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AMBIGUITY AVERSION, PORTFOLIO CHOICE, AND LIFE EXPECTANCY

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Ambiguity Aversion, Portfolio Choice, and Life Expectancy*

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Abstract

This paper studies how wealth and aging affect portfolio choices in a life-cycle model with ambiguity aversion. Ambiguity aversion implies that wealthier and older agents are endogenously more optimistic about risky asset returns, relative to poorer younger agents. As life expectancy grows, old agents become even more optimistic, while young agents become more pessimistic, amplifying the age gaps in portfolio composition, and implying further increases in intergenerational inequality. We find evidence for the mechanism in survey data on portfolios and subjective life expectancy. In a quantitative extension of the model, plausible life expectancy projections imply a 26% increase in the age-gradient of conditional risky asset shares between 2019 and 2100.

JEL codes: D84, E21, G11, J11

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1 Introduction

When faced with Knightian uncertainty, ambiguity averse agents over-weight the probability of the 'worst case' outcomes with low utility realizations (Gilboa and Schmeidler, 1989; Hansen and Sargent, 2008). These belief distortions differ according to the agent's environment, preferences, and characteristics: the worst case is very different, for example, for wealthy and poor households (Michelacci and Paciello, 2020, 2023), or between different policy regimes (Ferriere and Karantounias, 2019). As a consequence, when an economy goes through a structural shift, the belief distortions due to ambiguity aversion may change, with implications for a range of aggregate and distributional outcomes.

In this paper, we explore these effects for one of the key shifts taking place in developed economies today: population aging. In particular, we ask how increased lifespans affect ambiguity-averse beliefs about asset returns, and what that implies for portfolio choices and asset prices. This is likely to be important for policy in the coming decades, as savings behavior has been at the forefront of recent discussions of economic policy under demographic change (Goodhart and Pradhan, 2020; Auclert et al., 2021; Kopecky and Taylor, 2022). Moreover, there is substantial empirical evidence that ambiguity aversion is a key determinant of portfolio choices and asset prices (Antoniou et al., 2015; Dimmock et al., 2016; Collard et al., 2018; Corgnet et al., 2020).

To analyze the interaction between aging, portfolio choice, and ambiguity aversion, we build an overlapping-generations model with ambiguity over risky asset returns. We find that increases in life expectancy cause young agents to distort their beliefs more strongly towards pessimistic outcomes, while older agents in contrast become more optimistic. Population aging therefore increases the concentration of equity holdings among older households, as their relatively more optimistic beliefs drive them to choose higher risky asset shares than the young. This in turn implies older generations become relatively wealthier, as they earn greater asset returns than younger agents.

This prediction is not present in models with aging that do not consider ambiguity aversion. To test it, we turn to survey data from the US. Consistent with the model, we find that the *age gradient* of equity shares is strongly increasing in subjective life expectancy. That is, among young households, an expectation of a longer lifespan is associated with less risky portfolios. Among older households, that pattern is reversed. This mechanism is important: a quantitative version of the model predicts that plausible longevity increases over the next 80 years will cause the age-profile of risky asset shares to steepen by 26%.

As in Eggertsson et al. (2019) and Malmendier et al. (2020), we mostly focus on a simple case of the model with maximum three-period lifespans. This allows us to obtain analytic results, and inspect the mechanisms at work. We model ambiguity using so-called 'multiplier preferences', in which agents whose payoffs are more exposed to ambiguity distort their beliefs more strongly towards low-utility states (Hansen et al., 1999). Although all investors view low returns as the worst case, a given drop in returns is much more damaging to some investors than others, and those who would be particularly badly affected do more to insure themselves against that possibility.[1]

We begin our analysis in a small open economy aging alone, in which safe asset returns and the distribution of risky asset returns are held fixed. We find that ambiguity aversion endogenously generates return expectations which are increasing in wealth and age. This is consistent with survey evidence on return expectations (Giglio et al., 2021), experiments on biases in financial decision-making (Kovalchik et al., 2005), and the observation that young and poor households typically hold less risky portfolios than those further up the age and wealth distribution (Chang et al., 2018; Catherine, 2022). Note that we match this last result despite the fact that, in the absence of ambiguity, the model reduces to a simple Merton (1969)-style model in which risky asset shares are constant for all agents.

