retain stationarity; this correspondence is defined via the state-space representation*

$$w_t = \tilde{A}w_{t-1} + P\tilde{E}'(\tilde{E}P\tilde{E}')^{-1}\eta_t \qquad m_t^E = \tilde{E}w_t \qquad \eta_t \sim N(0, \tilde{E}P\tilde{E}') \tag{A.14}$$

Thus, for a square system, these alternative solutions for P lead to transfer functions from shocks to observables that differ by one or more Blaschke factors. However, what we need subsequently is a result that we can deduce from this lemma, which derives from the PMIC matrices associated with (A.14) *that arise from the general solution for P and the particular solution $P = \tilde{B}\tilde{B}'$, namely $\tilde{A} - \tilde{A}P\tilde{E}'(\tilde{E}P\tilde{E}')^{-1}\tilde{E}$ and $\tilde{A} - \tilde{A}\tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}$:*

[48] See Appendix H.

Corollary A.4. *If P is a symmetric solution of the Riccati equation of Lemma 3, then the eigenvalues of $\tilde{A} - \tilde{A}P\tilde{E}'(\tilde{E}P\tilde{E}')^{-1}\tilde{E}$ and $\tilde{A} - \tilde{A}\tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}$ are either identical or reciprocals of one another.*

A.5 The Econometrician's Innovations Process and the Riccati Equation

We now consider the general nature of the time series representation of the system that the econometrician can extract from the history of the observables. At this stage, we do need to make any assumptions about the number of observables vs shocks, other than to assume that $m \leq k$.

For any given set of observables, m_t^E, the econometrician's updating equation for state estimates, assuming convergence of the Kalman filtering matrices, is

$$\mathbb{E}_t s_{t+1} = \tilde{A}\mathbb{E}_{t-1}s_t + \tilde{A}P^E\tilde{E}'(\tilde{E}P^E\tilde{E}')^{-1}e_t, \qquad e_t = m_t^E - \tilde{E}\mathbb{E}_{t-1}s_t \qquad e_t \sim N(0, \tilde{E}P^E\tilde{E}') \tag{A.15}$$

where $\mathbb{E}_s$ denotes expectations conditioned on the econometrician's information set at time s, and $e_t \equiv m_t^E - \mathbb{E}_{t-1}m_t^E$, the innovations to the observables in period t, conditional upon information in period $t-1$.

The Riccati matrix $P^E = \text{cov}(s_t - \mathbb{E}_{t-1}s_t)$ for this Kalman filter is given in the limit by

$$P^E = Q^E P^E Q^{E'} + \tilde{B}\tilde{B}' \qquad \text{where } \; Q^E = \tilde{A} - \tilde{A}P^E\tilde{E}'(\tilde{E}P^E\tilde{E}')^{-1}\tilde{E} \tag{A.16}$$

To ensure stability of the solution P^E, it must satisfy the convergence condition that Q^E is a stable matrix, analogous to the requirement for Q^A above; a sufficient condition is either that $\tilde{A}$ is a stable matrix, or else the controllability of $(\tilde{A}, \tilde{B})$ and observability[49] of $(\tilde{E}, \tilde{A})$. We also have the following result that ensures uniqueness of the solution algorithm for II:

Lemma 5. *There is a unique positive semi-definite Riccati matrix P^E that has the property that Q^E is a stable matrix.*

Proof. Clearly $\tilde{A}$ must be stable, and the other PMIC condition discussed after (A.9) is that $\tilde{A} - \tilde{A}P^E\tilde{E}'(\tilde{E}P^E\tilde{E}')^{-1}\tilde{E}$ is stable. But if this latter condition does not hold then we have seen from (B.27) and the discussion following that P^E is not the appropriate solution of the Riccati equation. □

Note that if we subtract the first equation of (A.15) from the first equation of (30), we are able to evaluate $\text{cov}(s_{t+1} - \mathbb{E}_t s_{t+1},\ \varepsilon_{t+1}) = \tilde{B}$, from which it follows that the covariance between the innovations process and the shocks is given by $\text{cov}(e_t, \varepsilon_t) = \text{cov}(E(s_t -$

[49] Reduction to minimal form with these properties is fairly straightforward.

$\mathbb{E}_{t-1}s_t),\ \varepsilon_t) = \tilde{E}\tilde{B}$. We shall use this property later to evaluate how correlated are the residuals from a VAR to the structural shocks.

The Kalman Filter updated expectation of the state s_t given the extra information at time t is given by $\mathbb{E}_t s_t = \mathbb{E}_{t-1}s_t + P^E\tilde{E}'(\tilde{E}P^E\tilde{E}')^{-1}e_t$, and a little manipulation of (A.15) enables us to obtain the alternative steady state innovations representation as

$$\mathbb{E}_t s_t = \tilde{A}\mathbb{E}_{t-1}s_{t-1} + P^E\tilde{E}'(\tilde{E}P^E\tilde{E}')^{-1}e_t \qquad m_t^E = \tilde{E}\mathbb{E}_t s_t \tag{A.17}$$

This representation will be our main focus, but the representation of the innovations process in (A.15) is important in proving some of our theoretical results because it provides a means of evaluating the innovations process, and is essential for addressing approximate fundamentalness.

The innovations e_t to this representation have a dimension m equal to the number of observables, and the representation is valid given our general assumption as stated above that $m \leq k$.

The discussion up to now then leads to the following Lemma which applies for any general information set:

Lemma 6. *The innovations representation* (A.17) *applies for $m \leq k$ iff $\tilde{A}$ and Q^E has stable eigenvalues. Sufficient conditions for this to hold are the observability and controllability of* $(\tilde{A}, \tilde{B}, \tilde{E})$.

A.6 The Innovations Representation Under E-invertibility

When the structural shock system (30) is E-invertible, this means that $P^E = \tilde{B}\tilde{B}'$ is a stable solution to the Riccati equation, which in turn requires $Q^E = \tilde{A} - \tilde{A}\tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}$ to be a stable matrix. This is identical to the PMIC requirement and implies that the innovations process e_t from the filtering problem converges to $\tilde{E}\tilde{B}\varepsilon_t$ as $t \to \infty$. As a result, the state vector s_t is observable asymptotically by the econometrician.

B Proofs of Theorems, Lemmas and Corollaries

B.1 Proof of Theorem 1: Transformation of System to PCL Form

B.1.1 The Problem Stated

An important feature of the RE solution procedure of the seminal paper Blanchard and Kahn (1980) is that it provided necessary and sufficient conditions for the existence and uniqueness of a solution for linearized model. The only general results on II solutions to rational expectations models date back to PCL, who utilize the Blanchard-Kahn set-up, and generalize this result.

Theorem 1 states that Equation (9), re-expressed here

$$A_0 Y_{t+1,t} + A_1 Y_t = A_2 Y_{t-1} + \Psi \varepsilon_t \tag{B.1}$$

with agents' measurements

$$m_t^A = L Y_t \tag{B.2}$$

can be written in the form (10) and (11) originally used by PCL, re-stated here as

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{B.3}$$

with agents' measurements given by

$$m_t^A = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \tag{B.4}$$

To prove Theorem 1, the next section describes a completely novel algorithm for converting the state space (B.1), (B.2) under II to the form in (B.3) and (B.4). We assume that the system is 'proper', by which we mean the matrix A_1 is invertible; this precludes the possibility of a system that includes equations of the form $h^T Y_{t+1} = 0$, but it is fairly easy to take account of these as well.

B.1.2 An Iterative Algorithm

Although complicated, the basic stages for the conversion are fairly simple:

1. We first (Stages 1 to 3) find the singular value decomposition for the $n \times n$ matrix A_0 (which is typically of reduced rank $m < n$) which allows us to define a vector of m forward-looking variables that are linear combinations of the original Y_t.

2. We then introduce a novel iterative stage (Stage 4) which replaces any forward-looking expectations that use model-consistent updating equations. This reduces the number of equations with forward-looking expectations, while increasing the number of backward-looking equations one-for-one. But at the same time it introduces a dependence of the additional backward-looking equations on both state estimates $z_{t,t}$ $\left(\equiv E_t z_t | I_t^A\right)$ and estimates of forward-looking variables, $x_{t,t}$. This in turn implies that both (B.3) and (B.4) in general contain such terms.

3. A simple example may help to provide intuition for this iterative stage: suppose two of the equations in the system are of the form: $z_t = \rho z_t + \varepsilon_t,\ y_t = z_{t+1,t}$ (where

both y_t and z_t are scalars) i.e., we have one backward-looking (BL) equation and one forward-looking (FL) equation. However, using the first equation we can write $z_{t+1,t} = E_t z_{t+1} = \rho z_{t,t}$, hence substituting into the second equation, $y_t = \rho z_{t,t}$: i.e., we can use a model-consistent updating equation. Note, however, a crucial feature: since under II we cannot assume that z_t is directly observable, this updating equation is expressed in terms of the filtered state estimate $z_{t,t}$ rather than directly in terms of x_t. We thus now have two BL equations, but one of these is expressed in term of a state estimate.

4. The iterative Stage 4 may need to be repeated a finite number of times. In the case of PI, this is all that is needed, apart from defining what are the $t+1$ variables.

5. For II, we retain the same backward and forward looking variables as in the PI case, but the solution process is a little more intricate.

The detailed procedure for conversion of (B.1) and (B.2) to the form in (B.3) and (B.4) is as follows:
Stage 1: SVD and partitions of A_0. Obtain the singular value decomposition for the $n \times n$ matrix A_0: $A_0 = U_0 S_0 V_0^T$, where U_0, V_0 are unitary matrices. Assuming that only the first m values of the diagonal matrix S_0 are non-zero (rank$(A_0) = m < n$), we can rewrite this as $A_0 = U_1 S_1 V_1^T$, where U_1 are the first m columns of U_0, S_1 is the first $m \times m$ block of S_0 and V_1^T are the first m rows of V_0^T. In addition, U_2 are the remaining $n-m$ columns of U_0, and V_2^T are the remaining $n-m$ rows of V_0^T.
Stage 2: Extract FL subsystem from (B.3) using S_1 and U_1. Multiply (B.3) by $S_1^{-1} U_1^T$, which yields

$$V_1^T Y_{t+1,t} + S_1^{-1} U_1^T A_1 Y_t = S_1^{-1} U_1^T A_2 Y_{t-1} + S_1^{-1} U_1^T \Psi \varepsilon_t \tag{B.5}$$

We can now define an initial subdivision of Y_t into an (initially) m-vector of forward-looking variables $x_t = V_1^T Y_t$, and and an $(n-m)$-vector of backward-looking variables $s_t = V_2^T Y_t$ (noting that $Y_t = V_1 x_t + V_2 s_t$), and use the fact that $I = VV^T = V_1 V_1^T + V_2 V_2^T$ to rewrite (B.3) as

$$x_{t+1,t} + S_1^{-1} U_1^T A_1 (V_1 x_t + V_2 s_t) = S_1^{-1} U_1^T A_2 (V_1 x_{t-1} + V_2 s_{t-1}) + S_1^{-1} U_1^T \Psi \varepsilon_t \tag{B.6}$$

or simply

$$x_{t+1,t} + F_1 x_t + F_2 s_t = F_3 x_{t-1} + F_4 s_{t-1} + F_5 \varepsilon_t \tag{B.7}$$

where $F_1 = S_1^{-1} U_1^T A_1 V_1$, $F_2 = S_1^{-1} U_1^T A_1 V_2$, $F_3 = S_1^{-1} U_1^T A_2 V_1$, $F_4 = S_1^{-1} U_1^T A_2 V_2$ and $F_5 = S_1^{-1} U_1^T \Psi$. This is a set of m forward-looking equations. Note that in the iterative Stage 4, the definition of x_t will usually change further, and thus at this stage x_t is not usually equal to its final form in (B.3).

Stage 3: Extract BL subsystem from (B.3) using U_2. Multiply B.3 by U_2^T which yields

$$U_2^T A_1 Y_t = U_2^T A_2 Y_{t-1} + U_2^T \Psi \varepsilon_t \tag{B.8}$$

which can be rewritten as

$$U_2^T A_1 (V_1 x_t + V_2 s_t) = U_2^T A_2 (V_1 x_{t-1} + V_2 s_{t-1}) + U_2^T \Psi \varepsilon_t \tag{B.9}$$

or more simply

$$C_1 x_t + C_2 s_t = C_3 x_{t-1} + C_4 s_{t-1} + C_5 \varepsilon_t \tag{B.10}$$

where $C_1 = U_2^T A_1 V_1$, $C_2 = U_2^T A_1 V_2$, $C_3 = U_2^T A_2 V_1$, $C_4 = U_2^T A_2 V_2$ and $C_5 = U_2^T \Psi$. This is a set of $n - m$ backward-looking equations.

If C_2 is invertible then it is straightforward to multiply (B.3) by C_2^{-1}, and go straight to Stage 5. However if C_2 is not invertible we need to proceed to the next (iterative) stage.

Stage 4: Iterative transformation of FL equations using model-consistent updating. In this iterative stage we write (B.7) and (B.10) in the more general form

$$\begin{aligned} x_{t+1,t} + F_1 x_t + F_2 s_t &= F_3 x_{t-1} + F_4 s_{t-1} + F_5 \varepsilon_t & \text{(B.11)} \\ C_1 x_t + C_2 s_t + G_1 x_{t,t} + G_2 s_{t,t} &= C_3 x_{t-1} + C_4 s_{t-1} + C_5 \varepsilon_t & \text{(B.12)} \end{aligned}$$

where by comparison of (B.12) with (B.10) we have introduced two new matrices, G_1 and G_2 that must be zero in the first stage of iteration. However, at the end of the first iteration of this stage we shall increase the dimension of s_t, and reduce the dimension of x_t one-for-one, which will require us to re-define all the matrices in (B.11) and (B.12), such that, from the second iteration onwards, G_1 and G_2 will be non-zero. The whole of Stage 4 may then need to be iterated a finite number of times.

First find, a matrix J_2 such that $J_2^T(C_2+G_2)$=0, by using the SVD of C_2+G_2 (noting that in the first iterative stage, $G_2 = 0$) Then take forward expectations of (B.12) and pre-multiply by J_2^T to yield

$$J_2^T (C_1 + G_1) x_{t+1,t} = J_2^T C_3 x_{t,t} + J_2^T C_4 s_{t,t} \tag{B.13}$$

Then reduce the number of forward-looking variables by substituting for $x_{t+1,t}$ from (B.11). In addition find a matrix Q that has the same number of columns as $J_2^T(C_1+G_1)$ and is made up of rows that are orthogonal to it. Then we define the following subdivision of x_t

$$\begin{bmatrix} \bar{x}_t \\ \hat{x}_t \end{bmatrix} = \begin{bmatrix} Q \\ J_2^T (C_1 + G_1) \end{bmatrix} x_t \qquad x_t = M_1 \bar{x}_t + Q_2 \hat{x}_t \tag{B.14}$$

where $[Q_1 \;\; Q_2] = \begin{bmatrix} Q \\ J_2^T(C_1+G_1) \end{bmatrix}^{-1}$ From the substitution of $x_{t+1,t}$ into (B.13), we can then rewrite the system in terms of a new $\overline{m}$-vector of forward-looking variables $\bar{x}_t$, where $\overline{m}$ =rank$(C_2+G_2) \leq m$, and $n-\overline{m}$ backward-looking variables $(s_t, \; \hat{x}_t)$

$$\begin{aligned} & \bar{x}_{t+1,t} + QF_1Q_1\bar{x}_t + [QF_2 \;\; QF_1Q_2] \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \\ = \;\; & QF_3Q_1\bar{x}_{t-1} + [QF_4 \;\; QF_3Q_2] \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + QF_5\varepsilon_t \end{aligned} \tag{B.15}$$

$$\begin{aligned} & \begin{bmatrix} C_1Q_1 \\ J_2^T(C_1+G_1)F_1Q_1 \end{bmatrix} \bar{x}_t + \begin{bmatrix} C_2 & C_1Q_2 \\ J_2^T(C_1+G_1)F_2 & J_2^T(C_1+G_1)F_1Q_2 \end{bmatrix} \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \\ & + \begin{bmatrix} G_1Q_1 \\ J_2^TC_3Q_1 \end{bmatrix} \bar{x}_{t,t} + \begin{bmatrix} G_2 & G_1Q_2 \\ J_2^TC_4 & J_2^TC_3Q_2 \end{bmatrix} \begin{bmatrix} s_{t,t} \\ \hat{x}_{t,t} \end{bmatrix} \\ = \;\; & \begin{bmatrix} C_3Q_1 \\ J_2^T(C_1+G_1)F_3Q_1 \end{bmatrix} \bar{x}_{t-1} + \begin{bmatrix} C_4 & C_3Q_2 \\ J_2^T(C_1+G_1)F_4 & J_2^T(C_1+G_1)F_3Q_2 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} \\ & + \begin{bmatrix} C_5 \\ J_2^T(C_1+G_1)F_5 \end{bmatrix} \varepsilon_t \end{aligned} \tag{B.16}$$

The number of forward-looking states has now usually decreased from m to $\overline{m} \leq m$; while the number of backward-looking states $\bar{s}_t = \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix}$ has increased by the same amount. In addition the relationship $Y_t = V_1x_t + V_2s_t$ has changed to

$$Y_t = V_1Q_1\bar{x}_t + \begin{bmatrix} V_2 & V_1Q_2 \end{bmatrix} \bar{s}_t \tag{B.17}$$

Finally, we redefine $x_t = \overline{x}_t, s_t = \overline{s}_t$. Having done so, the system in (B.15) and (B.16) is now of the form of (B.11) and (B.12), subject to an appropriate redefinition of matrices. Thus, from (B.16), for G_1, and G_2, for example, we have an iterative scheme whereby, in the $(i+1)$th iteration,

$$G_1^{i+1} = \begin{bmatrix} G_1^iQ_1^i \\ (J_2^i)^T C_3^iQ_1^i \end{bmatrix}; \quad G_2^{i+1} = \begin{bmatrix} G_2^i & G_1^iQ_2^i \\ (J_2^i)^T C_4^i & (J_2^i)^T C_3^iQ_2^i \end{bmatrix}$$

where, e.g., G_1^i is the value of G_1 in the ith iteration, and $G_1^1 = 0, G_2^1 = 0$.

Repeat this stage until $C_2 + G_2$ is of full rank.

Proof of Theorem 1 for Perfect Information. In the PI case, the form (B.11), (B.12) with $s_t = s_{t,t}, x_t = x_{t,t}$ is generated after a finite number of iterations of Stage 3, where the number of iterations cannot exceed the number of variables. The forward looking variables are now x_t and the backward looking variables are s_t and x_{t-1}, and the system can be set up in Blanchard-Kahn form by defining $z_{t+1} = \begin{bmatrix} s_t \\ x_t \end{bmatrix}$. The only additional calculation is to invert C_2+G_2 to obtain the equation for s_t, and to substitute into (B.11). □

Proof of Theorem 1 for Imperfect Information. From this point, we eschew the details of matrix manipulations, as these are much more straightforward to understand conceptually compared with those above.

Stage 5: C_2 non-singular after Stage 4. First form expectations of (B.12), and invert $C_2 + G_2$ to obtain $s_{t,t}$ in terms of $x_{t,t}, x_{t-1,t}, s_{t-1,t}, \varepsilon_{t,t}$. Then substitute this back into (B.12), and invert C_2 to yield an expression for s_t in terms of the above expected values and also $x_t, x_{t-1}, s_{t-1}, \varepsilon_t$. This can be further substituted into (B.11) to yield an expression for $x_{t+1,t}$ in terms of these variables and their expectations. Similarly the measurement equations $m_t = LY_t$ can now be expressed in terms of all these variables. It follows that if we define $z_{t+1} = \begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \end{bmatrix}$, then the system can now be described by (B.3). Note that, since $\dim(s_t) + \dim(x_t) = n$, in this final form, $\dim(z_t) = n + rank\,(BB')$.

Stage 6: C_2 singular after Stage 4. We again start from (B.11) and (B.12), and regard x_t as the forward looking variable and (s_t, x_{t-1}) as the backward looking variables. Now advance these equations by changing t to $t+k$: $k = 1, 2, 3, ...$ and take expectations using information at time t, implying that $E_t s_{t+k} = E_t s_{t+k,t+k}$. Because $C_2 + G_2$ is invertible, we can rewrite these equations with just $x_{t+k+1,t}$ and $s_{t+k,t}$ on the LHS. Then the usual Blanchard-Kahn conditions for stable and unstable roots imply a saddlepath relationship of the form

$$x_{t+k+1,t} + N_1 s_{t+k,t} + N_2 x_{t+k,t} = 0 \tag{B.18}$$

where $[I\ N_1\ N_2]$ represents the eigenvectors of the unstable eigenvalues. In particular, this holds for $k = 0$, so if we substitute for $x_{t+1,t} = -N_1 s_{t,t} - N_2 x_{t,t}$ into (B.11), then together with (B.12) we obtain solutions for x_t, s_t in terms of $x_{t,t}, s_{t,t}, x_{t-1}, s_{t-1}, \varepsilon_t$. This is possible, because we have assumed the system is proper i.e., A_1 is invertible[50], and any manipulations of A_1 in the previous stages have been simple linear transformations of it to yield the matrices F_1, F_2, C_1, C_2. In addition, when we take expectations of (B.12) at time t, given that $C_2 + G_2$ is invertible, we obtain an equation for $s_{t,t}$ in terms of $x_{t,t}, s_{t-1,t}, x_{t-1,t}, \varepsilon_{t,t}$. It therefore follows that we can write s_t is terms of these latter variables as well as the variables above (excluding $s_{t,t}$). The same will be true of the

[50] The algorithm can be reworked without too much much difficulty if for example some of the forward looking equations in (B.1) are of the form $S_0 E_t Y_{t+1} = 0$.

measurements $m_t = LY_t$.

At this point, we have expressions for x_t and s_t, without any effect from $x_{t+1,t}$, so in principle we could solve the signal processing problem from this point onwards. However for consistency with the case of C_2 nonsingular, we can retrieve the representation of $x_{t+1,t}$ by substituting for s_t back into (B.11), and then the system has the same structure as that for the case C_2 nonsingular.

Finally, by defining $z_{t+1} = \begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \end{bmatrix}$, the converted form (B.3) becomes

$$\begin{aligned}
\begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ P_1 & G_{11} & G_{12} & G_{13} \\ 0 & 0 & 0 & I \\ P_3 & G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ FF_4 & FF_3 & FF_2 & FF_1 \\ 0 & 0 & 0 & 0 \\ FF_8 & FF_7 & FF_6 & FF_5 \end{bmatrix} \begin{bmatrix} \varepsilon_{t,t} \\ s_{t-1,t} \\ x_{t-1,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{t+1} \qquad \text{(B.19)}
\end{aligned}$$

where $G_{13} = -C_2^{-1}C_1$, $G_{12} = C_2^{-1}C_3$, $G_{11} = C_2^{-1}C_4$, $P_1 = C_2^{-1}C_5$, $G_{33} = -F_2G_{13} - F_1$, $G_{32} = -F_2G_{12}+F_3$, $G_{31} = -F_2G_{11}+F_4$, $P_3 = -F_2P_1+F_5$, $FF_1 = -C_2^{-1}G_1+C_2^{-1}G_2(C_2+G_2)^{-1}(C_1+G_1)$, $FF_2 = -C_2^{-1}G_2(C_2+G_2)^{-1}C_3$, $FF_3 = -C_2^{-1}G_2(C_2+G_2)^{-1}C_4$, $FF_4 = -C_2^{-1}G_2(C_2+G_2)^{-1}C_5$, $FF_5 = -F_2FF_1$, $FF_6 = -F_2FF_2$, $FF_7 = -F_2FF_3$ and $FF_8 = -F_2FF_4$. The C and F matrices are the reduction system matrices in (B.15) and (B.16) in the form of (B.11) and (B.12) (i.e., the iterative procedure that ensures invertibility to be achieved).

The measurements $m_t^A = LY_t$ can be written in terms of the states as $m_t^A = L(V_1x_t + V_2s_t)$, where V_1, V_2 have been updated by (B.17) through the same reduction procedure as above. Using (B.19), we show that m_t^A can be rewritten as

$$\begin{aligned}
m_t^A &= \begin{bmatrix} LV_2P_1 & LV_2G_{11} & LV_2G_{12} & LV_1 + LV_2G_{13} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} \\
&+ \begin{bmatrix} LV_2FF_4 & LV_2FF_3 & LV_2FF_2 & LV_2FF_1 \end{bmatrix} \begin{bmatrix} \varepsilon_{t,t} \\ s_{t-1,t} \\ x_{t-1,t} \\ x_{t,t} \end{bmatrix} \qquad \text{(B.20)}
\end{aligned}$$

So the observations (B.20) can now be cast into the form in (B.4)

$$m_t^A = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix}$$

where $M_1 = [LV_2P_1 \;\; LV_2G_{11} \;\; LV_2G_{12}]$ and $M_2 = LV_1 + LV_2G_{13}$. Similarly, $M_3 = [LV_2FF_4 \;\; LV_2FF_3 \;\; LV_2FF_2]$ and $M_4 = LV_2FF_1$. Thus the set-up is as required, with the vector of predetermined variables given by $[\varepsilon_t' \;\; s_{t-1}' \;\; x_{t-1}']'$, and the vector of jump variables given by x_t.

This completes the proof by construction for II. □

Example B.1 (Example of Stage 6 Being Needed for Imperfect Information)**.** Suppose that at the end of Stage 4, there is a system in scalar processes x_t and s_t,

$$x_{t+1,t} + \alpha x_t + s_t = \beta s_{t-1} + \varepsilon_t \qquad x_t - x_{t,t} + s_{t,t} = \gamma s_{t-1} \tag{B.21}$$

It is clear from examining these equations that they cannot be manipulated into BK form directly. However, if we now advance these equations by k periods and take expectations subject to I_t, one obtains two equations relating $x_{t+k+1,t}, s_{t+k,t}$ to $x_{t+k,t}, s_{t+k-1,t}$. Since this is true for all $k \geq 1$, and provided there is exactly one unstable eigenvalue corresponding to these dynamic relationships, it follows that there must be an expectational saddlepath relationship $x_{t+1,t} = -ns_{t,t}$. Substituting this into the first of the above equations allows us to solve in particular for s_t in terms of $x_t, s_{t,t}, s_{t-1}, \varepsilon_t$; from the second equation we can solve for $s_{t,t}$ in terms of $s_{t-1,t}$, so that we can replace the second equation by an equation for s_t in terms of $x_t, s_{t-1,t}, s_{t-1}, \varepsilon_t$. Redefining $z_{t+1} = s_t$, it is now straightforward to obtain the BK form for the first equation and the new second equation.

B.2 Proof of Lemma 1

The first step is to subtract the aggregate equations from the agent i equations, noting that under PI, $\mathbb{E}_{i,t}y_t = y_t, \mathbb{E}_{i,t}x_t = x_t$. In addition, since observation of the idiosyncratic shocks has no effect on one period ahead expectations, it follows that $\mathbb{E}_{i,t}y_{t+1} = \mathbb{E}_t y_{t+1}, \mathbb{E}_{i,t}x_{t+1} = \mathbb{E}_t x_{t+1}$. Thus the RE system in $y_{it} - y_t, x_{it} - x_t$ is driven solely by the shock ε_{it}, and standard saddlepath conditions apply.

B.3 Proof of Theorem 2

We conjecture that if a finite state solution to the aggregate variables z_t exists, then it will take the same structural form as that of the II-SA solution, with one difference: that a representation needs to be found for the matrix F or to be more precise, for the matrix

$Q = F - FP^A E'(EP^A E')^{-1}E$. The representation of Q for the II-HA case provides the proof of the theorem.

In order to distinguish the II-HA solution from the II-SA solution we replace the prediction error $\tilde{z}_t$ and forecast $z_{t,t-1}$ by $z_{1t},\ z_{2t}$ respectively.[51] Initially for simplicity we assume that there are no measurements directly dependent on aggregate jump variables x_t. This means that in the notation of (16) and (22) we have $J = E = M_1$, which we refer to as E. It follows that observations m_t^A of the aggregate variables can be rewritten as

$$m_t^A = EP^A J'(JP^A J')^{-1} Jz_{1t} + Ez_{2t} = EP^A E'(EP^A E')^{-1} Ez_{1t} + Ez_{2t} = E(z_{1t} + z_{2t}) \tag{B.22}$$

Since we assume that current aggregate shocks affect aggregate observations with their input/output relationship being full rank, it follows that we can re-normalize the observations as $m_t^A = \varpi_t + Sy_t$, so that $E = [I\ \ S]$.

We also introduce the saddle path relationship corresponding to (B.23):

$$\begin{bmatrix} N_a & N_y & I \end{bmatrix} \begin{bmatrix} R & 0 & 0 \\ A_{21} + I_{1a} & A_{22} + I_{1y} & A_{23} \\ A_{31} + H_a R + H_y(A_{21} + I_{1a}) + I_{2a} & A_{32} + H_y(A_{22} + I_{1y}) + I_{2y} & A_{33} + H_y A_{23} \end{bmatrix} = \Lambda \begin{bmatrix} N_a & N_y & I \end{bmatrix} \tag{B.23}$$

where $H = [H_a\ \ H_y], I_1 = [I_{1a}\ \ I_{1y}], I_2 = [I_{2a}\ \ I_{2y}]$, and denote $N = [N_a\ \ N_y]$. From now on, in order to reduce notation, we drop all terms in I_1, I_2; it is straightforward to show that the final representation of the HA solution at the end of the proof takes an identical form to that which includes these terms (and all that is needed to modify it is to use the saddle path relationship from (B.23) above that involves I_1, I_2).

We define the stable matrix A, which represents the dynamics of the saddle path solution in the PI case, as

$$A = \begin{bmatrix} R & 0 \\ A_{21} - A_{23}N_a & A_{22} - A_{23}N_y \end{bmatrix} \tag{B.24}$$

From the perspective of agent i, it follows after assuming that $\mathbb{E}_{i,t}x_{t+1} = -N\mathbb{E}_{i,t}z_{t+1}$ (to be verified later), and substituting for $\mathbb{E}_{i,t}z_{t+1}$ that the system can be written in the

[51] Although they play precisely the same role, in the proof, we need to calculate prediction errors and forcasts of each, meaning that the use of a double ˜ could be confusing.

form below, with $z_t = z_{1t} + z_{2t}$[52]:

$$\begin{bmatrix} \varepsilon_{i,t+1} \\ z_{1,t+1} \\ z_{2,t+1} \\ y_{i,t+1} \\ \mathbb{E}_{i,t}x_{i,t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 & 0 \\ 0 & AP^AE'(EP^AE')^{-1}E & A & 0 & 0 \\ A_{21} & [A_{21}\ \ 0] & [A_{21}\ \ 0] & A_{22} & A_{23} \\ W^{-1}A_{31} & W^{-1}([A_{31}\ \ 0]+Q_1) & W^{-1}([A_{31}\ \ 0]+Q_2) & W^{-1}A_{32} & W^{-1}A_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{i,t} \\ z_{1t} \\ z_{2t} \\ y_{i,t} \\ x_{i,t} \end{bmatrix} + \begin{bmatrix} 0 \\ B \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{t+1} + \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{i,t+1} \quad \text{(B.25)}$$

where

$$Q_1 = (H + (I - W)N)AP^AE'(EP^AE')^{-1}E \qquad Q_2 = (H + (I - W)N)A \qquad \text{(B.26)}$$

and P^A satisfies the Riccati equation

$$P^A = QP^AQ' + BB' \qquad B' = [I\ \ 0] \qquad \text{(B.27)}$$

Since Q now corresponds to $F - FP^AE'(EP^AE')^{-1}E$, an additional constraint on Q is that $QP^AE' = 0$. For the moment, we impose this condition, but later we verify that our representation below of Q satisfies this.

Note that we can now write the measurements of agent i as

$$\begin{bmatrix} m_t^A \\ m_{i,t}^A \end{bmatrix} = \begin{bmatrix} 0 & E & E & 0 & 0 \\ I & \begin{bmatrix} I & 0 \end{bmatrix} & \begin{bmatrix} I & 0 \end{bmatrix} & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{i,t} \\ z_{1t} \\ z_{2t} \\ y_{i,t} \\ x_{i,t} \end{bmatrix} \qquad \text{(B.28)}$$

The saddlepath relationship for this system corresponds to the row eigenvectors $[N_{\varepsilon_i}\ N_1\ \ N_2\ \ N_{y_i}\ \ I]$ of the unstable eigenvalues of the square matrix in (B.25). After some effort we can show that

$$N_1P^AE' = N_2P^AE' \qquad N_2 = [N_a\ \ N_y - N_{y_i}] \qquad \text{(B.29)}$$

[52]The states $z_{i,t},\ z_{2t}$ play the same role as $z_{t,t-1},\ \tilde{z}_t$ in the earlier II solution, because in this limiting case the observation m_{it} provides no information about z_t.

where N_{y_i} is obtained from two of these eigenvector equations:

$$N_{y_i}A_{22} + W^{-1}A_{32} = \Theta N_{y_i} \qquad N_{y_i}A_{23} + W^{-1}A_{33} = \Theta \tag{B.30}$$

where Θ is a square matrix whose eigenvalues are the unstable ones of the saddlepath. Thus for the HA problem, there are two saddle path conditions required for existence of a solution: the standard one, and this additional one. These precisely mirror the two Rondina and Walker (2021) saddle path conditions. In addition, we have

$$\Theta N_{\varepsilon_i} = N_{y_i}A_{21} + W^{-1}A_{31} \tag{B.31}$$

Since equations (B.29) to (B.31) are independent of the filtering problem, the solution for N that results must, as stated in Theorem 2, be identical to the solution under PI, i.e., as in the PI-HA case.

B.3.1 Solving for $y_{i,t}$

(B.25) and (B.28) are now in the form of (10) and (11), so we can now invoke the II results (B.69)–(18) in order to solve for $y_{i,t}$.

Since $m^A_{i,t} = \varepsilon_{i,t} + [I \ \ 0](z_{1t} + z_{2t})$ is observed by agent i, and by assumption $y_{i,t}$ and $x_{i,t}$ are known to agent i, it follows that there is no prediction error in any of these. In addition, since we are in the limiting case $\Sigma \to \infty$, which means that $m^A_{i,t}$ can provide no information on z_{1t} or z_{2t}, it is easy to show that the Riccati matrix for the agent's information problem is given by $\bar{P^A} = lim_{\Sigma\to\infty} diag(\Sigma, P^A, 0, 0)$, and therefore the F-matrix for this problem is given by

$$\bar{F} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & AP^AE'(EP^AE')^{-1}E & A & 0 \\ X & [X \ \ 0] - A_{23}A_{33}^{-1}Q_1 & [X \ \ 0] - A_{23}A_{33}^{-1}Q_2 & A_{22} - A_{23}A_{33}^{-1}A_{32} \end{bmatrix} \tag{B.32}$$

where $X = A_{21} - A_{23}A_{33}^{-1}A_{31}$. Similarly we obtain

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & AP^AE'(EP^AE')^{-1}E & A & 0 \\ A_{21} - A_{23}N_{\varepsilon_i} & [A_{21} \quad 0] - A_{23}N_1 & [A_{21} \quad 0] - A_{23}N_2 & A_{22} - A_{23}N_{y_i} \end{bmatrix} \tag{B.33}$$

Defining $\bar{E} = \begin{bmatrix} 0 & E & E & 0 \\ I & \begin{bmatrix} I & 0 \end{bmatrix} & \begin{bmatrix} I & 0 \end{bmatrix} & 0 \end{bmatrix}$, we can show after some effort that as $\Sigma \to$

∞

$$\bar{P^A}\bar{E}'(\bar{E}\bar{P^A}\bar{E}')^{-1}\bar{E} = \begin{bmatrix} I & [I\ \ 0]\,(I - P^AE'(EP^AE')^{-1}E) & [I\ \ 0]\,(I - P^AE'(EP^AE')^{-1}E) & 0 \\ 0 & P^AE'(EP^AE')^{-1}E & P^AE'(EP^AE')^{-1}E & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{B.34}$$

Hence

$$\bar{F}\bar{P^A}\bar{E}'(\bar{E}\bar{P^A}\bar{E}')^{-1}\bar{E} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & AP^AE'(EP^AE')^{-1}E & AP^AE'(EP^AE')^{-1}E & 0 \\ X & Z & Z & 0 \end{bmatrix} \tag{B.35}$$

where $Z = [X\ \ 0] - A_{23}A_{33}^{-1}HAP^AE'(EP^AE')^{-1}E$. Once one has calculated $\bar{F} - \bar{F}\bar{P^A}\bar{E}'(\bar{E}\bar{P^A}\bar{E}')^{-1}\bar{E}$, it becomes clear that $\tilde{z}_{1t} = z_{1t}, \tilde{z}_{2t} = 0, \tilde{y}_{i,t} = 0$ (where, in general, $\tilde{w}_t$ denotes $w_t - w_{t,t-1}$).

$$\bar{A}\bar{P^A}\bar{E}'(\bar{E}\bar{P^A}\bar{E}')^{-1}\bar{E} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & AP^AE'(EP^AE')^{-1}E & AP^AE'(EP^AE')^{-1}E & 0 \\ A_{21} - A_{23}N_{\varepsilon_i} & Y & Y & 0 \end{bmatrix} \tag{B.36}$$

where $Y = [A_{21}\ \ 0] - [A_{23}N_{\varepsilon_i}\ \ 0]\,(I - P^AE'(EP^AE')^{-1}E) - A_{23}N_2P^AE'(EP^AE')^{-1}E$.[53] Following on from our remark about $\tilde{z}_{1t}, \tilde{z}_{2t}, \tilde{y}_{i,t}$, it is clear that $z_{1t,t-1} = 0, z_{2t,t-1} = z_{2t}, y_{it,t-1} = y_{i,t}$.

Noting that $\int \varepsilon_{i,t}di = 0$, we can therefore summarize the system as follows:

$$\begin{aligned} z_{1,t+1} &= Qz_{1t} + B\varepsilon_{t+1} \\ z_{2t+1} &= AP^AE'(EP^AE')^{-1}Ez_{1t} + Az_{2t} \\ y_{i,t+1} &= Yz_{1t} + ([A_{21}\quad 0] - A_{23}N_2)z_{2t} + (A_{22} - A_{23}N_{y_i})y_{i,t} + (A_{21} - A_{23}N_{\varepsilon_i})\varepsilon_{i,t} \\ y_{t+1} &= Yz_{1t} + ([A_{21}\quad 0] - A_{23}N_2)z_{2t} + (A_{22} - A_{23}N_{y_i})y_t \end{aligned}$$

B.3.2 Requirements for Initial Conjecture to be Valid

The objective now is to show that we can pick a representation of the matrix Q such that

$$\bar{y}_t = [0\quad I](\tilde{z}_t + \bar{z}_t) \tag{B.37}$$

We first note that Q cannot be a full rank matrix since $QP^AE' = 0$. Since the number of measurements is equal to the number of shocks by assumption, it follows that the

[53] We have utilized the earlier result that $N_1P^AE' = N_2P^AE'$.

maximum rank of Q is the number of states in z_{1t} other than the shocks, n_{zm}. We can therefore write

$$Q = UV' = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} [V_1' \;\; V_2'] \tag{B.38}$$

where the number of columns of U and number of rows of V are equal to n_{zm}. We now address whether $[0 \quad I](z_{1,t+1} + z_{2,t+1}) - y_{t+1}$ is solely dependent on its previous value, and if this is the case, it follows that $[0 \quad I](z_{1t} + z_{2t}) = y_t$ in equilibrium.

$$\begin{aligned}
[0 \quad I](z_{1,t+1} + z_{2,t+1}) \;\; &- \;\; y_{t+1} \\
&= \;\; U_2[V_1' \;\; V_2']z_{1t} + [0 \;\; I]AP^A E'(EP^A E')^{-1}Ez_{1t} + [0 \;\; I]Az_{2t} \\
&- \;\; ([A_{21} \;\; 0] - [A_{23}N_{\varepsilon_i} \;\; 0]\,(I - P^A E'(EP^A E')^{-1}E) - A_{23}N_2 P^A E'(EP^A E')^{-1}E)z \\
&- \;\; (A_{22} - A_{23}N_{y_i})y_t - ([A_{21} \quad 0] - A_{23}N_2)z_{2t} \\
&= \;\; U_2[V_1' \;\; V_2']\tilde{z}_t + [A_{21} - A_{23}N_a \;\; A_{22} - A_{23}N_y](P^A E'(EP^A E')^{-1}Ez_{1t} + z_{2t}) \\
&- \;\; ([A_{21} \;\; 0] - [A_{23}N_{\varepsilon_i} \;\; 0]\,(I - P^A E'(EP^A E')^{-1}E))z_{1t} \\
&+ \;\; [A_{23}N_a \;\; A_{23}(N_y - N_{y_i})]P^A E'(EP^A E')^{-1}Ez_{1t} \\
&- \;\; [A_{21} \;\; 0]z_{2t} - (A_{22} - A_{23}N_{y_i})y_t + [A_{23}N_a \;\; A_{23}(N_y - N_{y_i})]z_{2t} \\
&+ \;\; [A_{21} - A_{23}N_a \;\; A_{22} - A_{23}N_y]z_{2t} \\
&= \;\; (A_{22} - A_{23}N_{y_i})([0 \quad I](z_{1t} + z_{2t}) - y_t) \\
&+ \;\; U_2[V_1' \;\; V_2']z_{1t} - [A_{21} - A_{23}N_{\varepsilon_i} \;\; A_{22} - A_{23}N_{y_i}](I - P^A E'(EP^A E')^{-1}E)z_{1t}
\end{aligned} \tag{B.39}$$

Since $(A_{22} - A_{23}N_{y_i})$ is a stable matrix by assumption, the theorem is proven if $U_2[V_1' \;\; V_2'] = [A_{21} + A_{23}N_{\varepsilon_i} \;\; A_{22} - A_{23}N_{y_i}](I - P^A E'(EP^A E')^{-1}E)$.