Importantly, the effects of age on responses to ambiguity are driven by changes in *life expectancy*, rather than the number of years an agent has lived. As a consequence, reductions in mortality rates for older agents cause changes in the age profile of risky asset shares. Specifically, as mortality rates fall, young and old agents adjust their return expectations in opposite directions. The young become more pessimistic, distorting beliefs more strongly towards low risky asset returns, and investing less as a result - while simultaneously older agents become more optimistic, and invest more. As populations age around the world, this has consequences for the future of intergenerational inequality and the composition of asset demand.

These endogenous changes in responses to ambiguity with wealth and life expectancy come about because of the interaction of two distinct channels. The first is the 'wealth channel', in which an agent who is saving more is more exposed in monetary terms to low asset returns. They have more 'skin in the game', and so have a greater desire to make their decisions robust to returns ambiguity, implying more pessimistic expectations.[2] The second is the 'marginal utility channel', in which an agent who expects to have a high marginal utility of consumption in the next period suffers a greater utility loss from a given fall in wealth. A poor agent may not lose much in monetary terms from a fall in returns, but due to the curvature of standard utility functions, they have a high marginal

[1]Several existing models of ambiguity aversion in portfolio choice instead assume all investors distort beliefs to the lowest return among a given set of possibilities, the range of which is typically specified as an exogenous aspect of preferences. This distinction is discussed further in the Related Literature section below.

[2]This echoes the literature arguing wealthier households have more incentive to process information about asset returns, for the same reason (Arrow, 1987; Lei, 2019; Macaulay, 2021).

utility of consumption, and so of wealth. A small monetary loss can therefore have large utility consequences for these agents.

As life expectancy increases, younger agents save more, to fund consumption in their now-extended old age. They also expect to do the same throughout their middle age, implying lower consumption in the immediate future. With a diminishing marginal utility of consumption, this means that through both channels they become more exposed to shortfalls in asset returns, so they distort their beliefs more towards low returns to make their decisions more robust to ambiguity. As a result they invest less in risky assets, and instead allocate more of their savings to the risk-free asset.

For agents in middle age, the wealth channel operates in the same way. However, the marginal utility channel is reversed. When the probability of surviving into old age is low, they do not save much for those potential future periods. Conditional on survival, their old-age consumption is therefore low. As the mortality rate falls and life expectancy rises, they save more for their old age, which implies greater consumption in that period. As such, rising life expectancy reduces the expected marginal utility of consumption in old age, implying utility is *less* exposed to a given fall in wealth. Through the marginal utility channel, middle-aged agents therefore become more optimistic about asset returns, and increase their portfolio share in risky assets.

Which of these channels dominates is regulated by a simple condition: with CRRA preferences, the marginal utility channel dominates if and only if the elasticity of intertemporal substitution (EIS) is less than 1. In that case marginal utility changes sharply with (future) wealth, and so outweighs the wealth channel. Intuitively, this is related to the standard result that income effects of interest rate changes dominate substitution effects when the EIS is less than 1 (see Flynn et al., 2023, for an extended discussion). Substitution effects are small when marginal utility is highly convex, and this is precisely when our marginal utility channel is large. Attempts to measure the EIS among households typically find values below 1 (Havránek, 2015), so we take this as our baseline.

A similar logic drives the effects of wealth on return expectations. An increase in wealth implies agents save more, making them more exposed to return fluctuations. At the same time, they expect to be wealthier in the future, which reduces their expected marginal utility of wealth. As with changes in life expectancy, the marginal utility channel dominates whenever the EIS is less than 1, in which case wealth reduces the extent to which agents distort return beliefs due to ambiguity aversion.

After characterizing these channels, we extend the model to a closed economy, in which the equity premium is endogenous. Initially, as life expectancy rises from a low level, the dominant force is the increasing optimism of the middle-aged agents. Aggregate demand for risk rises, and the equity premium falls. However, as life expectancy continues to grow, increases in middle-aged optimism slow down, and are eventually dominated by the greater pessimism of the young. Past a certain threshold, aggregate demand for risky assets begins to decline, and the equity premium rises as a result. This occurs because the effects of age on beliefs are smaller in the model for agents with more wealth. As mortality rates fall, young agents save more for their old age, middle-aged agents become wealthier, and so middle-aged agents become increasingly unresponsive to further increases in life expectancy. Interestingly, although the model is very stylized, this result is consistent with the U-shaped evolution of the equity premium observed across developed economies since 1950 (Kuvshinov and Zimmermann, 2020).