If SA-II is invertible, it is straightforward to show that this is equivalent to $A - AP^A E'(EP^A E')^{-1}E$ being a stable matrix where $P^A = diag(I, 0)$. In addition, this solution must correspond to $UV' = 0$, which implies from (B25) that $z_{1t} = \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix}$. In addition, with $P^A = diag(I, 0)$, it follows that

$$\begin{aligned}
& U_2[V_1' \;\; V_2']z_{1t} - [A_{21} - A_{23}N_{\varepsilon_i} \;\; A_{22} - A_{23}N_{y_i}](I - P^A E'(EP^A E')^{-1}E)z_{1t} \\
= \;\; & 0 - [A_{21} - A_{23}N_{\varepsilon_i} \;\; A_{22} - A_{23}N_{y_i}] \begin{bmatrix} 0 & -S \\ 0 & I \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix} = 0
\end{aligned}$$

implying that this is indeed the SA-PI solution.

B.3.3 Expressions for $Q, U_2[V_1' \;\; V_2']$ and $[0 \;\; A_{22} - A_{23}N_{y_i}](I - P^A E'(EP^A E')^{-1}E$

To find a representation of Q, we exploit the following:

1. The Riccati matrix P^A also satisfies the Lyapunov equation $P^A = QP^AQ' + BB'$;
2. $QP^AE' = 0$ (as we noted earlier);
3. Denoting the first subvector of each of the states z_{1t}, z_{2t} by a_{1t}, a_{2t}, so that $a_{1t}+a_{2t} = a_t$, represents the shocks, it must follow that $a_{1,t+1} + a_{2,t+1}$ must exactly equal $R(a_{1t} + a_{2t}) + \varepsilon_{t+1}$;
4. Under PI, we assumed that the system from the perspective of the econometrician is invertible. Given that the first subvectors above represent the shocks, it follows that the first m columns E_1 of the observation matrix $E = [E_1\ E_2]$ must be full rank. Since any linear combination of the observables will produce the same solution of the Riccati matrix, for ease of exposition we normalize to $E = [I\ S]$ where $S = E_1^{-1}E_2$.

An additional useful constraint on this setup is to note that the eigenvalues of Q are given by a number of zeros equal to the number of shocks, and in addition the eigenvalues of the matrix $\Lambda = V'U = V_1'U_1 + V_2'U_2$, and it is this matrix that will give us all the values U_1, U_2, V_1', V_2'. These eigenvalues will be associated with Blaschke factors that are at the heart of the solution.

We first note that the Riccati matrix satisfies $P^A = UV'P^AVU' + BB'$; multiplying this through by V' on the left and V on the right yields $V'P^AV = V'U(V'P^AV)U'V + V'BB'V$, so defining $Z = V'P^AV$ and recalling that $B' = [I\ \ 0]$, we can write this as

$$Z = \Lambda Z\Lambda' + V_1'V_1 \tag{B.40}$$

This eventually leads to the following:

$$\begin{aligned} U_1 &= RS && \text{(B.41)}\\ Z &= \Lambda Z\Lambda' + (\Lambda ZS'R' - ZS')(RSZ\Lambda' - SZ) && \text{(B.42)}\\ U_2 Z\Lambda' &= -Z && \text{(B.43)}\\ V_1' &= -\Lambda ZS'R' + ZS' && \text{(B.44)}\\ V_2' &= V_1'S - I && \text{(B.45)} \end{aligned}$$

In addition, we note that the Riccati matrix P^A is given by

$$P^A = UV'P^AVU' + BB' = UZU' + BB' = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} Z[U_1'\ \ U_2'] + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{B.46}$$

Hence, as required

$$QP^AE' = UV'P^AE' = U(\Lambda Z(U_1' + U_2'S') + V_1') = U(\Lambda ZS'R' - ZS' + V_1') = 0 \tag{B.47}$$

A crucial point to note is that a potential solution of (B.42) is $Z = 0$ and $P^A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; this is relevant for the PMIC for agents, and at the final stage of the proof.

We can see from the above that the matrices U_2, V_1', V_2' all depend on Z, with the latter dependent on the choice of matrix Λ. From this, it is easy to show that

$$EP^AE' = I + V_1\Lambda^{-T}Z^{-1}\Lambda^{-1}V_1' \tag{B.48}$$

and using (B.40), we can further show that

$$(EP^AE')^{-1} = I - V_1Z^{-1}V_1' \tag{B.49}$$

Further calculation gives

$$P^AE' = \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} -RS \\ Z\Lambda^{-T}Z^{-1} \end{bmatrix} \Lambda^{-1}V_1' \tag{B.50}$$

and hence

$$I - P^AE'(EP^AE')^{-1}E = \begin{bmatrix} SV_1' & -S(I - V_1'S) \\ -V_1' & I - V_1'S \end{bmatrix} = \begin{bmatrix} S \\ -I \end{bmatrix} [V_1' \quad -I + V_1'S] \tag{B.51}$$

and

$$\begin{aligned} &[A_{21} - A_{23}N_{\varepsilon_i} \quad A_{22} - A_{23}N_{y_i}](I - P^AE'(EP^AE')^{-1}E \\ &= ((A_{21} + A_{23}N_{\varepsilon_i})S - (A_{22} - A_{23}N_{y_i}))\,[V_1' \quad -I + V_1'S] \end{aligned} \tag{B.52}$$

Noting that $U_2[V_1' \ \ V_2'] = U_2[V_1' \quad -I+V_1'S]$, it immediately follows that $[0 \quad I](\tilde{z}_t+\bar{z}_t) = \bar{y}_t$ if

$$U_2 = (A_{21} - A_{23}N_{\varepsilon_i})S - (A_{22} - A_{23}N_{y_i}) \tag{B.53}$$

This analytic expression for U_2 then generates all the elements of Q that provide the equilibrium dynamics of the aggregate variables.

To actually obtain V_1 and V_2, we can rewrite (B.42) by pre and postmultiplying it by Z, and noting that $U_2 = -Z\Lambda^{-T}Z^{-1}$:

$$Z^{-1} = U_2^{-T}Z^{-1}U_2^{-1} + (U_2^{-T}S'R' + S')(RSU_2^{-1} + S) \tag{B.54}$$

This Lyapunov equation for Z^{-1} yields Λ via U_2, and hence V_1 and V_2. Clearly this is not a general solution as it is only valid when U_2^{-1} is a stable matrix, so we need some further steps.

If all the eigenvalues of U_2 are unstable then it is straightforward to check that $UV'$is indeed equal to $F(I - P^A E'(EP^A E')^{-1}E)$, where F is as given in the theorem statement.

B.3.4 PMIC for Agent i

We can now calculate the PMIC defining

$$\hat{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & AP^A E'(EP^A E')^{-1}E & A & 0 \\ A_{21} - A_{23}N_{\varepsilon_i} & Y & [A_{21} \quad 0] - A_{23}N_2 & A_{22} - A_{23}N_{y_i} \end{bmatrix} \tag{B.55}$$

$$\hat{B} = \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} I \\ 0 \end{bmatrix} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \hat{E} = \begin{bmatrix} I & \begin{bmatrix} I & 0 \\ I & S \end{bmatrix} & \begin{bmatrix} I & 0 \\ I & S \end{bmatrix} & 0 \\ 0 & & & 0 \end{bmatrix} \tag{B.56}$$

After some tedious algebra we can show that the PMIC matrix $\hat{A} - \hat{A}\hat{B}(\hat{E}\hat{B})^{-1}\hat{E}$ is block triangular, with diagonal blocks given by

$$Q \begin{bmatrix} 0 & -S \\ 0 & I \end{bmatrix}, \qquad A - AP^A E'(EP^A E')^{-1}E, \qquad A_{22} - A_{23}N_{y_i} \tag{B.57}$$

The last matrix is stable if, as we assumed above, there is saddle path stability for agent i. Consider now the first matrix

$$Q \begin{bmatrix} 0 & -S \\ 0 & I \end{bmatrix} = UV' \begin{bmatrix} 0 & -S \\ 0 & I \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} 0 & -I \end{bmatrix} = \begin{bmatrix} 0 & -U_1 \\ 0 & -U_2 \end{bmatrix} \tag{B.58}$$

So the PMIC for the agent requires $U_2 = (A_{21} - A_{23}N_{\varepsilon_i})S - (A_{22} - A_{23}N_{y_i})$ to be a stable matrix. It is easy to see that this is consistent with $Z = 0$, $P^A = BB'$, meaning that the second matrix of (B.57) is equal to $A - AB(EB)^{-1}E$, which is the earlier form of the PMIC in the SA case.

B.3.5 Solution When U_2 Has Both Stable and Unstable Eigenvalues

An obvious conjecture is that once U_2 is diagonalized into stable and unstable blocks, then the stable block will be associated with a transformation of Z equal to 0, while the unstable block will generate a solution for some transformation of Z similar to (B.54); we show that this is indeed the case.

Assume therefore that we diagonalize U_2 as

$$U_2 = T^{-1} \begin{bmatrix} U_2^1 & 0 \\ 0 & U_2^2 \end{bmatrix} T \tag{B.59}$$

where U_1^2 has all eigenvalues greater than 1 in modulus, and U_2^2 all less than 1. Then (B.43) can be rewritten as

$$-TZT^T = \begin{bmatrix} U_2^1 & 0 \\ 0 & U_2^2 \end{bmatrix} TZT^T T^{-T} \Lambda^T T^T \tag{B.60}$$

Assume that $TZT^T = diag(Z_1, 0)$. For (B.60) to be consistent with this, we require

$$T^{-T} \Lambda^T T^T = \begin{bmatrix} \Lambda_{11}^T & 0 \\ \Lambda_{21}^T & \Lambda_{22}^T \end{bmatrix} \qquad \text{and} \qquad - Z_1 = U_2^1 Z_1 \Lambda_{11}^T \tag{B.61}$$

A simple calculation then shows that (B.42) reduces to

$$Z_1 = \Lambda_{11} Z_1 \Lambda_{11}^T + (\Lambda_{11} Z_1 X_1' - Z_1 Y_1')(X_1 Z_1 \Lambda_{11} - Y_1 Z_1) \tag{B.62}$$

where X_1, Y_1 are defined conformably via $(RST^{-1})^T = [X_1' \;\; X_2']$, $(ST^{-1})^T = [Y_1' \;\; Y_2']$. Multiplying through on both sides by Z_1^{-1}, we obtain an equation analogous to (B.54):

$$Z_1^{-1} = (U_2^1)^{-T} Z_1^{-1} (U_2^1)^{-1} + ((U_2^1)^{-T} X_1^T + Y_1^T)(X_1 (U_2^1)^{-1} + Y_1) \tag{B.63}$$

Finally, compute $Z = T^{-1} diag(Z_1, 0) T^{-T}$, and $P^A = UZU' + BB'$. Note that the linearity of the solution for Z_1 (given T) implies that its solution is unique. Furthermore, although the choice of matrix T is non-unique, it is trivial to demonstrate that the matrices UV' and P^A are independent of which of the T matrices are used to diagonalize U_2, so that they in turn are unique. It follows that in this case the HA-II(∞) solution is the same as SA-PI.

B.3.6 Summary of Proof

To summarize, the solution is obtained as follows:

1. Find the eigenvectors $[N_{\varepsilon_i} \; N_1 \;\; N_2 \;\; N_{y_i} \;\; I]$ of the unstable eigenvalues of the square matrix in (B.25);

2. Compute U_2 as in (B.53);

3. If U_2 is a stable matrix then $P^A = BB'$, and the solution is equivalent to that under PI and to II-SA (PI $\equiv$ II-SA $\equiv$ II-HA(∞) RE solutions) ;

4. Otherwise:

    (a) If all eigenvalues of U_2 are unstable, compute the solution Z^{-1} to the Lyapunov equation (B.54);

    (b) If there are stable and unstable eigenvalues, diagonalize U_2, and generate the solution as in the previous subsection;

5. Compute $P^A = UZU' + BB'$;

6. Compute Q via (B.38), noting that in the case of U_2 having mixed stable and unstable eigenvalues that $V_1 = [(X_1(U_2^1)^{-1} + Y_1)Z_1 \quad 0]T^{-T}$.

We finally note that all of this analysis leads to a saddle path that represents a linear relationship between $x_{i,t}$ and $\varepsilon_{i,t}, z_{1t}, z_{2t}, y_{i,t}$, which in turn implies a linear relationship between x_t and z_{1t}, z_{2t}. But consistency with the saddle path under PI implies that $\mathbb{E}_{i,t}x_{t+1} = -N\mathbb{E}_{i,t}z_{t+1} = -N\mathbb{E}_{i,t}(z_{1,t+1} + z_{2,t+1}) = 0 - Nz_{2,t+1}$, which was our initial assumption.

B.3.7 Connection Between II-SA and II-HA (Limiting Case) Solutions

We know from (B.51) that

$$I - P^A E'(EP^A E')^{-1}E = \begin{bmatrix} S \\ -I \end{bmatrix} [V_1' \;\; V_1'S - I] \tag{B.64}$$

To show that

$$F = \begin{bmatrix} R & 0 \\ A_{21} - A_{23}N_{\varepsilon_i} & A_{22} - A_{23}N_{y_i} \end{bmatrix} \tag{B.65}$$

turns the solution into one that corresponds to SA-II, we note from (B.64) that

$$\begin{aligned} F(I - P^A E'(EP^A E')^{-1}E) &= \begin{bmatrix} R & 0 \\ A_{21} - A_{23}N_{\varepsilon_i} & A_{22} - A_{23}N_{y_i} \end{bmatrix} \begin{bmatrix} S \\ -I \end{bmatrix} [V_1' \;\; V_1'S - I] \\ &= \begin{bmatrix} RS \\ U_2 \end{bmatrix} [V_1' \;\; V_1'S - I] = UV' \end{aligned} \tag{B.66}$$

as in the initial assumption of the theorem.

B.3.8 Application to the GW Model of (1)–(5)

For this example we have $S = \frac{\kappa_1}{\kappa_1+\kappa_2} - 1, A_{21} = \kappa_1 + \kappa_2, A_{22} = \kappa_1, A_{23} = 1 - A_{21}, A_{31} = A_{32} = 0, A_{33} = 1$. It follows that $\Theta = \kappa_1$, $N_{y_i} = \frac{\kappa_1 - 1}{1-\kappa_1-\kappa_2}$, $N_{\varepsilon_i} = \frac{1-\frac{1}{\kappa_1}}{\frac{1}{\kappa_1+\kappa_2}-1}$. Hence

$U_2 = -\frac{\kappa_1+\kappa_2}{\kappa_1}$, and therefore $\Lambda = \frac{\kappa_1}{\kappa_1+\kappa_2}$. The latter leads to the Blaschke factor in (C.114). In addition, recalling that $\kappa_1 = \frac{1}{\beta}$, $\kappa_2 = \frac{1-\alpha}{\alpha\beta}(1-\beta(1-\delta))$. For the Rondina and Walker (2021) case

B.3.9 Proof with Observations dependent on Aggregate Jump Variables

We now assume that measurements are given by $m_t^A = M_1 z_t + M_2 x_t$, which leads to the conjecture that $m_t^A = EP^A J'(JP^A J')^{-1} J z_{1t} + E z_{2t}$, where

$$J = M_1 - M_2 A_{33}^{-1} A_{32} \qquad E = M_1 - M_2 N \tag{B.67}$$

The dynamic equations for z_{1t}, z_{2t} are given as in (B.25), but with E replaced by J, from which it follows that in lag operator form we can derive the expression

$$m_t^A = E(I-AL)^{-1} P^A J'(JP^A J')^{-1} J(I-QL)^{-1} B\varepsilon_t \tag{B.68}$$

The full rank input/output requirement for contemporaneous shocks then implies that JB is of full rank; since JB is square and $B' = [I \;\; 0]$, it follows that if we write $J = [J_1 \;\; J_2]$ conformably with B' then J_1 is full rank. After some effort we can then show that the only change to the proof above is in the definition of the matrix S, which is now given as $S = J_1^{-1} J_2$ with the expression for Y in (B.36) having E replaced by J throughout.

B.3.10 Solution of the Illustrative Model in the Non-Limiting Case

Assume that the solution for the backward looking **aggregate** variables $z_t = z_{1t} + z_{2t}$ takes the form

$$\begin{aligned} \text{Predictions :} \quad z_{2,t+1} &= Az_{2t} + APJ'(JPJ')^{-1}Jz_{1t} && \text{(B.69)} \\ \text{Prediction Errors :} \quad z_{1,t+1} &= F[I - PJ'(JPJ')^{-1}J]z_{1t} + B\varepsilon_{t+1} && \text{(B.70)} \end{aligned}$$

The matrix A is the dynamic system matrix for the perfect information solution, while the matrix F is a matrix for which we have to find a fixed point solution. The measurements are given by

$$m_t = Jz_t = J(z_{1t} + z_{2t}) \qquad m_{it} = a_t + v_{it} \qquad v_{it} \sim N(0, V) \tag{B.71}$$

Each agent i takes the aggregate variables z_{1t}, z_{2t} as given, and any expectations of these will take into account the measurements m_t, m_{it}. Note that because z_{2t} directly depends on Jz_t, there will be no Kalman filtering necessary for this part of the aggregate variable. Kalman filtering is only used for z_{1t}, and this is the advantage of the PCL formulation of the II solution.

In order to ensure that the structure of the shock processes is always preserved, the matrices, the vectors z_{1t}, z_{2t} are ordered so that the shock processes are first, so that F, A is the lower block traingular. In particular, for the R&W model we have $F = \begin{bmatrix} \rho & 0 \\ \mu & \phi \end{bmatrix}$ where ϕ, μ are to be determined.

For the special case $\rho = 0$, we need to solve the Riccati equation

$$P = FPF' - FPJ'(JPJ')^{-1}JPF' + BB' \qquad \varepsilon_t \sim N(0,1) \tag{B.72}$$

where

$$F = \begin{bmatrix} 0 & 0 \\ \mu & \phi \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad J = [1 \quad \alpha - 1] \tag{B.73}$$

It is easy to verify that $P = diag(1, p)$, where $p = \frac{(\phi-(1-\alpha)\mu)^2-1}{(\alpha-1)^2}$. It then follows that

$$F[I - PJ'(JPJ')^{-1}J] = \begin{bmatrix} 0 & 0 \\ -\frac{(\phi-(1-\alpha)\mu)^2-1}{(1-\alpha)(\phi-(1-\alpha)\mu)} & \frac{1}{(\phi-(1-\alpha)\mu)} \end{bmatrix} \tag{B.74}$$

C;early this matrix does not not depend on ϕ and μ individually, but only on $(\phi-(1-\alpha)\mu)$, which we define as λ. Hence

$$P = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1/\lambda^2-1}{(\alpha-1)^2} \end{bmatrix} \qquad F[I - PJ'(JPJ')^{-1}J] = \begin{bmatrix} 0 & 0 \\ \frac{\lambda-1/\lambda}{\alpha-1} & \lambda \end{bmatrix} \tag{B.75}$$

In addition, define the unstable eigenvalue of the system as $1/\zeta$ and therefore the stable eigenvalue of the system is ζ/β, which implies that we can write $\eta = \frac{(\zeta-1)(1-\zeta/\beta)}{\zeta(1-\alpha)(1-\frac{1}{\alpha\beta})}$. It follows that

$$A = \begin{bmatrix} 0 & 0 \\ \frac{\zeta}{\alpha\beta} & \frac{\zeta}{\beta} \end{bmatrix} \qquad APJ'(JPJ')^{-1}J = \begin{bmatrix} 0 & 0 \\ \frac{\zeta(\alpha-\lambda^2)}{\alpha\beta(\alpha-1)} & \frac{\zeta(\alpha-\lambda^2)}{\alpha\beta} \end{bmatrix} \tag{B.76}$$

Because of the position of the 0s in these last two matrices, it follows that the third element of the system vector is 0 for all time. We can therefore write the system as

$$\begin{aligned} \varepsilon_{t+1} &= \varepsilon_{t+1} && \text{(B.77)} \\ k_{1,t+1} &= \lambda k_{1t} + \frac{1}{\alpha-1}(\lambda - \frac{1}{\lambda})(1-\alpha)\varepsilon_t && \text{(B.78)} \\ k_{2,t+1} &= \frac{\zeta}{\beta}k_{2t} + \frac{\zeta(\alpha-\lambda^2)}{\alpha\beta}k_{1t} + \frac{\zeta(\alpha-\lambda^2)}{\alpha\beta(\alpha-1)}(1-\alpha)\varepsilon_t && \text{(B.79)} \\ v_t &= m_t = (1-\alpha)(\varepsilon_t - (k_{1t} + k_{2t})) && \text{(B.80)} \end{aligned}$$

This implies that

$$v_t = \frac{(1-\frac{1}{\lambda}L)(1-\frac{\lambda^2\zeta}{\alpha\beta}L)}{(1-\frac{\zeta}{\beta}L)(1-\lambda L)}(1-\alpha)\varepsilon_t \tag{B.81}$$

Agent i needs to calculate $\mathbb{E}_{it}r_{t+1} = \mathbb{E}_{it}[(1-\alpha)k_{t+1}+\varepsilon_{t+1}] = \mathbb{E}_{it}[(1-\alpha)(k_{1,t+1}+k_{2,t+1})]$, using observations of r_t and $\varepsilon_t + v_{it}$. It is easy to verify that the Riccati matrix P_1 for this problem is given by $P_1 = diag(1, p_1, 0)$ where

$$p_1 = \frac{v(\frac{1}{\lambda^2}-1)}{(1+v)(1-\alpha)^2} \tag{B.82}$$

Defining $\hat{k}_{1,t+1}$ as the best prediction by agent i of $k_{1,t+1}$, and $\tilde{k}_{1,t+1}$ as its prediction error, we can now rewrite the system as

$$\begin{aligned}
v_{i,t+1} &= v_{i,t+1} && \text{(B.83)}\\
\varepsilon_{t+1} &= \varepsilon_{t+1} && \text{(B.84)}\\
\tilde{k}_{1,t+1} &= \lambda\tilde{k}_{1t} + \frac{v}{(1+v)(\alpha-1)}(\lambda-\frac{1}{\lambda})(1-\alpha)\varepsilon_t - \frac{1}{(1+v)(\alpha-1)}(\lambda-\frac{1}{\lambda})v_{it} && \text{(B.85)}\\
\hat{k}_{1,t+1} &= \lambda\hat{k}_{1t} + \frac{1}{(1+v)(\alpha-1)}(\lambda-\frac{1}{\lambda})(1-\alpha)\varepsilon_t + \frac{1}{(1+v)(\alpha-1)}(\lambda-\frac{1}{\lambda})v_{it} && \text{(B.86)}\\
k_{2,t+1} &= \frac{\zeta}{\beta}k_{2t} + \frac{\zeta(\alpha-\lambda^2)}{\alpha\beta}(\tilde{k}_{1t}+\hat{k}_{1t}) + \frac{\zeta(\alpha-\lambda^2)}{\alpha\beta(\alpha-1)}(1-\alpha)\varepsilon_t && \text{(B.87)}\\
v_t &= m_t = (1-\alpha)(\varepsilon_t - (\tilde{k}_{1t}+\hat{k}_{1t}+k_{2t})) && \text{(B.88)}
\end{aligned}$$

Note that we should strictly denote that both $\tilde{k}_{1t}$ and $\hat{k}_{1t}$ are particular to agent i, but have dropped this for convenience. However we will need average values for each of these $\bar{\tilde{k}}_{1t}$ and $\bar{\hat{k}}_{1t}$ across all agents. Scrutiny of the relevant equations shows that $\bar{\tilde{k}}_{1t} = v\bar{\hat{k}}_{1t}$.

We now have to solve the system for agent i who is making decisions as in (D.2)-(D.3),which we rewrite as :

$$\begin{aligned}
k_{i,t+1} &= \frac{1}{\beta}k_{it} + (1-\frac{1}{\alpha\beta})c_{it} + \frac{1}{\alpha\beta}((1-\alpha)\varepsilon_t + v_{it}) && \text{(B.89)}\\
\mathbb{E}_{it}c_{i,t+1} &= c_{it} - \frac{(\zeta-1)(1-\zeta/\beta)}{\zeta(1-\frac{1}{\alpha\beta})}[\hat{k}_{1,t+1}+k_{2,t+1}] && \text{(B.90)}\\
&= c_{it} - \frac{(\zeta-1)(1-\zeta/\beta)}{\zeta(1-\frac{1}{\alpha\beta})}[(\lambda+\frac{\zeta}{\beta}-\frac{\lambda^2\zeta}{\alpha\beta})\hat{k}_{1t} + \frac{\zeta}{\beta}k_{2t} + (\frac{\zeta}{\beta}-\frac{\lambda^2\zeta}{\alpha\beta})\tilde{k}_{1t}\\
&\quad + \frac{\lambda-\frac{1}{\lambda}}{(\alpha-1)(1+v)}v_{it} + (\frac{\lambda-\frac{1}{\lambda}}{(\alpha-1)(1+v)} + \frac{\frac{\zeta}{\beta}-\frac{\lambda^2\zeta}{\alpha\beta}}{\alpha-1})(1-\alpha)\varepsilon_t] && \text{(B.91)}
\end{aligned}$$

It is straightforward to see that (i) the unstable eigenvalue of the system is $\frac{1}{\beta}$, (ii) the best predictor of $k_{i,t+1}$ is $k_{i,t+1}$ itself. This is because both $(\varepsilon_t + v_{it})$ and $[\hat{k}_{1,t+1}+k_{2,t+1}]$ are observable by agent i. Now apply the results of PCL to obtain k_{it}; for space reasons, we just write down the average value of this, denoted $\bar{k}_{it}$, by merely omitting the response to v_{it}:

$$\begin{aligned}\bar{k}_{i,t+1} &= \bar{k}_{it} + (\frac{\zeta}{\beta} - 1)k_{2t} - \frac{\beta-\zeta}{1-\beta\lambda}(\frac{\alpha-\lambda^2}{\alpha\beta} + \frac{\lambda(1-\zeta}{\zeta})\hat{k}_{1t} - \frac{(\alpha-\lambda^2)(\beta-\zeta)}{\alpha\beta}\tilde{k}_{1t} \\ &+ \left(\frac{1}{\alpha} - \frac{(\alpha-\lambda^2)(\beta-\zeta)}{\alpha\beta(\alpha-1)} - \frac{(\lambda-\frac{1}{\lambda})(\beta-\zeta)(1-\frac{\lambda^2\zeta}{\alpha})}{(\alpha-1)(1-\beta\lambda)(1+v)\zeta}\right)(1-\alpha)\varepsilon_t \qquad \text{(B.92)}\end{aligned}$$

We now need to ascertain that $\bar{k}_{i,t+1} - k_{2,t+1} - \bar{\hat{k}}_{1,t+1} - \bar{\tilde{k}}_{1,t+1} = 0$. After a great deal of algebra, the condition for this is

$$(1+v)(\lambda - \alpha) = \frac{(\beta-\zeta)(\alpha-\zeta\lambda^2)}{\zeta(1-\beta\lambda)} \qquad \text{(B.93)}$$

For the limiting case, as $v \to \infty$, we have $\lambda = \alpha$. For other values of v, we can rearrange this as

$$v(\lambda-\alpha)(1-\beta\lambda) = (1-\lambda\zeta)(\lambda - \alpha\beta/\zeta) \qquad \text{(B.94)}$$

and it is fairly straightforward to show that a solution for $0 < \lambda < 1$ always exists.

B.4 Proof of Theorem 3

Proof. Using the expressions (32)–(31) for II, and the invertibility requirement that $\tilde{A} - \tilde{A}\tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}$ has stable eigenvalues, we calculate the latter as the matrix

$$\begin{bmatrix} A - AP^AJ'(EP^AJ')^{-1}E & 0 \\ -F(I - P^AJ'(JP^AJ')^{-1}J)(JB)^{-1}JP^AJ'(EP^AJ')^{-1}E & F(I - B(JB)^{-1}J) \end{bmatrix} \qquad \text{(B.95)}$$

If $F(I - B(JB)^{-1}J)$ has eigenvalues outside the unit circle, it immediately follows that II is not E-invertible. If its the eigenvalues are inside the unit circle, it follows that the solution to (23) is $P^A = BB'$; this is because the Convergence Condition for P^A is that $F - FP^AJ'(JP^AJ')^{-1}J = F(I - B(JB)^{-1}J)$ is a stable matrix. Furthermore it follows that $A - AP^AJ'(EP^AJ')^{-1}E = A(I - B(EB)^{-1}E)$, so that (B.95) is a stable matrix as required for invertibility.

To show that invertibility implies that II and PI are equivalent, we note that (18) now implies that $\tilde{z}_t = B\varepsilon_t + (F(I - B(JB)^{-1}J))^t\tilde{z}_0$, which in dynamic equilibrium implies $\tilde{z}_t = B\varepsilon_t$. This implies that $z_{t+1,t} = Az_{t,t-1} + AB\varepsilon_t$, and hence that $z_{t+1} = \tilde{z}_{t+1} + z_{t+1,t} = Az_{t,t-1} + AB\varepsilon_t + B\varepsilon_{t+1} = Az_t + B\varepsilon_{t+1}$ as in the PI case. In addition, from (20), $m_t^A = Ez_{t,t-1} + E\tilde{z}_t = Ez_t$, also as in the PI case. If $F(I - B(JB)^{-1}J)$ is not a stable matrix, then $P^A \neq BB'$, and the overall dynamics of (B.69)-(20) are of a higher dimension than under PI.

Finally, for the HA(∞) case, we have $JB = EB = I$, so clearly invertible, and given the representation of F the proof of Theorem 2 it follows that

$$F - FB(EB)^{-1}E = \begin{bmatrix} 0 & -RS \\ 0 & (A_{21} - A_{23}N_{\varepsilon_i})S - (A_{22} - A_{23}N_{y_i}) \end{bmatrix} \tag{B.96}$$

This is evidently a stable matrix provided that the A-invertibility condition holds for HA(∞) □

B.5 Proof of Corollary 3.1

Proof. Writing (20) in terms of lagged state variables and shocks yields a coefficient matrix on the latter given by $EP^A J'(JP^A J')^{-1}JB$, and the rank of this is $\leq rank(JB) \leq rank(J)$. This immediately implies that the system is E-non-invertible. □

B.6 Proof of Theorem 4

In order to show the existence of Blaschke factors, we need to show that a subset of the eigenvalues of the matrix for the PMIC condition are the inverses of the eigenvalues of those of the system dynamics.

Recall from the proof of Theorem 2 that the matrices defining the aggregate system under II-HA are given by

$$\hat{E} = [E \;\; E] \qquad \hat{A} = \begin{bmatrix} UV' & 0 \\ APE'(EPE')^{-1}E & A \end{bmatrix} \qquad \hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{B.97}$$

We first note that the eigenvalues of $\hat{A}$ are precisely 0, those of $V'U = \Lambda$ and those of A. In addition, inverting the system to describe shocks in terms of observables yields the PMIC matrix

$$\begin{aligned} \hat{A} - \hat{A}\hat{B}(\hat{E}\hat{B})^{-1}\hat{E} &= \begin{bmatrix} UV' & 0 \\ APE'(EPE')^{-1}E & A \end{bmatrix} - \begin{bmatrix} UV'B \\ APE'(EPE')^{-1}EB \end{bmatrix} (EB)^{-1}[E \;\; E] \\ &= \begin{bmatrix} UV' - UV'B(EB)^{-1}E & -UV'B(EB)^{-1}E \\ 0 & A - APE'(EPE')^{-1}E \end{bmatrix} \end{aligned} \tag{B.98}$$

Again, from the proof of Theorem 2, we have $E = [I \;\; S]$, $B' = [I \;\; 0]$ (so $EB = I$) and $V_2' - V_1'S = -I$. It therefore follows that

$$UV' - UV'B(EB)^{-1}E = \begin{bmatrix} 0 & -U_1 \\ 0 & -U_2 \end{bmatrix} \tag{B.99}$$

Finally, from the proof of Theorem 2, we also know that $-U_2 = Z\Lambda^{-T}Z^{-1}$, which means that the eigenvalues of $-U_2$ are the inverses of those of Λ. Hence the aggregate system has Blaschke factors.

B.7 Proof of Theorem 5

Proof. We first solve the steady state Riccati equation (B.27) corresponding to the matrices (32)–(34). It is easy to verify that $\tilde{P^E} = diag(M, P^A)$ where $M = Z - P^A J'(JP^A J')^{-1} JP^A$ and Z satisfies

$$Z = AZA' - AZE'(EZE')^{-1}EZA' + P^A J'(JP^A J')^{-1} JP^A \tag{B.100}$$

For the innovations representation, we use the notation $s_t = [s'_{1t}\ s'_{2t}]'$, rather than $s_t = [z'_{t,t-1}\ \tilde{z}'_t]'$ as the notation for one-step ahead predictors of the latter will lead to confusion. We can then show that the steady state innovations representation corresponding to (A.15) is given by

$$\mathbb{E}_t s_{t+1} = \begin{bmatrix} A & AP^A J'(JP^A J')^{-1} J \\ 0 & F - FP^A J'(JP^A J')^{-1} J \end{bmatrix} \mathbb{E}_{t-1} s_t + \begin{bmatrix} AZE'(EZE')^{-1} \\ 0 \end{bmatrix} e_t \qquad e_t = m_t^E - \tilde{E}\mathbb{E}_{t-1} s_t \tag{B.101}$$

or more succinctly

$$\mathbb{E}_t s_{1,t+1} = A\mathbb{E}_{t-1} s_{1,t} + AZE'(EZE')^{-1} e_t \qquad e_t = m_t^E - E\mathbb{E}_{t-1} s_{1t} \tag{B.102}$$

The corresponding VARMA representation arises from defining $\xi_t = \mathbb{E}_{t-1} s_{1t} + ZE'(EZE')^{-1} e_t$ which yields

$$\xi_{t+1} = A\xi_t + ZE'(EZE')^{-1} e_{t+1} \qquad m_t^E = E\xi_t \qquad e_t \sim N(0, EZE') \tag{B.103}$$

The final step follows from comparing (B.103) with (B.69)–(20); clearly the dynamics of the RE saddle-path solution explained by the innovations process e_t are of smaller dimension that the dynamics yielding the impulse responses. □

B.8 Proof of Corollary 5.1

Proof. From the proof of Theorem 2 we have seen that the MA roots of the VARMA process include the eigenvalues of $F(I - B(JB)^{-1}J)$, while from (B.69)–(20), the AR roots include the eigenvalues of $F(I - P^A J'(JP^A J')^{-1} J)$. By Corollary A.4, it follows that one or more of these are reciprocals of one another. Hence the transfer function from shocks to observables incorporates at least one Blaschke factor. It follows that IRFs of structural shocks from the latter cannot be linear combinations of IRFs from VAR residuals, which will only mimic the IRFs from the innovations process. □

B.9 Proof of Corollary 5.2

Proof. The state-space equations describing the system, (B.69)–(20), will be unchanged, as these depend on the measurements made by the agents. However, if the information set of the econometrician is a subset of that of the agents, this means that in the notation of (9), we have $L^E = WL^A$ for some matrix W. It then follows that the measurement equation of the econometrician, following from (20), is given by $m_t = W(Ez_{t,t-1} + EPD'(DPD')^{-1}D\tilde{z}_t)$. Thus the innovations process and the VARMA as shown in the proof of Theorem 4 are changed merely by replacing E by WE, with the Riccati matrix Z also obtained with the same replacement of E. □

B.10 Proof of Theorem 7

Proof. Both of these results follow from finding the best fit of a linear combination of structural shocks and residuals, which can be expressed as

$$min_{a,b}\mathbb{E}(a'\varepsilon - b'e)^2 \;\; s.t. \;\; a'a = 1 \tag{B.104}$$

Given a, one obtains b via standard OLS techniques, and the problem reduces to minimizing $a'\mathbb{F}^{PI}a$ s.t. $a'a = 1$, with solution a equal to the eigenvector of the minimum eigenvalue of $\mathbb{F}^{PI}$. □

C More on the Illustrative Analytical Example

In this section, we provide more detail on the illustrative model first discussed in the Introduction. In Appendix E.4, we show that the illustrative model is a special case of the full RBC model considered in Section 6 below.

We first show the derivations of each of the reduced form representations of the single observable, the rental rate of capital, and then provide more detail on the responses of the economy to a aggregate technology shock. For this model, given that there is only a single shock, we are also able to compare the solution for the limiting case of extreme heterogeneity with intermediate cases, using a solution technique that matches the solution of Rondina & Walker (2021). We show that the limiting case is both quantitatively similar to intermediate cases for empirically plausible degrees of heterogeneity, but also provides qualitative insights for a much wider range of values, even for cases close to heterogeneity.

C.1 The Single Agent Framework

We can derive a single agent version of the model the model as set out in equations (1) to (5) in the Introduction by setting idiosyncratic shocks to zero. The system then has a state-space form

$$\begin{bmatrix} \varepsilon_{a,t+1} \\ k_{t+1} \\ c_{t+1,t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \kappa_2 & \kappa_1 & 1-\kappa_1-\kappa_2 \\ 0 & 0 & 1+\kappa_4(\kappa_1+\kappa_2-1) \end{bmatrix} \begin{bmatrix} \varepsilon_{a,t} \\ k_t \\ c_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\kappa_4\kappa_2 & -\kappa_4\kappa_1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{a,t,t} \\ k_{t,t} \\ c_{t,t} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{a,t+1} \tag{C.105}$$

For the II-SA case, we simply assume (without justification) the censored information set which is the history of the single observable, the rental rate on capital

$$m_t^A = m_t^E = [1 \quad -1 \quad 0] \begin{bmatrix} \varepsilon_{a,t} \\ k_t \\ c_t \end{bmatrix} \qquad \varepsilon_{a,t} \sim N(0, \sigma_a^2) \tag{C.106}$$

where $\kappa_4 \equiv (1-\beta(1-\delta)(1-\alpha)$. Note that observing the rental rate $v_t = (1-\alpha)(\varepsilon_{a,t} - k_t)$ is equivalent to the measurement assumption $m_t^A = m_t^E = \varepsilon_{a,t} - k_t$.

Using our earlier notation from the general solution, we obtain (after a little effort for matrix A)

$$F = \begin{bmatrix} 0 & 0 \\ \kappa_2 & \kappa_1 \end{bmatrix} \qquad J = E = [1 \quad -1] \qquad A = \begin{bmatrix} 0 & 0 \\ \frac{\kappa_2}{\kappa_1}\mu & \mu \end{bmatrix} \tag{C.107}$$

where μ is the stable eigenvalue of the system.

C.2 The PI Solution

If agents have PI it is straightforward to show that the L-operator representation of the single observable is an ARMA(1,1) process given by

$$m_t^E = m_t^A = E(I-AL)^{-1}B\varepsilon_t = \left(\frac{1 - \frac{(\kappa_1+\kappa_2)\mu L}{\kappa_1}}{1-\mu L} \right) \varepsilon_{a,t} \tag{C.108}$$

By exploiting the properties of the linearization constants and the stable eigenvalue, μ in Appendix E.6) it can be show that the MA parameter $\frac{(\kappa_1+\kappa_2)\mu}{\kappa_1}$ is non-negative, but, for different values of the risk aversion parameter σ, it may lie either below or above unity.

After substituting for κ_1 and κ_2 the condition for fundamentalness is

$$\begin{aligned}\frac{\kappa_1}{(\kappa_1+\kappa_2)\mu} &= \frac{1}{(1+\frac{(1-\alpha)}{\alpha}(1-\beta(1-\delta)))\mu} \geq 1 \\ &\Rightarrow \mu \leq \frac{1}{(1+\frac{(1-\alpha)}{\alpha}(1-\beta(1-\delta)))}\end{aligned} \tag{C.109}$$

The RHS of (C.109) lies in the interval $(0,1)$ for all $\delta \in [0,1]$ so in principle, the PI solution for both the Rondina and Walker (2021) and GW models can be either fundamental or non-fundamental. However, it can be shown that $\mu = \mu(\sigma)$ where $\mu'(\sigma) > 0$ for $\sigma > 0$ so there exists a threshold for $\sigma > 0$ below which condition (C.109) holds. Figure 8 illustrates this result. It shows that the condition only holds for $\sigma < 0.5$ approximately. For empirically plausible values of σ, therefore the representation $(C.108)$ will be non-fundamental. This is the basis for the representation given by (6) and (7) in the Introduction.

C.3 The II-SA Solution

Under II, the stable solution to the Ricatti equation is given by $P^A = \sigma_a^2 diag(1, \ \ (\kappa_1 + \kappa_2)^2 - 1)$ and the Kalman gain is given by

$$P^A J'(JP^A J')^{-1} J = \begin{bmatrix} \frac{1}{(\kappa_1+\kappa_2)^2} \\ \frac{1}{(\kappa_1+\kappa_2)^2} - 1 \end{bmatrix} [1 \ \ -1] \tag{C.110}$$

Stability of the solution to the Ricatti equation is given by the stability of

$$Q^A = F(I - P^A J'(JP^A J')^{-1} J) = \begin{bmatrix} 0 & 0 \\ \kappa_1 + \kappa_2 - \frac{1}{\kappa_1+\kappa_2} & \frac{1}{\kappa_1+\kappa_2} \end{bmatrix} \tag{C.111}$$

which is a stable matrix since $1 < (\kappa_1 + \kappa_2)$.[54]

Thus, despite the fact that the PMIC may at least in principle sometimes be satisfied under PI, the system can *never* be A-invertible: II does not replicate PI. Hence, from Theorem 3, the system is not E-invertible.