To test this mechanism, we use survey data to explore one of the most striking model predictions: that greater life expectancy is associated with a steeper age-gradient of risky asset shares (assuming an EIS below 1). This prediction is not present in other models in this literature, so offers a useful way to distinguish our model mechanism from others in the literature. In the Household Finance module of the US Survey of Consumer Expectations, households are asked about their portfolios and their subjective life expectancy. Using simple regressions of risky asset shares on age, interacted with subjective life expectancy, we find strong support for the model's prediction. The age-gradient of risky asset shares is close to zero for those with low subjective life expectancy, but is large and positive for those with high life expectancy.

Finally, we end with an illustration of how these effects might play out with plausible degrees of demographic change in the coming decades. We embed our model of ambiguity aversion into an otherwise-standard quantitative portfolio choice model, based on Gomes and Michaelides (2005), which includes risky age-dependent labor income, Epstein-Zin preferences, and equity market participation costs. We calibrate the model to data on mortality rates in the US in 2019, and to our regression results from the Survey of Consumer Expectations. The model produces risky portfolio shares with a similar level and age-gradient as those observed in the data, despite us not targeting these moments in the calibration. With this model, we simulate the effect of an increase in life expectancy, replacing the calibrated 2019 mortality rates with projected mortality rates for 2100. As in the analytical model, young agents become more pessimistic about asset returns relative to older agents, so the gap between the risky asset shares of old and young widens. Comparing agents at ages 80 and 35, the increase in life expectancy increases the gap in their risky asset shares by 26%. The simulated demographic changes to 2100 therefore have substantial consequences for portfolio decisions.

Related Literature: We principally contribute to the literature on ambiguity aversion, demographic change, and life-cycle portfolio choice in macroeconomics and finance.

Ambiguity aversion has been successful in providing theoretical explanations for a number of phenomena in macroeconomics and finance (see reviews in Ilut and Schneider (2022) and Epstein and Schneider (2010)). Our work is particularly related to models in which agents can endogenously adjust their response to ambiguity based on their own exposure to the variable(s) in question (Hansen et al., 1999; Cagetti et al., 2002; Bidder and Smith, 2012).[3] In particular, Michelacci and Paciello (2020, 2023) show that with ambiguity aversion wealthy and poor households hold systematically different expectations of aggregate variables. This explains several features of survey data on expectations, and influences macroeconomic dynamics. Our model extends these insights to portfolio choice, and shows that demographic changes therefore affect inequality and the equity premium.

Several of our results for how the belief distortions driven by ambiguity change with age and wealth depend on whether the elasticity of intertemporal substitution is greater or less than 1. In this we therefore add to the insights of Ferriere and Karantounias (2019) and Balter et al. (2022), who show that the same condition determines the outcome of models of optimal fiscal policy and momentum in asset return expectations respectively, once ambiguity aversion is present. Recent empirical evidence has tended to favor an EIS substantially below 1 (Havránek, 2015; Crump et al., 2022), so we take this as the baseline case for our analysis.

In addition, our results are relevant for the literature on how demographic change will affect asset markets and inequality. The link between demography and asset markets was made prominent by a series of papers attempting to forecast what would happen as the large 'baby boomer' generation aged (the so-called 'asset market meltdown hypothesis', e.g. Poterba, 2001; Abel, 2001, 2003; Geanakoplos et al., 2004; DellaVigna and Pollet, 2007). This literature focuses largely on the consequences of changes in the relative size of older and younger investor populations. In contrast, the mechanisms we study concern how portfolio decisions *conditional on age* may change as expected lifespans grow.

This is important, because a variety of papers have studied demographic effects on aggregate asset demand by holding age profiles of asset holdings or savings rates fixed,

[3]These examples, like us, use so-called 'multiplier preferences', in which agents distort beliefs towards a worst-case model, subject to a penalty of belief distortions which is linear in the relative entropy between the central and worst-case model used. An alternative approach that also allows endogenous continuous adjustment of belief distortions is the smooth ambiguity preferences of Klibanoff et al. (2005, 2009), adapted to macroeconomic settings in Altug et al. (2020). Chen et al. (2014) use this in a model of portfolio choice, but do not study changes over the life-cycle or with age. Multiple-priors models (as in Gilboa and Schmeidler, 1989) also feature endogenous belief distortions, but the optimal distortion is typically a corner solution. In Michelacci and Paciello (2020, 2023), for example, distortions to the perceived probability of certain shocks are either positive or negative, depending on a household's characteristics. Within those for whom a negative shock would harm utility, there are no variations in beliefs. In models like ours where the ambiguity is over asset returns, lower returns are always worse for utility, so multiple-priors models deliver a constant belief distortion for all agents.

and changing the proportions of households within each age group in line with past demographic data, or future projections (e.g. Mankiw and Weil, 1989; Mian et al., 2021).[4] This approach only yields sufficient statistics for aggregate asset demand if household decision rules depend on that household's age, but are otherwise independent of aggregate demographic change (Auclert et al., 2021). We show that under ambiguity aversion, that is not the case, as changes in life expectancy affect decision rules.