It is easy to show that the L-operator representation of the interest rate under II is then given by

$$\begin{aligned} m_t^E = m_t^A &= E(I - AL)^{-1} P^A J'(JP^A J')^{-1} J(I - Q^A L)^{-1} B\varepsilon_t \\ &= \left(\frac{1 - \frac{\mu L}{(\kappa_1+\kappa_2)\kappa_1}}{1 - \mu L} \right) \left(\frac{1 - (\kappa_1 + \kappa_2)L}{1 - \frac{L}{(\kappa_1+\lambda_2)}} \right) \varepsilon_{a,t} \end{aligned} \tag{C.112}$$

[54]The alternative solution of the Riccati equation is $P^A = diag(1, 0)$ but this is not a stable solution since it implies that $Q^A = diag(0, \kappa_1)$, which is an unstable matrix.

$$= \left(\frac{1 - \frac{\mu L}{(\kappa_1+\lambda_2)\kappa_1}}{1-\mu L} \right) e_t \tag{C.113}$$

which implies a representation as in (6) and (7) in the Introduction.

C.4 The PI-HA Solution

For the full heterogeneous agent version of the model as set out in equations (1) to (5) in the introduction, it is straightforward to show that if PI is simply provided as an endowment, the solution for the aggregate economy is identical to the PI case. At the level of each heterogeneous agent the solution is supplemented by the saddlepath responses to both the idiosyncratic component in technology, and to each agent's idiosyncratic capital; but these responses cancel out in the aggregate.

C.5 The II-HA(∞) Solution

When agents have imperfect, market-consistent information sets, for the general case they exploit information from the markets they trade in, hence the histories of both the aggregate observable, the rental rate on capital, and of the local wage. But as heterogeneity becomes extreme, we apply Theorem 2, so that the solution to any agent's filtering problem for the aggregate economy takes the same form as for the II-SA case, but with a different F matrix, where κ_1 and κ_2 are replaced with values shifted by the saddlepath responses to pure idiosyncratic shocks. In Appendix B.3.8, we show that this implies that the reduced form ARMA process for the single observable takes the form

$$\begin{aligned}
\textbf{II-HA}(\infty) \ : \quad m_t^A = m_t^E &= \frac{\left(1 - \frac{\mu_1\kappa_1 L}{(\kappa_1+\kappa_2)}\right)}{(1-\mu_1 L)} \frac{\left(1 - \frac{(\kappa_1+\kappa_2)}{\kappa_1}L\right)}{\left(1 - \frac{\kappa_1 L}{(\kappa_1+\kappa_2)}\right)} \varepsilon_{a,t} \\
&= \frac{\left(1 - \frac{\mu_1\kappa_1 L}{(\kappa_1+\kappa_2)}\right)}{(1-\mu_1 L)} e_t
\end{aligned} \tag{C.114}$$

which again matches the representation in (6) and (7) of the Introduction.

D Time Versus Frequency Domain Finite-Space Solution

An important development in the recent literature on diffuse information are finite-space solutions that avoid the high-order beliefs in the famous "beauty contest" models emphasized by Keynes (1936). Our solution has the same structure as the single agent solution (see Theorem 2) a feature Huo and Pedroni (2020) refer to as a 'single-judge' outcome of the beauty contest. It holds for the limiting case of a very general set-up set out in Section 3. Here we show that for the analytical RBC example our time domain approach yields the same solution as the frequency domain approach of Rondina and Walker (2021) (henceforth RW) once we have corrected the minor errors in the RW theorems. Moreover we would emphasize that our solution method is simpler.

In their notation, the RW model is given by

$$
\begin{aligned}
a_{t+1} &= \rho a_t + \epsilon_{t+1} && \text{(D.1)}\\
k_{i,t+1} &= \frac{1}{\beta}k_{it} + (1-\frac{1}{\alpha\beta})c_{it} + \frac{1}{\alpha\beta}(a_t + v_{it}) && \text{(D.2)}\\
\mathbb{E}_{it}c_{i,t+1} &= c_{i,t} + \frac{1}{\sigma}\mathbb{E}_{it}r_{t+1} && \text{(D.3)}\\
r_t &= (\alpha-1)k_t + a_t && \text{(D.4)}
\end{aligned}
$$

where $k_t = \int \mu_i k_{it} di$. Measurements are given by

$$
m_t = r_t \qquad m_{it} = a_t + v_{it} \qquad v_{it} \sim N(0, v) \tag{D.5}
$$

The model differs from our illustrative example (1)-(5) in two respects: RW assume 100% depreciation so $\delta = 1$ which implies that the real rate of interest r_t equals the rental rate v_t. The other difference is that the technology shock in RW is total factor productivity (TFP) as oppose to labour productivity in our example.[55] In what follows we stick to the simpler case of $\rho = 0$ as in the text.

RW assume that the solution to the problem in lag operator form is given by

$$
r_t = (L-\lambda)G(L)\varepsilon_t \equiv Y(L)\varepsilon_t \qquad x_{it} = K(L)\varepsilon_t + V(L)v_{it} \tag{D.6}
$$

where x_{it} is lagged capital stock of the ith firm. In addition aggregate TFP is given by $a_t = A(L)\varepsilon_t$. In the very simple case of TFP being equal to white noise, it follows that $A(L) = 1$ and

$$
r_t = (1-(1-\alpha)LK(L))\varepsilon_t \tag{D.7}
$$

To obtain the solution to x_{it}, RW subtract (D.2) from its forward-looking version, and then replace $\mathbb{E}_{it}c_{i,t+1} - c_{i,t}$ by $\frac{1}{\sigma}\mathbb{E}_{it}r_{t+1}$. This leads to an equation solely in terms of k_{it} and its forward leads, or equivalently an equation in x_{it}, its lag and its forward expectation. RW then apply the Wiener-Kolmogorov formula to this expectation, based on the assumed representation of the measurements and the representations in (D.6), and substitute into the equation for x_{it}.

As RW point out, $A(\lambda)V(\lambda) = K(\lambda)$, so that in our case we have $V(\lambda) = K(\lambda)$. In addition for our limiting case, as the variance of the idiosyncratic shock $\to \infty$, it follows

[55] It follows that a_t in RW becomes $(1-\alpha)a_t$ and v_{it} becomes $(1-\alpha)\epsilon_{it}$ in our example, (1)-(5).

that $\tau(\lambda) = 0$ in their equation (A.69), $\tau(\lambda)$ being a signal to noise measure.

The equation involving x_{it} can now be expressed as lag operator expressions multiplying each of ε_t and v_{it}, as in their (A.69), respectively as follows:[56]

$$\begin{aligned}\alpha\beta[K(L) - K(0)] - \alpha\beta\tfrac{(1-\lambda^2)L}{\lambda(1-\lambda L)}[V(0) - K(0)] - \tfrac{\alpha(1-\zeta)(\beta-\zeta)}{\zeta}LK(L) \\ + \tfrac{\alpha(1-\zeta)(\beta-\zeta)}{\zeta(1-\alpha)}\tfrac{(1-\lambda^2)L}{\lambda(1-\lambda L)} = \alpha L(1+\beta-L)K(L) - L\end{aligned} \tag{D.8}$$

$$\alpha\beta[V(L) - V(0)] = \alpha L(1+\beta-L)V(L) - L \tag{D.9}$$

(D.9) can be rewritten as $\alpha(L-1)(L-\beta)V(L) = \alpha\beta V(0) - L$; potentially this means that V(L) is represented by an unstable ARMA process unless the term on the RHS also has a factor $(L-\beta)$. Thus to avoid this unstable autoregressive root for $V(L)$, $\alpha\beta V(0) - L = 0$ at $L = \beta$. This implies that $V(0) = 1/\alpha$. With this value in place, we can rewrite (D.8) as

$$\begin{aligned}\alpha(L-\zeta)(L-\frac{\beta}{\zeta})K(L) = \quad & \alpha\beta K(0)\left[1 - \tfrac{(1-\lambda^2)L}{\lambda(1-\lambda L)}\right] - L + \tfrac{\beta(1-\lambda^2)L}{\lambda(1-\lambda L)} \\ & - \tfrac{\alpha(1-\zeta)(\beta-\zeta)}{\zeta(1-\alpha)}\tfrac{(1-\lambda^2)L}{\lambda(1-\lambda L)}\end{aligned} \tag{D.10}$$

Here too $K(L)$ potentially contains an unstable autoregressive root unless the RHS of this equation also contains a factor $(L-\zeta)$; so the RHS must equal zero when $L = \zeta$. This in turn implies that

$$K(0) = \frac{\lambda\zeta(1-\lambda\zeta)}{\alpha\beta(\lambda-\zeta)} - \frac{(1-\lambda^2)(\beta\zeta - \alpha\beta + \alpha\zeta - \alpha\zeta^2)}{\alpha\beta(1-\alpha)(\lambda-\zeta)} \tag{D.11}$$

and hence

$$K(L) = \frac{1}{\alpha(L-\beta/\zeta)(1-\lambda L)(\lambda-\zeta)}\left[\lambda(\lambda(\zeta+L) - 1 - \zeta L) + \frac{(1-\lambda^2)(\beta\zeta - \alpha\beta + \alpha\zeta - \alpha\zeta^2)}{(1-\alpha)\zeta}\right] \tag{D.12}$$

The final step is to calculate the expression for r_t, namely $Y(L)\varepsilon_t = (1-(1-\alpha)LK(L))\varepsilon_t$. Since we require the numerator of $Y(L)$ to have a factor $(L-\lambda)$ (as in (D.6)), this implies

$$Y(\lambda) = 0 = 1 - (1-\alpha)\lambda K(\lambda) = 1 - \frac{\lambda(-\lambda\zeta(1-\alpha) + \beta\zeta - \alpha\beta + \alpha\zeta - \alpha\zeta^2)}{\zeta\alpha(\lambda-\beta/\zeta)(\lambda-\zeta)} \tag{D.13}$$

The RW II-HA solution is then characterized by the value of λ that satisfies this equation. It is easy to show by direct substitution that $Y(\lambda) = 0$ when $\lambda = \alpha$. This is exactly the value of Λ that we obtain when addressing this example at the end of our proof of Theorem 2 in Subsection B.3.8.

[56](A.69) has an error, that the term $(1-\alpha\beta)$ should be multiplied by η. We replace that product in (D.8) by $\frac{\alpha(1-\zeta)(\beta-\zeta)}{\zeta(1-\alpha)}$.

This is not the solution that would be obtained using by utilizing (A.70) in RW Theorem 1. This is because of an elementary error in RW, where it is easy to see that (A.38) does not follow from (A.36) because the function $\Phi(L)$ is incorrectly defined.

D.1 The Non-Limiting Case:

When the ratio Σ of the variance of the idiosyncratic shock to the aggregate shock is finite, then equations (D.8) and (D.9) become

$$\alpha\beta[K(L)-K(0)]-\alpha\beta(1-\tau)\tfrac{(1-\lambda^2)L}{\lambda(1-\lambda L)}[V(0)-K(0)]-\tfrac{\alpha(1-\zeta)(\beta-\zeta)}{\zeta}LK(L)$$
$$+(1-\tau)\tfrac{\alpha(1-\zeta)(\beta-\zeta)}{\zeta(1-\alpha)}\tfrac{(1-\lambda^2)L}{\lambda(1-\lambda L)}=\alpha L(1+\beta-L)K(L)-L \qquad \text{(D.14)}$$

$$\alpha\beta[V(L)-V(0)]=\alpha L(1+\beta-L)V(L)-L-\alpha\beta\tau\frac{(1-\lambda^2)L}{\lambda(1-\lambda L)}[V(0)-K(0)]+\frac{\tau\alpha(1-\zeta)(\beta-\zeta)}{\zeta(1-\alpha)}\frac{(1-\lambda^2)L}{\lambda(1-\lambda L)} \qquad \text{(D.15)}$$

where $\tau=\frac{1}{1+\Sigma}$, As above, all terms in (D.14) not involving $K(L)$ must have a factor $L-\zeta$, and all terms in (D.15) not involving $V(L)$ must have a factor $L-\beta$. This implies

$$\alpha\beta K(0)-\zeta+\alpha\beta(1-\tau)\frac{(1-\lambda^2)\zeta}{\lambda(1-\lambda\zeta)}[V(0)-K(0)]-(1-\tau)\frac{\alpha(1-\zeta)(\beta-\zeta)}{\zeta(1-\alpha)}\frac{(1-\lambda^2)\zeta}{\lambda(1-\lambda\zeta)}=0 \qquad \text{(D.16)}$$

$$\alpha\beta V(0)-\beta-\alpha\beta\tau\frac{(1-\lambda^2)\beta}{\lambda(1-\lambda\beta)}[V(0)-K(0)]+\tau\frac{\alpha(1-\zeta)(\beta-\zeta)}{\zeta(1-\alpha)}\frac{(1-\lambda^2)\beta}{\lambda(1-\lambda\beta)}=0 \qquad \text{(D.17)}$$

If we now subtract (D.16) from (D.14), then we can directly remove the factor $L-\zeta$ from the whole expression to yield

$$\alpha(L-\frac{\beta}{\zeta})K(L)=-1+\frac{\alpha\beta(1-\tau)(1-\lambda^2)}{\lambda(1-\lambda L)(1-\lambda\zeta)}[V(0)-K(0)]-(1-\tau)\frac{\alpha(1-\zeta)(\beta-\zeta)}{\zeta(1-\alpha)}\frac{(1-\lambda^2)}{\lambda(1-\lambda L)(1-\lambda\zeta)} \qquad \text{(D.18)}$$

Then, incorporating the assumption that the interest rate $r_t=(1-(1-\alpha)LK(L))\varepsilon_t$ has a factor $L-\lambda$, it follows that

$$\lambda-\frac{\alpha\beta}{\zeta}-(1-\alpha)\lambda[\frac{\alpha\beta(1-\tau)}{\lambda(1-\lambda\zeta)}[V(0)-K(0)]-(1-\tau)\frac{\alpha(1-\zeta)(\beta-\zeta)}{\zeta(1-\alpha)\lambda(1-\lambda\zeta)}] \qquad \text{(D.19)}$$

Eliminating $V(0)$ and $K(0)$ from equations (D.16), (D.17) and (D.18) yields an equation for λ[57]:

$$\Sigma(1-\lambda\beta)(\lambda-\alpha)=(\frac{\alpha\beta}{\zeta}-\lambda)(1-\lambda\zeta) \qquad \text{(D.20)}$$

The paths of the roots λ of this equation as Σ changes are shown on the root-locus diagram, Figure 7. Given that $\frac{\alpha\beta}{\zeta}<1$, there is a unique value of $\lambda<1$ for each Σ, with

[57]This equation is derived much more simply within a state space setting, and is part of a currently uncompleted follow-up paper to this one.

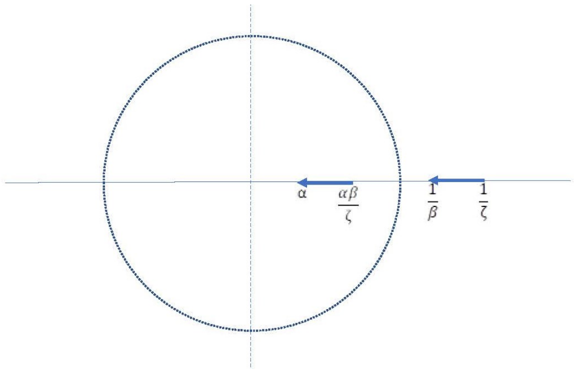

Figure 7: Values of λ as Σ changes. The unit circle is depicted as is usual in root-locus diagrams.

the limiting value $\lambda = \alpha$ as $\Sigma \to \infty$.

Since the structure of this model is identical to that of GW, even though the underlying assumptions slightly differ, the solution of this Section provides the basis for the impulse responses in Figure 2.

E The RBC Model

We first consider the standard RBC model with a zero-growth steady state. We distinguish supply of capital and hours by households from demand for these factors of production by firms. Then we consider a simplified special case without investment adjustment costs suitable for an analytical solution.

E.1 The Full Aggregate Model

The household has a budget constraint in period t

$$B_{t+1} = R_{t-1}B_t + V_tK_t + W_tH_t - C_t - I_t - T_t \tag{E.1}$$

where B_t is the given net stock of financial assets at the beginning of period t, V_t is the gross rental rate, W_t is the wage rate and R_t is the gross real interest rate paid on bonds held at the beginning of period t, C_t is consumption, I_t is investment and T_t are lump-sum taxes. Beginning of period capital stock K_t accumulates according to

$$K_{t+1} = (1-\delta)K_t + I_t \tag{E.2}$$

The household at time t maximizes a value function $\sum_{\tau=0}^{\infty} \beta^\tau U(C_{t+\tau}, L_{t+\tau})$ where $\beta \in (0,1)$ is a discount factor, C_t is real consumption, $L_t = 1 - H_t$ is leisure and H_t is the proportion of available hours worked.

First-order conditions are

$$\text{Euler Consumption} \;: 1 = R_t \mathbb{E}_t \left[\Lambda_{t,t+1}\right] \tag{E.3}$$

$$\text{Euler Capital Supply} : 1 = \mathbb{E}_t \left[R^K_{t+1} \Lambda_{t,t+1}\right] \tag{E.4}$$

$$\text{Stochastic Discount Factor} : \Lambda_{t,t+1} \equiv \beta \frac{U_{C,t+1}}{U_{C,t}} \tag{E.5}$$

$$\text{Labour Supply} : \frac{U_{H,t}}{U_{C,t}} = -\frac{U_{L,t}}{U_{C,t}} = -W_t \tag{E.6}$$

$$\text{Leisure and Hours} : L_t \equiv 1 - H_t \tag{E.7}$$

$$\text{Gross Return on Capital} : R^K_t = V_t + 1 - \delta \tag{E.8}$$

The Euler consumption equation, (E.3), where $U_{C,t} \equiv \frac{\partial U_t}{\partial C_t}$ is the marginal utility of consumption and $\mathbb{E}_t[\cdot]$ denotes rational expectations based on the agents' information set, describes the optimal consumption-savings decisions of the household. It equates the marginal utility from consuming one unit of income in period t with the discounted marginal utility from consuming the gross income acquired, R_t, by saving the income. (E.4) is essentially an arbitrage condition for bond and capital investment. (E.6) equates the real wage with the marginal rate of substitution between consumption and leisure. Equations (E.15)–(E.6) determine consumption, the supply by households of capital, K^s_t and hours H^s_t, and aggregate demand $C_t + I_t + G_t$ where G_t are exogenous government services in a balanced government budget constraint with $G_t = T_t$.

Output and the firm behaviour is summarized by:

$$\text{Output} : Y^s_t = F(A_t, H^d_t, K^d_t) \tag{E.9}$$

$$\text{Labour Demand} : F_{H,t} = W_t \tag{E.10}$$

$$\text{Capital Demand} : F_{K,t} = V_t \tag{E.11}$$

where (E.9) is a production function. Equation (E.10), where $F_{H,t} \equiv \frac{\partial F_t}{\partial H_t}$, equates the marginal product of labour with the real wage. (E.11), where $F_{K,t} \equiv \frac{\partial F_t}{\partial K_t}$, equates the marginal product of capital with the cost of capital. The model is completed with an output, capital and labour market equilibrium conditions:

$$Y^s_t = Y^d_t = C_t + G_t + I_t = Y_t \tag{E.12}$$

$$H^s_t = H^d_t = H_t \tag{E.13}$$

$$K^s_t = K^d_t = K_t \tag{E.14}$$

For our quantitative analysis using a numerical solution, we now generalize the model by adding the Smets and Wouters (2007) form of investment adjustment costs to the

RBC model. The law of motion for the household supply of capital becomes

$$\begin{aligned} K^s_{t+1} &= (1-\delta)K^s_t + (1-S(X_t))I_t\,;\;\; S',\, S'' \geq 0\,;\;\; S(1) = S'(1) = 0 \\ X_t &\equiv \frac{I_t}{I_{t-1}} \end{aligned}$$

Households at time t convert I_t of output into $(1 - S(X_t))I_t$ of new capital sold at a real price Q_t and then maximize with respect to $\{I_t\}$ expected discounted profits. The first-order condition for investment

$$Q_t(1 - S(X_t) - X_t S'(X_t)) + E_t\left[\Lambda_{t,t+1}\, Q_{t+1} S'(X_{t+1}) X^2_{t+1}\right] = 1$$

and the net return on capital becomes

$$R^K_t \equiv \frac{V_t + (1-\delta)Q_t}{Q_{t-1}} \tag{E.15}$$

Note that without investment costs, $S = 0$, $Q_t = 1$ (E.15) reduces (E.8). We complete this set-up with the functional form for investment adjustment, $S(X) = \phi_X(X_t - 1)^2$, which completes the RBC model with investment adjustment costs.

We now specify functional forms for production and utility and AR(1) processes for exogenous variables A_t and G_t. For production we assume a Cobb-Douglas function. The consumers' utility function is non-separable and consistent with a balanced growth path when the inter-temporal elasticity of substitution, $1/\sigma$ is not unitary. These functional forms, the associated marginal utilities and marginal products, and exogenous processes are given (in equilibrium) by

$$F(A_t, H_t, K_t) = (A_t H_t)^{1-\alpha} K^\alpha_t \tag{E.16}$$

$$F_H(A_t, H_t, K_t) = \frac{(1-\alpha)Y_t}{H_t} \tag{E.17}$$

$$F_K(A_t, H_t, K_t) = \frac{\alpha Y_t}{K_t} \tag{E.18}$$

$$\log A_t - \log \bar{A}_t = \rho_A(\log A_{t-1} - \log \bar{A}_{t-1}) + \varepsilon_{A,t} \tag{E.19}$$

$$\log G_t - \log \bar{G}_t = \rho_G(\log G_{t-1} - \log \bar{G}_{t-1}) + \varepsilon_{G,t} \tag{E.20}$$

$$U_t = \frac{(C_t^{(1-\varrho)} L_t^{\varrho})^{1-\sigma} - 1}{1-\sigma} \tag{E.21}$$

$$U_{C,t} = (1-\varrho) C_t^{(1-\varrho)(1-\sigma)-1} (1-H_t)^{\varrho(1-\sigma)} \tag{E.22}$$

$$U_{H,t} = -\varrho C_t^{(1-\varrho)(1-\sigma)} (1-H_t)^{\varrho(1-\sigma)-1} \tag{E.23}$$

(E.15)–(E.23) describe an equilibrium in aggregates C_t, W_t, V_t, Y_t, H_t, K_t, I_t, Q_t, R_t, R^K_t and T_t, given A_t and G_t where for the latter we assume AR(1) processes about steady states $\bar{A}, \bar{G}$ driven by zero mean iid shocks $\varepsilon_{A,t}$ and $\varepsilon_{G,t}$.

E.2 The Zero-Growth Steady State

We assume a zero-growth steady state with $\bar{A}_t = \bar{A}_{t-1} = A$ say and $\bar{G}_t = \bar{G}_{t-1} = G$. $K_t = K_{t-1} = K$, etc. Then the full steady state of the standard RBC model is given by:

$$
\begin{aligned}
Q &= 1 \\
X &= 1 \\
S &= 0 \\
R &= \frac{1}{\beta} \\
R^K &= R = V + 1 - \delta \\
V &= \frac{\alpha Y}{K} \\
Y &= (AH)^{1-\alpha} K^{\alpha} \\
\frac{\varrho C}{(1-\varrho)(1-H)} &= W \\
\frac{\alpha Y}{H} &= W \\
\frac{K}{Y} &= \frac{\alpha}{R - 1 + \delta} \\
I &= \delta K \\
Y &= C + I + G \\
G &= T \\
U &= \frac{(C^{(1-\varrho)}(1-H)^{\varrho})^{1-\sigma} - 1}{1-\sigma} \\
&\to (1-\varrho)\log C_t + \varrho \log(1-H_t) \text{ as } \sigma \to 1 \\
U_C &= (1-\varrho) C^{(1-\varrho)(1-\sigma)-1}((1-H)^{\varrho(1-\sigma)}) \\
U_H &= -\varrho C^{(1-\varrho)(1-\sigma)}(1-H)^{\varrho(1-\sigma)-1}
\end{aligned}
$$

Given A and G, the steady state above gives 8 equations in 8 stationary variables R, C, Y, W, H, I, K , T. This describes the zero-growth steady-state equilibrium.

In *recursive form* this steady state can be written

$$
\begin{aligned}
R &= \frac{1}{\beta} \\
R^K &= R \\
V &= R^K - 1 + \delta \\
\frac{K}{Y} &= \frac{\alpha}{V} = \frac{\alpha}{R - 1 + \delta} \\
\frac{I}{Y} &= \frac{\delta K}{Y} = \frac{\alpha \delta}{R - 1 + \delta}
\end{aligned}
$$

$$
\begin{aligned}
\frac{C}{Y} &= 1 - \frac{I}{Y} - \frac{G}{Y} = 1 - \frac{I}{Y} - g_y \\
\frac{H\varrho}{(1-H)(1-\varrho)} &= \frac{WH}{C} = \frac{WH/Y}{C/Y} = \frac{1-\alpha}{C/Y} \\
&\Rightarrow\ H = \frac{(1-\alpha)(1-\varrho)}{\varrho C/Y + (1-\alpha)(1-\varrho)} \\
Y &= (AH)^{1-\alpha}K^{\alpha} = (AH)^{1-\alpha}\left(\frac{K}{Y}\right)^{\alpha}(Y)^{\alpha} \Rightarrow\ Y = AH(K/Y)^{\frac{\alpha}{1-\alpha}} \\
G &= g_y Y \\
W &= (1-\alpha)\frac{Y}{H} \\
I &= \frac{I}{Y}Y \\
C &= \frac{C}{Y}Y \\
K &= \frac{K}{Y}Y
\end{aligned}
$$

E.3 Linearization of the Aggregate Model

The linearized form of this RBC model with investment adjustment costs about a balanced zero-growth steady state with $R = R^K = \frac{1}{\beta}$ and $c_y = \frac{C}{Y}$, $i_y = \frac{I}{Y}$ and $g_y = \frac{G}{Y}$ then takes the state-space form

$$
\begin{aligned}
a_t &= \rho_A a_{t-1} + \varepsilon_{A,t} \\
g_t &= \rho_G g_{t-1} + \varepsilon_{G,t} \\
k_t^s &= (1-\delta)k_{t-1}^s + \delta i_t \\
\mathbb{E}_t[u_{C,t+1}] &= u_{C,t} - r_t \\
\mathbb{E}_t r_{t+1}^K &= r_t \\
r_t^K &= \frac{(R-1+\delta)v_t + (1-\delta)q_t}{R} - q_{t-1} \\
\left(1 + \frac{1}{R}\right) i_t &= \frac{1}{R}E_t i_{t+1} + i_{t-1} + \frac{1}{S''(1)}q_t
\end{aligned}
$$

with further outputs defined in terms of the dynamic state variables by

$$
\begin{aligned}
u_{C,t} &= -(1 + (\sigma-1)(1-\varrho))c_t + (\sigma-1)\varrho\frac{H}{1-H}h_t^s \\
u_{L,t} &= u_{C,t} + c_t + \frac{H}{1-H}h_t^s \\
w_t &= u_{L,t} - u_{C,t} \\
y_t^s &= (1-\alpha)(a_t + h_t^d) + \alpha k_t^d \\
y_t^d &= c_y c_t + i_y i_t + g_y\, g_t = y_t^s = y_t
\end{aligned}
$$

$$
\begin{aligned}
k_t^s &= k_t^d = k_t \\
h_t^s &= h_t^d = h_t \\
g_t &= t_t \\
w_t &= y_t^s - h_t^d \\
v_t &= y_t^s - k_t^d
\end{aligned}
$$

E.4 A Special Case of the Aggregate Model in Linearized Form

The analytical example in Section C, taken from GW , is a linearized form of a special case of the full RBC model for which hours H_t are constant and normalized at unity, $\varrho = 0$, $G_t = 0$ leaving only one technology shock process and there are no investment adjustment costs so $S_t(X_t) = S_t'(X_t) = 0$ and $Q_t = 1$. Hence $q_t = h_t = g_t = 0$ and $u_{C,t} = -c_t$.

Then the linearized aggregate model above becomes:

$$
\begin{aligned}
k_{t+1} &= (1-\delta)k_t + \delta i_t && \text{(E.24)} \\
y_t &= (1-\alpha)a_t + \alpha k_t = c_y c_t + i_y i_t && \text{(E.25)} \\
\mathbb{E}_t c_{t+1} &= c_t + \frac{1}{\sigma} r_t && \text{(E.26)} \\
w_t &= y_t && \text{(E.27)} \\
v_t &= y_t - k_t && \text{(E.28)} \\
r_t &= \mathbb{E}_t r_{t+1}^K && \text{(E.29)} \\
r_t^K &= \frac{(R-1+\delta)v_t}{R} = (1-\beta(1-\delta))v_t && \text{(E.30)} \\
v_t &= (1-\alpha)(a_t - k_t) && \text{(E.31)}
\end{aligned}
$$

where the steady state ratios are given in E.2. Combining (E.24)–(E.25) gives

$$
\begin{aligned}
k_{t+1} &= \kappa_1 k_t + \kappa_2 a_t + (1-\kappa_1-\kappa_2)c_t && \text{(E.32)} \\
\mathbb{E}_t c_{t+1} &= c_t + \frac{1}{\sigma} r_t = c_t + \kappa_3 \mathbb{E}_t v_{t+1} && \text{(E.33)}
\end{aligned}
$$

where

$$
\kappa_1 = \frac{1}{\beta}; \quad \kappa_2 = \frac{(1-\alpha)}{\alpha\beta}(1-\beta(1-\delta)); \quad \kappa_3 = \frac{(1-\beta(1-\delta))}{\sigma}
$$

This now gives us the aggregate form of the model used for the illustrative model in the introduction of the main text.

A further specialization of the RBC model is provided by Rondina and Walker (2021) who assume 100% capital depreciation. Then $\delta = 1$ and (E.30) gives $r_t^K = v_t$. Also technical change in their production function is Hicks-neutral rather than labour-augmenting so that A_t becomes total factor productivity and $(1-\alpha)a_t$ above is then replaced with a_t.

E.5 The Heterogeneous Agent Model

Following GW, we consider a standard islands model with a large number of households and firms in each island i. The heterogeneous agent island-specific counterpart of the aggregate model above is

$$\begin{aligned} k^s_{i,t+1} &= \kappa_1 k^s_{i,t} + \kappa_2(a_t + \varepsilon_{i,t}) + \left(1 - \frac{1}{\alpha\beta}\right)(1 - \kappa_1 - \kappa_2)c_{i,t} && \text{(E.34)} \\ \mathbb{E}_{i,t}c_{i,t+1} &= c_{i,t} + \kappa_3 \mathbb{E}_{i,t}v_{t+1} && \text{(E.35)} \\ y_{i,t} &= (1-\alpha)(a_t + \varepsilon_{i,t}) + \alpha k^d_{i,t} && \text{(E.36)} \\ w_{i,t} &= y_{i,t} && \text{(E.37)} \\ v_t &= y_{i,t} - k^d_{i,t} && \text{(E.38)} \end{aligned}$$

Combining (E.36) - (E.38) we arrive at

$$w_{i,t} = a_t + \varepsilon_{i,t} - \frac{\alpha}{1-\alpha}v_t \tag{E.39}$$

According to the principle of market-consistent information both the rental rate v_t and the island-specific wage $w_{i,t}$ are assumed to be observed by households. It follows from (E.39) that the composite shock $a_t + \varepsilon_{i,t}$ is also observed as assumed in the information set (5) in the main text. Note that $k^s_{i,t} \neq k^d_{i,t}$ since capital is free to flow from less to more productive islands.

As in GW we set up the same RBC model without the restriction $\delta = 1$. As in Rondina and Walker (2021) households and firms are located in I islands each of which there are a large number of both types of agents and firms on island i only employ labour from households on the same island in which case the wage $W_{i,t}$ is island-specific. There are aggregate and island-specific shocks. In island i for the model these are a composite productivity shock process $A_t \exp(\varepsilon_{A,it})$. As in our more general model we can add a government spending shock process $G_t \exp(\varepsilon_{G,it}$ where the aggregate components A_t and G_t are AR1 processes as before and $\varepsilon_{A,it}$ and $\varepsilon_{G,it}$ are i.i.d mean zero shocks.[58]

We solve this heterogeneous agent (HA) model under imperfect information with these informational assumptions and refer to the solution as II-HA. We also solve for the single-agent (SA) aggregate model under II assuming only v_t is observed and refer to the solution as II-SA. In all these cases agents are individually rational in arriving at decision rules (E.34) and (E.35). What distinguishes the II-SA and II-HA(∞) solutions

[58]This is more restrictive than GW who have AR1 idiosyncratic processes as well.

is a general equilibrium effect, namely in the former agents do not use the fact they are single; i.e., do not use $a_{i,t} = a_t$.

E.6 Stability Analysis

The dynamic properties of the aggregate model under PI with $k_{t,t} = k_t$, $c_{t,t} = c_t$ and $a_{t,t} = a_t$ are driven by

$$\begin{bmatrix} k_{t+1} \\ \mathbb{E}_t c_{t+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} k_t \\ c_t \end{bmatrix} + \text{ terms in } a_t \tag{E.40}$$

where

$$a_{11} = \kappa_1 = R > 0 \tag{E.41}$$
$$a_{12} = 1 - \kappa_1 - \kappa_2 = -\frac{1}{\alpha}\left(R - 1 + (1-\alpha)\delta)\right) < 0 \tag{E.42}$$
$$a_{21} = -\frac{1}{\sigma}\left(R - 1 + \delta\right)(1-\alpha) < 0 \tag{E.43}$$
$$a_{22} = 1 - \frac{a_{12}}{\sigma R}(R - 1 + \delta)(1-\alpha) > 0 \tag{E.44}$$

and $R = \frac{1}{\beta}$.

The eigenvalues of (E.40) are given by

$$\mu^2 - (a_{11} + a_{22})\mu + a_{11}a_{22} - a_{12}a_{21} = 0 \tag{E.45}$$

which we write as

$$\mu^2 - \text{tr}(A)\mu + \det(A) = 0 \tag{E.46}$$

with solution

$$\mu = \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2} \tag{E.47}$$

The necessary condition for real roots is therefore $\text{tr}(A)^2 \geq \det(A)$ which can be written

$$(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) = (a_{11} - a_{22})^2 + 4a_{12}a_{21} \geq 0 \tag{E.48}$$

Since $a_{12}a_{21} > 0$ in our model we conclude that both roots are real.

Given real roots and following the approach of Woodford (2003), Appendix C, we can show that a necessary and sufficient condition for one root to be greater than unity, and one within the unit circle is that

$$-\text{tr}(A) - 1 < \det(A) < \text{tr}(A) - 1 \tag{E.49}$$

From (E.41)–(E.44) a little algebra gives

$$\det(A) \;=\; R = \frac{1}{\beta} > 1 \tag{E.50}$$

$$\mathrm{tr}(A) \;=\; R + 1 + \frac{(R-1+(1-\alpha)\delta)(R-1+\delta)(1-\alpha)}{\sigma\alpha R} > R+1 \tag{E.51}$$

Hence the condition (E.49) holds and the model is saddle-path stable for all permitted parameter values.

E.7 The Fundamentalness of the PI Solution

The PI Solution for the observable rental rate $m_t^E = v_t$ is

$$m_t^E = v_t = \left(\frac{1 - \frac{(\kappa_1+\kappa_2)\mu_1 L}{\kappa_1}}{1-\mu_1 L} \right) \varepsilon_{a,t} \tag{E.52}$$

where $\kappa_1 = a_{11}$ and $\kappa_1 + \kappa_2 = 1 - a_{12}$ and μ_1 is the stable eigenvalue of (E.47). From (E.41)-(E.44) we have

$$\frac{\kappa_1+\kappa_2}{\kappa_1} = \frac{1-a_{12}}{a_{11}} = \frac{R + \frac{1-\alpha}{\alpha}(R-1+\delta)}{R} \tag{E.53}$$

where $R = \frac{1}{\beta}$. The special case of the model in Rondina and Walker (2021) with $\delta = 1$ then gives

$$m_t^E = v_t = \left(\frac{1 - \frac{\mu_1}{\alpha} L}{1-\mu_1 L} \right) \varepsilon_{a,t} \tag{E.54}$$

noting that with $\delta = 1$, the stable eigenvalue $\mu_1 = \frac{1}{\beta\mu_2}$ where μ_2 is the unstable eigenvalue.

The condition for fundamentalness is therefore

$$\begin{aligned} \frac{\kappa_1}{(\kappa_1+\kappa_2)\mu_1} &= \frac{R}{(R + \frac{(1-\alpha)}{\alpha}(R-1+\delta))\mu_1} \geq 1 \\ &\Rightarrow \;\; \mu_1 \leq \frac{R}{(R + \frac{(1-\alpha)}{\alpha}(R-1+\delta))} \end{aligned} \tag{E.55}$$

The RHS of (E.55) lies in the interval $(0,1)$ for all $\delta \in [0,1]$ so, in principle, the PI solution for both the Rondina and Walker (2021) and GW models can be non-fundamental. In fact, it can be shown that $\mu_1 = \mu_1(\sigma)$ where $\mu_1'(\sigma) < 0$ for $\sigma > 0$ so there exists a threshold for $\sigma > 0$ below which condition (E.55) holds. Figure 8 illustrates this result. Only for the risk aversion parameter $\sigma < 0.5$ approximately and for an empirically plausible calibration of δ do we have a fundamental MA process for the rental rate v_t.

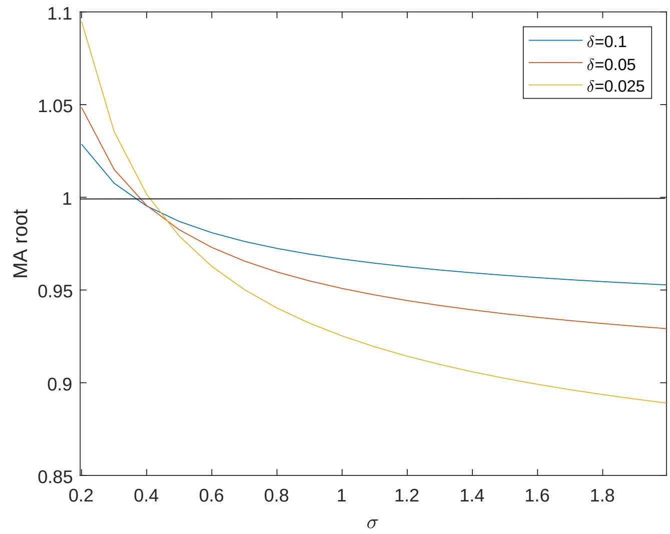

Figure 8: Simple RBC Model. Condition for the Fundamentalness of the MA process for the Rental Rate v_t. Parameter Values: $\alpha = 0.33$, $\beta = 0.985$, $\delta = 0.025, 0.05, 0.1$; $\sigma \in [0.2, 2]$

F A RBC Model with News Shocks

We now introduce fiscal policy and news shocks into the model of Section E.

F.1 Households

To highlight the role of nominal interest rates and the interaction of monetary and fiscal policy we now express this budget in *nominal* terms. In fact, as we show, it leads to the same constraint in real terms as in Section E. The household budget constraint is then

$$P_t^B B_t^n = B_{t-1}^n + P_t(1-\tau_{k,t})r_t^K K_{t-1} + P_t(1-\tau_{w,t})W_t H_t - P_t C_t - P_t I_t \tag{F.1}$$

where B_t^n is the number of 1-period nominal bonds held by the household at the end of period t with face value unity (i.e., each paying one unit of currency in the next period), $P_t^B = \frac{1}{R_{n,t}}$ is the price of bonds where $R_{n,t}$ is the nominal interest rate, r_t^K is the rental rate on capital received from firms, W_t is the real wage rate I_t is real investment, $\tau_{k,t}$ and $\tau_{w,t}$ are capital and labour distortionary tax rates.

The first-order conditions for the household optimization problem are

$$\begin{aligned} \text{Euler Consumption} &: \quad U_{C,t} = \beta R_t \mathbb{E}_t\left[U_{C,t+1}\right] \\ \text{Labour Supply} &: \quad \frac{U_{H,t}}{U_{C,t}} = -\frac{U_{L,t}}{U_{C,t}} = -W_t(1-\tau_{w,t}) \\ \text{Leisure and Hours} &: \quad L_t \equiv 1 - H_t \end{aligned} \tag{F.2}$$

$$
\begin{aligned}
\text{Investment FOC} \quad &: \quad Q_t(1 - S(X_t) - X_t S'(X_t)) \\
&+ \quad E_t\left[\Lambda_{t,t+1}\, Q_{t+1} S'(X_{t+1}) X_{t+1}^2\right] = 1 \\
\text{Capital Supply} \quad &: \quad \mathbb{E}_t\left[\Lambda_{t,t+1} R_{t+1}^K\right] = 1
\end{aligned}
$$

where $\Lambda_{t,t+1} \equiv \beta \frac{U_{C,t+1}}{U_{C,t}}$ is the *real stochastic discount factor* over the interval $[t, t+1]$, $X_t = I_t/I_{t-1}$ is the rate of change of investment and R_t^K is the gross return on capital net of tax is given by

$$
R_t^K = \frac{\left[r_t^K(1 - \tau_{k,t}) + (1-\delta)Q_t\right]}{Q_{t-1}} \tag{F.3}
$$

The only change from Section E are (F.2) and (F.3) where the supply of labour and capital by the household is lowered by the existence of distortionary taxes. The rest of the model is as before: firms still face a pre-tax real wage and rental rate of capital since it is the households who pay these taxes.