Finally, we also relate to the large literature on portfolio choice over the life cycle (see Gomes et al., 2021, for a review). Within this literature, a number of papers have proposed mechanisms to explain why older households typically invest the same or greater shares of their wealth in risky assets than young households. This pattern, while not present in standard life-cycle models (Cocco et al., 2005; Gomes, 2020), can be generated by declines in labor market uncertainty as households age (Chang et al., 2018), or the cyclicality of return skewness (Catherine, 2022). Indeed, like us, Campanale (2011) and Peijnenburg (2018) offer explanations of the data based on ambiguity aversion.

We view the mechanism in this paper as complementary to these other forces. The key conceptual distinction between us and the existing literature is that in many of these previous papers, portfolio choices depend explicitly on the number of years an agent has lived to date. In contrast, the endogenous responses to ambiguity in our model imply that return expectations and portfolio decisions depend on the number of years an agent *expects* to live in the future.[5] In Peijnenburg (2018), for example, savers face Knightian uncertainty over a bounded interval of possible mean asset returns. With every period of life, they observe some returns data, and so are able to shrink that interval. Ambiguity-averse agents in that model set return expectations to the lower bound of the plausible interval, so the learning results in expected risky asset returns that rise with age.[6] In our model, we instead consider a constant preference for robustness, and abstract from reductions in ambiguity through learning.[7] In this environment, we show that older households are typically less vulnerable to return shortfalls, and so choose to react less to their ambiguity. This mechanism therefore implies ambiguity aversion can generate an upward-sloping age profile of risky asset shares, even if young poor households learn

[4]An alternative approach analyses quantitative life-cycle models with rational expectations (e.g. Carvalho et al., 2016; Kopecky and Taylor, 2022), which also abstract from the mechanisms we study.

[5]Note that this implies the same mechanisms operate at the individual level for any increase in subjective survival probabilities, even if that is not reflected in objective reality (such deviations are studied in e.g. Grevenbrock et al. (2021)). We abstract from this in this model to highlight the effects through returns ambiguity, though we make use of it in testing the model predictions in survey data.

[6]Similarly, in Chang et al. (2018) labor market uncertainty declines as households age, which enables them to take more risk in their asset portfolios.

[7]A related model with multiplier preferences is Maenhout (2004). However, in that model the preference for robustness is normalized by wealth. While this normalization makes the model more tractable, it also assumes away many of the changes in belief distortions we study, and so abstracts from the mechanisms in our model.

nothing about asset markets, as would be the case if they choose not to pay attention to them (as in e.g. Lei, 2019; Macaulay, 2021). Importantly, our mechanism also means that the age-profile of risky asset shares changes with life expectancy: among those with greater (subjective) life expectancy, older households are even more optimistic about asset returns relative to the young, and so the age gradient of risky asset shares is steeper. This accounts for the facts we document in survey data, which cannot be explained with other models in this literature.

The rest of the paper is organized as follows. Section 2 sets out the model environment with a general maximum lifespan. Section 3 characterizes the effects of ambiguity aversion and demographic change analytically in the special case where the maximum lifespan is 3 periods. Section 4 tests predictions of the model in survey data. Section 5 returns to the model, adds standard features of quantitative portfolio-choice models, and explores the implications for the age profile of risky asset shares both now and in the future. Section 6 concludes.

2 Model

2.1 Environment

Demographics: Time is discrete. In each period t, a continuum of agents with measure 1 are born with age $j = 0$. An agent of age j survives to age $j+1$ in the next period with exogenous probability ϕ_j. We set $\phi_J = 0$, implying a maximum age of J. There is no population growth.