F.2 Government Budget Constraint

Following Leeper *et al.* (2013) we assume a government balanced budget constraint:

$$
B_t = 0 = G_t - (\tau_{w,t} W_t H_t + \tau_{k,t} r_t^K K_{t-1}) \tag{F.4}
$$

This gives a zero-growth steady state

$$
\tau_w = \frac{G - \tau_k r^K K}{WH} = \frac{g_y - \tau_k r^K K/Y}{(1-\alpha)} \tag{F.5}
$$

F.3 Fiscal Policy and News Shocks

We now introduce tax news shocks along the lines of Leeper *et al.* (2013). We model information flows about tax rates with the follow policy rules

$$
\tau_{w,t} = \rho_w \sum_{j=0}^{J} [\sigma_w \varepsilon_{\tau,t-j}^w + \xi \sigma_k \varepsilon_{\tau,t-j}^k] \tag{F.6}
$$

$$
\tau_{k,t} = \rho_k \sum_{j=0}^{J} [\sigma_w \varepsilon_{\tau,t-j}^w + \xi \sigma_k \varepsilon_{\tau,t-j}^k] \tag{F.7}
$$

where ξ allows labour and capital tax rates to be correlated. News shocks $\{\varepsilon_{\tau,t-j}^w, \varepsilon_{\tau,t-j}^k\}$ enter the information set of agents and $\sum_j \phi_j = 1$ imposes information flows as moving averages.

We report results for the labour tax news shocks only with $J = 2$, $\rho_w = \xi = 0$, $\phi_1 = \theta$,

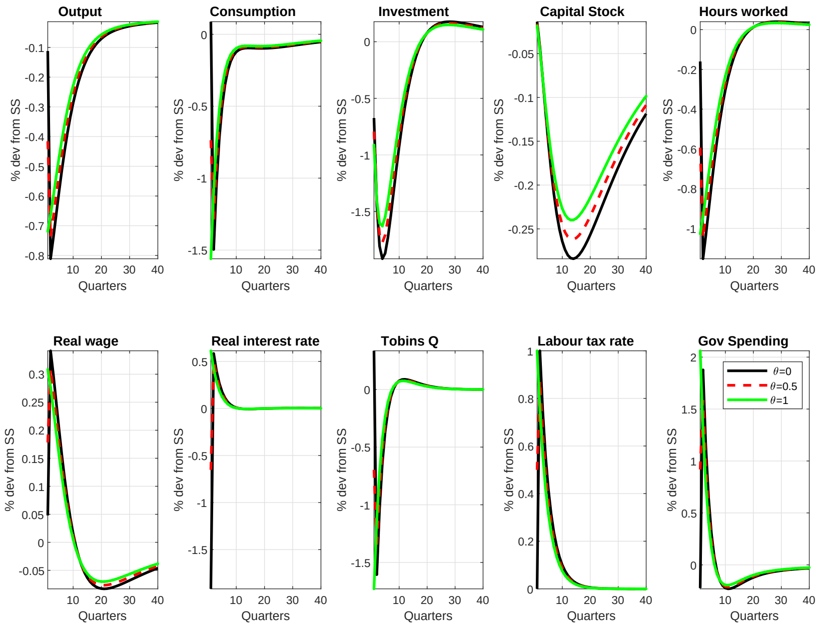

Figure 9: Impulse Responses to the News Shock $\varepsilon_{w,t}$

$\phi_2 = 1 - \theta$, $\tau_{k,t} = \tau_k$, $\sigma_w = 1$. Then

$$\tau_{w,t} = \theta\varepsilon_{w,t} + (1-\theta)\varepsilon_{w,t-1} \tag{F.8}$$

where $\theta \in (0, 1)$. If $\theta = 0$ then agents have perfect foresight because they observe $\tau_{w,t+1}$ perfectly. If $\theta = 1$ then agents have no foresight and receive news only about the current tax rate. As θ goes from 1 to 0 agents receive more news about next period's tax rate.

The model is solved with agents having PI in that they observe enough current values of variables and the news shocks $\varepsilon_{w,t}$ and $\varepsilon_{w,t-1}$ to achieve A-invertibility. Figure 9 shows the impulse responses to the tax news shock $\tau_{w,t}$ as θ goes from 1 to 0 and receive more news about the next period's tax rate. Thus we see a corresponding increase in the response of real variables such as output and investment.

Single agent models that include news shocks imply that there is a common news shock that is observed by all agents and not by the econometrician. For this to be consistent with our assumptions in the HA case, we extend this approach to news shocks by assuming that agents all observe the news shock, but with idiosyncratic noise. In the context of our paper, with signal to noise ratio tending to zero, agents react solely in the current period to this noisy news shock. Thus our theoretical results can include this form of news shocks.

G Simple NK Partial Equilibrium Model

Consider a New Keynesian Phillips curve dependent on the real marginal cost mc_t and a mark-up shock $\varepsilon_{1,t}$ *assumed exogenous*

$$\pi_t = \beta\pi_{t+1,t} + \lambda mc_t + \sigma_1\varepsilon_{1,t} \tag{G.1}$$

$$mc_{t+1} = \rho mc_t + \sigma_2\varepsilon_{2,t+1} \tag{G.2}$$

where $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}$ and $(1-\theta)$ is the constant per period probability that the Calvo contract is reset and $\varepsilon_{i,t} \sim N(0,1)$. This of the Blanchard-Kahn state-space form:

$$\begin{bmatrix} \varepsilon_{1,t+1} \\ mc_{t+1} \\ \mathbb{E}_t[\pi_{t+1}] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ -1/\beta & -\lambda/\beta & 1/\beta \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ mc_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \\ 0 \end{bmatrix}$$

G.1 PI Solution

Consider first the solution under agents' PI. To solve this, we need to first go back (B.23) below from the paper and the saddle path satisfying

$$x_t + Nz_t = 0 \quad \text{where} \quad \begin{bmatrix} N & I \end{bmatrix}(G+H) = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{G.3}$$

where Λ^U is a matrix with unstable eigenvalues. If the number of unstable eigenvalues of $(G+H)$ is the same as the dimension of x_t, then the system will be determinate.

To find N, consider the matrix of eigenvectors W satisfying

$$W(G+H) = \Lambda^U W \tag{G.4}$$

Then, as for G and H, partitioning W conformably with z_t and x_t, from PCL we have

$$N = -W_{22}^{-1}W_{21} \tag{G.5}$$

In our example

$$G+H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ -1/\beta & -\lambda/\beta & 1/\beta \end{bmatrix} \tag{G.6}$$

which has eigenvalues 0, ρ both less than unity and $\frac{1}{\beta} > 1$. Now write the ij element of

W as w_{ij}, $i, j \in 1, 3$. Then corresponding to the eigenvalue $1/\beta$ we have the eigenvector

$$[w_{31}\, w_{32}\, w_{33}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ -1/\beta & -\lambda\beta & 1/\beta \end{bmatrix} = \frac{1}{\beta}[w_{31}\, w_{32}\, w_{33}] \tag{G.7}$$

leaving w_{31}, w_{32}, w_{33} to satisfy

$$\begin{aligned} -w_{33} &= w_{31} \\ \rho w_{32} - \frac{\lambda}{\beta} w_{33} &= \frac{1}{\beta} w_{32} \\ w_{33}\frac{1}{\beta} &= \frac{1}{\beta} w_{33} \end{aligned}$$

Without loss of generality, we can put $w_{33} = 1$. Hence $w_{31} = -1$ and $w_{32} = \frac{\lambda\beta}{\beta\rho - 1}$ giving $N = \left[\beta \ \frac{\lambda}{1-\beta\rho}\right]$.

From our general solution procedure above, the following matrices are defined

$$A = F = \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}; \quad E = -N = -\left[\beta \ \frac{\beta}{1-\beta\rho}\right]; \quad J = [\beta \ \beta]\,; \quad BB' = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

It follows that under PI that

$$\pi_t = \beta\varepsilon_{1,t} + \frac{\lambda}{1-\beta\rho} mc_t \equiv \pi_t^{PI} \tag{G.8}$$

Along with (G.2) we then have a **VAR(1)** process in $[\pi_t \ mc_t]'$ and $[\varepsilon_{1,t} \ \varepsilon_{2,t}]'$. In case of Nimark (2008), where $\varepsilon_{1,t} = 0$, this becomes

$$\pi_t = \frac{\lambda}{1-\beta\rho} mc_t \tag{G.9}$$

which is Equation (11) in Nimark (2008).

G.2 Agents' Imperfect Information

We consider agents' information sets

1. Perfect Information (PI): $[\varepsilon_{1,t} \ mc_t \ \pi_t]'$

2. Imperfect Information (II): π_t

3. Imperfect Information (II): π_{t-1}

Case (1), PI solution is above. Next consider Case (2) where agents have II with π_t observed. Following our PI solution in the main text, we arrive at

$$
\begin{aligned}
mc_t &= \rho mc_{t-1} + \varepsilon_{2,t} \\
\tilde{mc}_t &\equiv mc_t - mc_{t,t-1} = \frac{\rho}{\sigma_1^2 + p}(\sigma_1^2 \tilde{mc}_{t-1} - p\varepsilon_{1,t-1}) + \varepsilon_{2,t} \qquad (G.10) \\
\pi_t &= \beta\left(1 + \frac{\beta\rho p}{(1-\beta\rho)(\sigma_1^2+p)}\right)\varepsilon_{1,t} + \frac{\lambda}{1-\beta\rho} mc_t \\
&- \frac{\beta\rho\sigma_1^2}{(1-\beta\rho)(\sigma_1^2+p)}\tilde{mc}_t \qquad (G.11)
\end{aligned}
$$

where, from the main text, the agents' steady-state Ricatti equation is given by

$$
P^A = FP^AF' - FP^AJ'(JP^AJ')^{-1}JP^AF' + BB' = Q^AP^A(Q^A)' + BB' \qquad (G.12)
$$

This has a solution

$$
P^A = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & p \end{bmatrix} \qquad \text{where } p = \frac{\rho^2 p \sigma_1^2}{\sigma_1^2 + p} + \sigma_2^2
$$

noting that $N - G_{22}^{-1}G_{21} = \begin{bmatrix} 0 & \frac{\beta\lambda\rho}{1-\beta\rho} \end{bmatrix}$, This is an **VARMA(1,1)** process in $[\pi_t \; mc_t \; \tilde{mc}_t]'$ and $[\varepsilon_{1,t} \; \varepsilon_{2,t}]'$.

Figure 10 shows the impulse response function following a negative marginal cost shock $\varepsilon_{2,t}$. The greater is σ_1^2, the greater is the difference between II and PI.

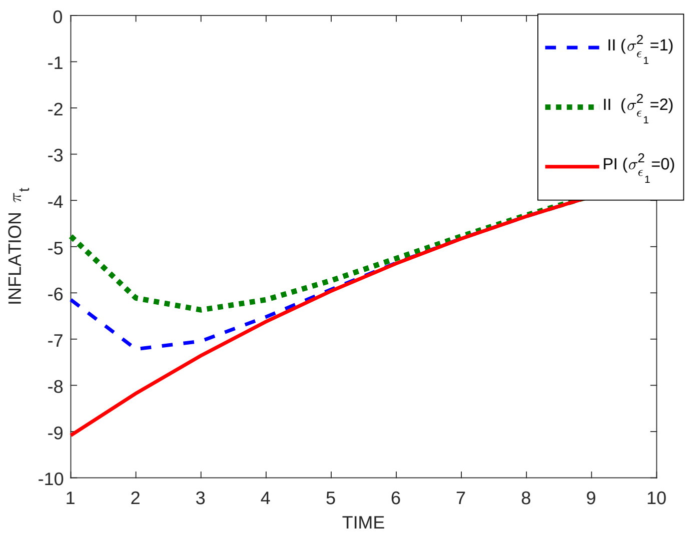

Figure 10: Inflation Dynamics under PI and II

To obtain the innovations representation, we first solve for Z in (B.100); it is easy to verify that Z is given by

$$Z = P^E J'(JP^E J')^{-1} JP^E = \frac{1}{\sigma_1^2 + p} \begin{bmatrix} \sigma_1^2 \\ p \end{bmatrix} \begin{bmatrix} \sigma_1^2 & p \end{bmatrix} \tag{G.13}$$

The innovations process that provides the VARMA for π_t, corresponding to (G.3) is then

$$\begin{aligned} \tilde{s}_{1,t} &= \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix} \tilde{s}_{1,t-1} + \frac{1}{\beta\sigma_1^2 + \frac{\beta}{1-\beta\rho}p} \begin{bmatrix} \sigma_1^2 \\ p \end{bmatrix} \hat{\varepsilon}_t \\ \pi_t &= \begin{bmatrix} \beta & \frac{\lambda}{1-\beta\rho} \end{bmatrix} \tilde{s}_{1,t} \end{aligned}$$

from which it is readily seen that the system is back to a **VAR(1)** process as under PI. This illustrates Theorem 4 of our paper: even though II adds more persistence than under PI, the innovations process dynamics has the same dimensions in each case.

G.3 Nimark (2008)

Now consider the Nimark (2008) example of Section 3.2. Defining $\pi_t = p_t - p_{t-1}$, it is easy to see that $\pi_t = (1-\theta)(p_t^* - p_{t-1})$. Correspondingly, defining $\pi_{i,t} = (1-\theta)(p_{i,t}^* - p_{t-1})$, it follows that one can derive the equation

$$\pi_{i,t} = \beta\theta \mathbb{E}_{i,t}\pi_{i,t+1} + (1-\theta)\mathbb{E}_{i,t}\pi_t + \lambda\theta(mc_t + \varepsilon_{i,t}) \tag{G.14}$$

where $\lambda = (1-\theta)(1-\beta\theta)/\theta$, with information set $m_{1t}^A = \pi_{t-1}, m_{2t}^A = mc_t + \varepsilon_{i,t}$. Does Nimark's solution (with higher order expectations), in the limit as the variance of idiosyncratic shocks dominates the aggregate component, tend to our solution which is an II solution?

We first write a candidate representation for the aggregate solution in the limiting case as the Phillips curve above, but without any idiosyncratic shocks

$$\pi_t = \beta\theta\mathbb{E}_t\pi_{t+1} + (1-\theta)\mathbb{E}_t\pi_t + \lambda\theta mc_t$$

The state-space form is now:

$$\begin{bmatrix} mc_{t+1} \\ \pi_t \\ \mathbb{E}_t[\pi_{t+1}] \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{\lambda}{\beta} & 0 & \frac{1}{\beta\theta} \end{bmatrix} \begin{bmatrix} mc_t \\ \pi_{t-1} \\ \pi_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\theta-1}{\beta\theta} \end{bmatrix} \begin{bmatrix} mc_{t,t} \\ \pi_{t-1,t} \\ \pi_{t,t} \end{bmatrix} + \begin{bmatrix} \sigma_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{2,t+1} \\ 0 \\ 0 \end{bmatrix} \tag{G.15}$$

with observation $m_t^A = [0 \;\; 1 \;\; 0] \begin{bmatrix} mc_t \\ \pi_{t-1} \\ \pi_t \end{bmatrix}$. The saddle path is associated with the unstable eigenvalue $\frac{1}{\beta} > 1$, and the associated eigenvector yields $N = [\frac{\lambda}{\rho\beta - 1}\ 0]$.

The agent's PI solution is therefore

$$\pi_t^{PI} = \frac{\lambda}{1-\rho\beta} mc_t \tag{G.16}$$

For II, we need the matrices

$$F \equiv G_{11} - G_{12}G_{22}^{-1}G_{21} \qquad J \equiv M_1 - M_2 G_{22}^{-1} G_{21} \tag{G.17}$$

$$A = G_{11} + H_{11} - (G_{12} + H_{12})N \qquad E = M_1 + M_3 - (M_2 + M_4)N \tag{G.18}$$

capturing intrinsic dynamics in the system:

$$F = \begin{bmatrix} \rho & 0 \\ \lambda\theta & 0 \end{bmatrix} \qquad E = J = [0\ 1] \qquad A = \begin{bmatrix} \rho & 0 \\ \frac{\lambda}{1-\beta\rho} & 0 \end{bmatrix} \tag{G.19}$$

It is easy to show that the solution to the Riccati equation (G.12) is $P^A = \begin{bmatrix} 1+\rho^2 & \rho\lambda\theta \\ \rho\lambda\theta & \lambda^2\theta^2 \end{bmatrix}$ and hence

$$Q^A = \begin{bmatrix} \rho & -\frac{\rho^2}{\lambda\theta} \\ \lambda\theta & -\rho \end{bmatrix} \qquad P^A J'(JP^A J')^{-1} J = \begin{bmatrix} 0 & \frac{\rho}{\lambda\theta} \\ 0 & 1 \end{bmatrix} \qquad AP^A J'(JP^A J')^{-1} J = \begin{bmatrix} 0 & \frac{\rho^2}{\lambda\theta} \\ 0 & \frac{\rho}{\theta(1-\beta\rho)} \end{bmatrix} \tag{G.20}$$

In lag operator form, it then easy to verify that $\tilde{mc}_t = (1+\rho L)v_t, \tilde{\pi}_{t-1} = \pi_{t-1} - \pi_{t-1,t-1} = \lambda\theta L v_t$, and therefore $mc_{t,t-1} = \frac{\rho^2 L^2}{1-\rho L} v_{t-1}, \pi_{t,t} = \frac{\lambda}{1-\beta\rho}\frac{\rho L}{1-\rho L} v_t$. Finally

$$\pi_t = \tilde{\pi} + \pi_{t,t} = \lambda(\theta + \frac{1}{1-\beta\rho}\frac{\rho L}{1-\rho L} v_t) \tag{G.21}$$

as in Nimark (2008).

A full check that this does represent the aggregate solution in the limiting case requires the setting up of (G.14), which requires the calculation of $\mathbb{E}_{i,t}\pi_t$. If the variance of the idiosyncratic shock tends to ∞, then the relevant information set by Lemma 2(a) is the same as for case studied above, i.e., $\mathbb{E}_{i,t}\pi_t = \mathbb{E}_t\pi_t = \pi_{t,t}$. The system setup therefore involves $\tilde{mc}_t, \tilde{\pi}_{t-1}, mc_{t,t-1}, \pi_{t-1,t-1}$

$$\begin{bmatrix} \tilde{mc}_{t+1} \\ \tilde{\pi}_t \\ mc_{t+1,t} \\ \pi_{t,t} \\ \mathbb{E}_{i,t}\pi_{i,t+1} \end{bmatrix} = \begin{bmatrix} \rho & -\frac{\rho^2}{\lambda\theta} & 0 & 0 & 0 \\ \lambda\theta & -\rho & 0 & 0 & 0 \\ 0 & \frac{\rho^2}{\lambda\theta} & \rho & 0 & 0 \\ 0 & \frac{\rho}{\theta(1-\beta\rho)} & \frac{\lambda}{1-\beta\rho} & 0 & 0 \\ -\frac{\lambda}{\beta} & -\frac{(1-\theta)\rho}{\beta\theta^2(1-\beta\rho)} & -\frac{(1-\theta)\lambda}{\beta\theta(1-\beta\rho)} - \frac{\lambda}{\beta} & 0 & \frac{1}{\beta\theta} \end{bmatrix} \begin{bmatrix} \tilde{mc}_t \\ \tilde{\pi}_{t-1} \\ mc_{t,t-1} \\ \pi_{t-1,t-1} \\ \pi_{i,t} \end{bmatrix} + \begin{bmatrix} v_{t+1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\lambda}{\beta}\varepsilon_{i,t} \end{bmatrix} \tag{G.22}$$

Although one can work through this to show that the aggregate of the $\pi_{i,t}$ is equal to π_t, it follows from the proof of the theorem.

H Recoverability

A recent innovation in the economics literature by Chahrour and Jurado (2022) is the notion of recoverability, which they point out is a generalization of much earlier work by Kolmogorov (see Shiryayev, 1992), and which relates to situations for which the shocks are non-fundamental, so that the system of dynamic equations is non-invertible. We shall be calling on this notion subsequently because when the II solution differs from that of the PI solution, then the former will be characterized by non-invertibility (or non-fundamentalness of the shocks). The main point that they make is that if the VARMA is known, then it is possible (under mild conditions) to recover the values of all the shocks to have affected the VARMA process using the data, assuming observations over *all time*, as opposed to data only up to time t as available to economic agents in the model. In particular what this means is that for a finite set of data, one can obtain an accurate estimate of shocks that have taken place around the middle of the dataset.

To be more specific, suppose that the VARMA process is fully invertible, then the residuals as calculated above will converge to the true values of the shocks, so that the estimate of a shock at time t will be calculated using all past values of the observations. We illustrate with an example.

H.1 Fundamental and Non-fundamental MA Processes

For example, if measurements $\{m_t^E : t \geq -\infty\}$ are generated by the MA(1) process

$$m_t^E = \varepsilon_t - \theta\varepsilon_{t-1} = (1-\theta L)\varepsilon_t, \quad -1 < \theta < 1, \;\; \varepsilon_t \sim N(0, \sigma^2) \tag{H.1}$$

where L is the lag operator, then the root of $(1-\theta L)$ lies outside the unit circle and the process is fundamental.[59] Then $\varepsilon_t = \sum_{s=0}^{\infty} \theta^s m_{t-s}^E$. For a finite number of observations starting at $t = 0$, truncating this sum at $s = t$ will achieve a very close approximation (with probability 1) for values of t that are large enough to ensure that the variance of the untruncated terms, which equals $\theta^{2t}\sigma^2/(1-\theta^2)$ is below a certain threshold. However if $\theta > 1$, then the above representation is non-fundamental and cannot converge. If instead we write the lag operator representation of ε_t as $\varepsilon_t = m_t^E/(1-\theta L)$ as $\varepsilon_t =$

[59]An MA process $m_t^E = \Phi(L)\varepsilon_t$ is a fundamental representation if the roots of $\Phi(L)$ lie outside the complex unit circle (see, for example, Lippi and Reichlin, 1994 and Kilian and Lutkepohl, 2017).

$-\theta^{-1}L^{-1}m_t^E/(1-\theta^{-1}L^{-1})$, then we can rewrite the representation of the shocks as

$$\varepsilon_t = -\sum_{s=1}^{\infty} \theta^{-s} m_{t+s}^E \tag{H.2}$$

Thus recovering the shocks requires summing over *future* values of the observations. Clearly, for a finite sample of length T, one cannot obtain an accurate approximation to the most recent shock ε_T, but one can obtain a good approximation to the earliest shocks provided that T is large enough.

One can readily extend this to the MA(2) case $m_t^E = (1-\theta_1 L)(1-\theta_2 L)\varepsilon_t$ when $-1 < \theta_2 < \theta_1 < 1$. Then the process is fundamental and we have

$$\varepsilon_t = \frac{1}{\theta_1-\theta_2}\left(\frac{\theta_1}{1-\theta_1 L} - \frac{\theta_2}{1-\theta_2 L}\right) m_t^E = \frac{1}{\theta_1-\theta_2}\left(\sum_{s=0}^{\infty} \theta_1^{s+1} m_{t-s}^E - \sum_{s=0}^{\infty} \theta_2^{s+1} m_{t-s}^E\right) \tag{H.3}$$

When however $-1 < \theta_1 < 1 < \theta_2$, we can rewrite the expression for the shock as

$$\varepsilon_t = \frac{1}{\theta_2-\theta_1}\left(-\frac{\theta_1}{1-\theta_1 L} + \frac{L^{-1}}{1-\theta_2^{-1} L^{-1}}\right) m_t^E = \frac{1}{\theta_2-\theta_1}\left(\sum_{s=0}^{\infty} \theta_1^{s+1} m_{t-s}^E - \sum_{s=1}^{\infty} \theta_2^{-s+1} m_{t+s}^E\right) \tag{H.4}$$

so that recovering the shocks requires summing over *both past and future* values of the observations. For finite samples the approximating values of shocks at the beginning and end of the sample will be a poor fit to the true values.

Similarly, when $-1 < \theta_2 < 1 < \theta_1$, we have

$$\varepsilon_t = \frac{1}{\theta_1-\theta_2}\left(-\sum_{s=0}^{\infty} \theta_2^{s+1} m_{t-s}^E - \sum_{s=1}^{\infty} \theta_1^{-s+1} m_{t+s}^E\right) \tag{H.5}$$

Then when $\theta_2,\ \theta_1$ lie outside $[-1,1]$, we can rewrite the expression for the shock as

$$\varepsilon_t = \frac{1}{\theta_1-\theta_2}\left(-\frac{L^{-1}}{1-\theta_1^{-1} L^{-1}} - \frac{L^{-1}}{1-\theta_2^{-1} L^{-1}}\right) m_t^E = \frac{1}{\theta_1-\theta_2}\left(-\sum_{s=0}^{\infty} \theta_1^{-s+1} m_{t+s}^E - \sum_{s=1}^{\infty} \theta_2^{-s+1} m_{t+s}^E\right) \tag{H.6}$$

so that recovering the shocks requires summing over *only future* values of the observations. Again for finite samples the approximating values of shocks at the end of the sample will be a poor fit to the true values.

Finally, consider an ARMA(1,1) process $m_t^E = \frac{(1-\theta L)}{(1-\frac{L}{\theta})}\varepsilon_t$ for a Blaschke factor. If $\theta > 1$ this is non-fundamental. But we can write

$$\varepsilon_t \;=\; \frac{(1-\frac{L}{\theta})}{(1-\theta L)} m_t^E = \frac{(L^{-1}-\frac{1}{\theta})}{(L^{-1}-\theta)} m_t^E$$

$$= \left(1 + \frac{\theta - \frac{1}{\theta}}{L^{-1} - \theta}\right) m_t^E = \left(1 + \frac{\frac{1}{\theta^2} - \theta}{1 - \theta^{-1} L^{-1}}\right) m_t^E \tag{H.7}$$

Hence solving forward from time t we can recover the structural shock from the convergent summation

$$\varepsilon_t = m_t^E + \left(\frac{1}{\theta^2} - 1\right) \sum_{s=1}^{\infty} \theta^{-s} m_{t+s}^E \tag{H.8}$$

H.2 Blaschke Factors and Spectral Factorization

If a square non-invertible system of n stationary measurements and n shocks in each period is estimated, then although the parameters of the system can be consistently estimated using maximum likelihood, the innovations process (i.e., the residuals) will nevertheless correspond to those of the statistically equivalent invertible system. They cannot therefore be matched to a linear transformation of the structural shocks, and the same will automatically hold true when a VAR approximation to the system is estimated, since by definition the latter is invertible. The literature, summarized by Kilian and Lutkepohl (2017) suggests using Blaschke factors on the lag operator representation of the VAR in order to 'flip' roots of the MA process from invertible to non-invertible.

To see how this works, first consider the general MA process $m_t^E = \Phi(L)\varepsilon_t$ assumed to be fundamental and write

$$m_t^E = \Phi(L)\varepsilon_t = \Phi(L)B(L)B(L)^{-1}\varepsilon_t \equiv \Phi(L)^* \varepsilon_t^* \tag{H.9}$$

where $\varepsilon_t^* = B(L)^{-1}\varepsilon_t$ and $\Phi(L)^* = \Phi(L)B(L)$. Then Lippi and Reichlin (1994) show that Φ^* has roots inside the complex unit circle (so that $m_t^E = \Phi(L)^*\varepsilon_t^*$ is non-fundamental) if $B(L)$ is chosen to be a 'Blaschke matrix' which has two properties (i) all roots inside the complex unit circle and (ii) $B(L)^{-1} = B^*(L^{-1})$ where the asterik denotes the conjugate transpose. Then corresponding to our MA(2) fundamental example $\Phi(L) = (1 - \theta_1 L)(1 - \theta_2 L)$ above with $-1 < \theta_1,\ \theta_2 < 1$ we have three non-fundamental representations $\Phi(L)B(L)$ corresponding to the Blaschke factors:

$$-1 < \theta_1 < 1 < \theta_2 \quad : \quad B(L) = \frac{L - \theta_1}{1 - \theta_1 L} \tag{H.10}$$

$$-1 < \theta_2 < 1 < \theta_1 \quad : \quad B(L) = \frac{L - \theta_2}{1 - \theta_2 L} \tag{H.11}$$

$$-1 < \theta_1, \theta_2 < 1 \quad : \quad B(L) = \left(\frac{L - \theta_1}{1 - \theta_1 L}\right)\left(\frac{L - \theta_2}{1 - \theta_2 L}\right) \tag{H.12}$$

For the four possible combinations of θ_1 and θ_2 one MA(2) representation will be fundamental and the other three non-fundamental. Only the fundamental one will be captured by the data VAR estimation. If the econometricians are estimating θ_1, θ_2 they will be confronted with three non-fundamental and one fundamental processes with identical statistical properties (i.e., the same first and second moments). It therefore follows that one can only use recoverability to obtain the structural shock unambiguously if the four cases (H.3)–(H.6) can be separated by the econometrician by prior information on the location of θ_1 and θ_2.

H.3 A Further Test of Fundamentalness

Lippi and Reichlin (1994), Fernandez-Villaverde *et al.* (2007), Kilian and Lutkepohl (2017) and others, have pointed out that non-invertibility is a missing information problem arising from econometricians not using the appropriate measurements. Choosing the right measurements may then alleviate the problem. Closely related to this idea and also to recoverability is a recent paper by Canova and Sahneh (2017), that shows how to test the residuals of a VAR model for fundamentalness. Suppose that a VARMA process m_t^E in shocks ε_t is estimated in the VAR form $\Phi(L)m_t^E = u_t$, where u_t are the residuals; then a linear transformation is applied to u_t in order to attempt to recover an approximation e_t to the structural shocks ε_t. However in principle there is no way that one can determine whether e_t is a linear transformation of the structural shocks ε_t using the VAR alone.

But suppose that there is an additional measurement m_{2t}^E available to the econometrician of the form $m_{2t}^E = \Theta_1(L)\varepsilon_t + \Theta_2(L)\varepsilon_{2t}$, which is dependent on the same shocks ε_t as the main variables m_t^E, and some additional shocks ε_{2t}. If there is no invertibility problem for m_t^E estimated as a VAR, then m_{2t}^E can be rewritten (as $t \to \infty$) as

$$m_{2t}^E = \Theta_1(L)e_t + \Theta_2(L)\varepsilon_{2t} \tag{H.13}$$

If there is an invertibility problem then (H.13) no longer applies, because at least one element of ε_t depends on future values of e_t via one or more Blaschke factors[60]. Thus conducting a standard Granger causality test of whether m_{2t}^E depends on future values of the recorded residuals e_t is sufficient to deduce whether the latter are fundamental or not.

I "Noisy News" Models

This section reviews two single agent models which explore the econometric implications of information assumptions in DSGE models: Blanchard *et al.* (2013) and Forni *et al.* (2017). The former model begins by writing productivity a_t as a sum of a permanent

[60]Suppose for example that $y_t = (1-\alpha^{-1}L)\varepsilon_t$, where $\alpha < 1$, so that it is non-invertible. After this is estimated as a finite VAR, it can then be approximately written as $y_t = (1-\alpha L)e_t$. It follows that $\varepsilon_t = \frac{(1-\alpha L)}{(1-\alpha^{-1}L)}e_t = \frac{-\alpha L^{-1}(1-\alpha L)}{(1-\alpha L^{-1})}e_t$, so that it is dependent on future values of e.

component x_t following a root process and a AR1 transitory component z_t as follows

$$\begin{aligned} a_t &= x_t + z_t && \text{(I.1)} \\ \Delta x_t &= \rho_x \Delta x_{t-1} + \varepsilon_t && \text{(I.2)} \\ z_t &= \rho_y z_{t-1} + \eta_t && \text{(I.3)} \end{aligned}$$

Then with the assumptions that $\rho_x = \rho_z \equiv \rho$ and $\rho\sigma_\varepsilon^2 = (1-\rho)^2\sigma_\eta^2$ it can be shown that $\mathbb{E}[a_{t+1}|a_t, a_{t-1}, \cdots] = a_t$; i.e., a_t follows a random walk.

The information assumptions for agents are that they observe a_t and receive a noisy signal about the permanent component x_t given by

$$s_t = x_t + \nu_t\,;\nu_t \sim n.i.i.d(0, \sigma_\nu^1) \tag{I.4}$$

Consumers are assumed to set c_t equal to long-run productivity expectations

$$c_t = \lim_{j\to\infty} \mathbb{E}[a_{t+j}|I_t] = \lim_{j\to\infty} \mathbb{E}[x_{t+j}|I_t] \tag{I.5}$$

(I.2) and (I.5) lead to

$$\begin{aligned} \lim_{j\to\infty} \mathbb{E}_t[x_{t+j} - x_t] &= \frac{\rho}{1-\rho}\mathbb{E}_t[x_t - x_{t-1}] \\ &= c_t - \mathbb{E}_t x_t \\ &\Rightarrow\; c_t = \frac{1}{1-\rho}(\mathbb{E}_t[x_t] - \rho\mathbb{E}_t[x_{t-1}]) && \text{(I.6)} \end{aligned}$$

The model in now in state-space form with a state vector $[a_t,\, x_t,\, z_t,\, c_t]'$, $m_t^A = [a_t,\, s_t]'$ and shock $[\varepsilon_t,\, \eta_t,\, \nu_t]'$. It is in the form given by our general procedure in Section (2.4) to give a II-SA solution. Clearly with more shocks than observables it is not A-invertible. For the econometrician, Blanchard *et al.* (2013) consider $m_t^E = [c_t,\, a_t]'$ or $m_t^E = [c_t,\, a_t,\, s_t]'$ but from Theorem 3 neither can be E-invertible and have a VAR representation for the RE solution.

Forni *et al.* (2017) replace the exogenous shock component of the model (I.1)-(I.3) with simply

$$a_t = a_{t-1} + \varepsilon_{t-1} \tag{I.7}$$

and (I.5) becomes

$$c_t = \mathbb{E}[a_{t+1}|I_t] \tag{I.8}$$

The rest of the model is unchanged. This simpler set-up is more tractable. In fact from (I.7) and (I.8) we have

$$c_t = \mathbb{E}[a_t] + \mathbb{E}[\varepsilon_t] \tag{I.9}$$

so there are no intrinsic dynamics as defined in Theorem 6.

The state-space form of the RE solution then gives the **II-SA solution**:

$$\begin{bmatrix} \Delta a_t \\ \Delta c_t \\ s_t \end{bmatrix} = \begin{bmatrix} L & 0 \\ \gamma + (1-\gamma)L & \gamma(1-L) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ v_t \end{bmatrix} \tag{I.10}$$

In the absence of noise, $v_t = \sigma_v^2 = 0$, $\gamma = 1$ and agents observe the shock and we have PI. Then the **PI solution** is

$$\Delta c_t = \varepsilon_t \tag{I.11}$$

and after a shock consumption jumps immediately to its new long-run level. But with II consumption jumps to $c_t = \gamma\varepsilon_t$ in the first period and reaches $c_{t+1} = c_t + (1-\gamma)\varepsilon_t = c_{t-1} + \varepsilon_t$.

Returning to II-SA, the spectrum of the two process $\Delta a_t, s_t$ is given by

$$\mathbb{E}\left[\begin{bmatrix} L\varepsilon_t \\ \varepsilon_t + \nu_t \end{bmatrix} [L^{-1}\varepsilon_t \;\; \varepsilon_t + \nu_t]\right] = \begin{bmatrix} \sigma_\varepsilon^2 & L\sigma_\varepsilon^2 \\ L^{-1}\sigma_\varepsilon^2 & \sigma_\varepsilon^2 + \sigma_\nu^2 \end{bmatrix}$$

(See Appendix A.3.) It is easy to show that an alternative spectral factorization of this joint process is

$$\begin{bmatrix} 1 & L\frac{\sigma_\varepsilon^2}{\sigma_s^2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_s^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L^{-1}\frac{\sigma_\varepsilon^2}{\sigma_s^2} & 1 \end{bmatrix}$$

where $\sigma_u^2 = \sigma_\varepsilon^2\sigma_\nu^2/(\sigma_\varepsilon^2 + \sigma_\nu^2)$.

So starting with

$$\begin{bmatrix} \Delta a_t \\ s_t \end{bmatrix} = \begin{bmatrix} L & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ v_t \end{bmatrix} \tag{I.12}$$

we arrive at the representation

$$\begin{bmatrix} \Delta a_t \\ s_t \end{bmatrix} = \begin{bmatrix} 1 & L\frac{\sigma_\varepsilon^2}{\sigma_s^2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ s_t \end{bmatrix} \tag{I.13}$$

where it is easy to show that

$$\begin{bmatrix} u_t \\ s_t \end{bmatrix} = \begin{bmatrix} L\frac{\sigma_v^2}{\sigma_s^2} & -L\frac{\sigma_\varepsilon^2}{\sigma_s^2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ v_t \end{bmatrix} \tag{I.14}$$

gets us back to (I.12).

(I.12) and (I.14) have a root $r = 0$ and are non-fundamental. But the MA representation (I.13) has a determinant equal to 1 and is therefore fundamental. In estimating a VAR for Δa_t and s_t the econometrician can generate IRFs for u_t and the signal s_t but not the structural shocks ε_t and v_t. However, these shocks are recoverable in the sense of the term proposed by Chahrour and Jurado (2022) To see this use (I.14) to obtain

$$\begin{bmatrix} \varepsilon_t \\ s_t \end{bmatrix} = \begin{bmatrix} L\frac{\sigma_v^2}{\sigma_s^2} & -L\frac{\sigma_\varepsilon^2}{\sigma_s^2} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} u_t \\ s_t \end{bmatrix} = \begin{bmatrix} L^{-1} & \frac{\sigma_\varepsilon^2}{\sigma_s^2} \\ -L^{-1} & \frac{\sigma_v^2}{\sigma_s^2} \end{bmatrix}^{-1} \begin{bmatrix} u_t \\ s_t \end{bmatrix} \tag{I.15}$$

Thus the structural shocks at time t within the sample can be recovered by the econometrician using future data at times $t+1$, $t+2\ldots$ which is available to her within sample, but not of course available to the agents in the model. Thus recoverability is possible in this particular simple example, but as Theorem 5 shows, this results depends on the absence of intrinsic dynamics.

A Blaschke factor features in this representation as follows. Consider a general specification $\Delta a_t = C(L)\varepsilon_t$ where $C(L)$ is a rational function with $C(0) = 0$. In our model above $C(L) = L$. Then (I.12) becomes

$$\begin{bmatrix} \Delta a_t \\ s_t \end{bmatrix} = \begin{bmatrix} C(L) & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ v_t \end{bmatrix} \tag{I.16}$$

Let $r_j, j-1, ..., n$ be those roots of $C(L)$ within the unit circle and let r_j^* be the complex conjugate of r_j. Then generalize (I.13) to

$$\begin{bmatrix} \Delta a_t \\ s_t \end{bmatrix} = \begin{bmatrix} \frac{C(L)}{B(L)} & C(L)\frac{\sigma_\varepsilon^2}{\sigma_s^2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ s_t \end{bmatrix} \tag{I.17}$$

where

$$\begin{bmatrix} u_t \\ s_t \end{bmatrix} = \begin{bmatrix} B(L)\frac{\sigma_v^2}{\sigma_s^2} & -B(L)L\frac{\sigma_\varepsilon^2}{\sigma_s^2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ v_t \end{bmatrix} \tag{I.18}$$

where $B(L)$ is a Blaschke factor

$$B(L) = \prod_{j=1}^{n} \frac{L - r_j}{1 - r_j^* L} \tag{I.19}$$

Then (I.17) is fundamental because $\frac{C(L)}{B(L)} = 0$ only for $|L| \geq 1$. Note in our simple model $r_1 = 0$ and $B(L) = L$.

We can now estimate the shock processes in a DSGE model which is not E-invertible, owing to the failure of A-invertibility. Maximum likelihood estimation of the parameters will generate an innovations process, equal in the limit to the residuals from the estimation of a VAR.[61] The theoretical econometrician will, at least in our simple examples, be able to

[61] The a-theoretical econometrician will mistake VAR estimation for a VAR in the reduced-form structural shocks and make a misleading comparison with the IRFs of the assumed DSGE model, even if the algorithm of Forni *et al.* (2017) is used.

work out the Blaschke factors and convert the innovations process into structural shocks, assuming data from $-\infty$ to $+\infty$. While the Blaschke factors cannot be directly estimated because their second moments are the same as white noise, they can be calculated from the estimated parameters.

For Blanchard *et al.* (2013), these conclusions demonstrate the limits of SVAR estimation and the need for the estimation of the structural (DSGE) model. They estimate a medium-sized NK model similar to Smets and Wouters (2007) by Bayesian-maximum-likelihood methods. The estimation uses seven US time series (GDP, consumption, investment, employment, the federal funds rate, inflation and wages) and eight shocks. The RE solution is of the form II-SA described in our paper and is not A-invertible. But since SVARs are avoided altogether this is of no consequence. Validation in such an exercise would compare second moments in the model with those in the data rather than the impulse responses of the estimated model and an estimated SVAR.

J Dynare Based Toolkit

Levine *et al.* (2020) describes the working and use of the Imperfect Information (Partial Information)[62] software that solves, simulates and estimates DSGE RE models in Dynare under II. The software is a MATLAB based toolbox and is integrated into Dynare version 4.6.1. The solution techniques adopted are based on the work by Pearlman *et al.* (1986). In particular, the software

1. Transforms Dynare's linearized model solutions into the Blanchard-Kahn form which is solved to yield a reduced-form system. See Theorem 1 of the paper.

2. Provides the conditions for invertibility under which II is equivalent to PI. See Theorem 3 of paper.

3. Implements multivariate measures of goodness of fit of the innovation residuals to the fundamental shocks, and provides information as to how well VAR residuals correspond to the fundamentals in DSGE models. See Theorem 7 of paper.

4. Simulates the model and uses the resulting reduced-form solution to obtain theoretical moments and IRFs

5. Evaluates the reduced-form system via the Kalman filter to obtain the likelihood function for estimation purposes and results from an identified DSGE-VAR.

[62] Different terminologies are found in the literature (see the discussion in the Introduction). Most DSGE models are solved on the assumption that agents have PI of the current state as an endowment. This is the default option in Dynare. Under II, this assumption is relaxed.