Preferences: An agent of age j chooses consumption and their portfolio allocation to maximize expected discounted lifetime utility:

$$U_{j,t} = E_{j,t} \sum_{k=j}^{J} \left[\beta^{k-j} \Phi_{j,k} \frac{c_{k,t+k-j}^{1-\gamma}}{1-\gamma} \right] \tag{1}$$

where β is the discount factor, $c_{j,t}$ is the consumption of an agent with age j in period t, and γ is relative risk aversion. $\Phi_{j,k}$ is the cumulative probability of surviving to age k, conditional on having survived to age j, defined as:

$$\Phi_{j,k} = \begin{cases} 1 & \text{for } k = j \\ \prod_{x=j}^{k-1} \phi_x & \text{for } k > j \end{cases} \tag{2}$$

Ambiguity aversion affects choices because it distorts the expectations operator $E_{j,t}$ away from the expectations calculated under the objective probability distribution of future outcomes. Ambiguity averse agents overweight the probabilities of future states with low utility and underweight states with high utility (Gilboa and Schmeidler, 1989; Hansen and Sargent, 2008).

Endowment and Savings: Agents are born with no financial assets. They receive a stream of deterministic labor income $\{y_{j,t}\}_j^J$.

There are two assets available for savings: a risk-free bond with gross interest rate R^f, and a risky asset with a gross return of R_t. The return on the risky asset is such that $\log(R_t)$ has an i.i.d. Normal distribution with mean $\tilde{\mu}$ and variance σ^2. For the results below, it will be helpful to define $\mu = \tilde{\mu} + \sigma^2/2$, where μ is the logarithm of $E_t R_{t+1}$.

Denote $s_{j,t}$ and $d_{j,t}$ as the amount of risky assets and risk-free bonds purchased by an agent of age j in period t. The budget constraint therefore reads:

$$c_{j,t} + s_{j,t} + d_{j,t} = R_t s_{j-1,t-1} + R^f d_{j-1,t-1} + y_{j,t} \tag{3}$$

Define financial wealth $x_{j,t}$ as the right hand side of equation (3):

$$x_{j,t} \equiv R_t s_{j-1,t-1} + R^f d_{j-1,t-1} + y_{j,t} \tag{4}$$

Furthermore, define human wealth $h_{j,t}$ as the present discounted value of future labor income:

$$h_{j,t} \equiv \sum_{k=j+1}^{J} \frac{y_{k,t+k-j}}{(R^f)^{k-j}} \tag{5}$$

Following Angeletos (2007), we define the agent's 'effective wealth' $w_{j,t}$ as the sum of financial and human wealth:

$$w_{j,t} \equiv x_{j,t} + h_{j,t} \tag{6}$$

Finally, we impose that $w_{J+1,T+1} \geq 0$, where T denotes the time period in which the agent will be age J. This prevents agents from borrowing when they know for certain they are in their final period. There are no bequests, so by assumption if an agent dies with positive asset holdings those assets are destroyed.

This setup is somewhat simpler than standard life-cycle portfolio choice models, such as Gomes and Michaelides (2005). In particular, we abstract from income risk, bequests, and risky asset participation costs. These simplifications allow us to obtain an analytic solution for portfolio choices. We view this as crucial for understanding the mechanisms behind the interactions between aging and ambiguity aversion.[8] We include these standard quantitative model features in Section 5, and find that our results from the analytic model are qualitatively unchanged.

2.2 Value Functions

Let $\alpha_{j,t} \in [0, 1]$ be the share of the agent's saving out of period-t effective wealth invested in risky assets:[9]

$$\alpha_{j,t} = \frac{s_{j,t}}{w_{j,t} - c_{j,t}} = \frac{s_{j,t}}{s_{j,t} + d_{j,t} + h_{j,t}} \tag{7}$$

Given this, we can express the budget constraint in terms of effective wealth only:

$$\begin{aligned} w_{j+1,t+1} &= R_{t+1}s_{j,t} + R^f d_{j,t} + y_{j+1,t+1} + h_{j+1,t+1} \\ &= (w_{j,t} - c_{j,t})[\alpha_{j,t}R_{t+1} + (1 - \alpha_{j,t})R^f] \end{aligned}$$

where we use $y_{j+1,t+1} = R^f h_{j,t} - h_{j+1,t+1}$, which follows from the definition of $h_{j,t}$.

The agent's utility maximization problem can now be written as:

$$V_j(w_{j,t}) = \max_{c_{j,t},\alpha_{j,t}} \left\{ \frac{c_{j,t}^{1-\gamma}}{1-\gamma} + \beta\phi_j E[V_{j+1}(w_{j+1,t+1})] \right\} \tag{8}$$

subject to:

$$w_{j+1,t+1} = (w_{j,t} - c_{j,t})R^p_{j,t+1} \tag{9}$$

$$w_{J+1,T+1} \geq 0 \tag{10}$$

The initial condition is that $w_{0,t_0} = y_{0,t_0} + h_{0,t_0} = \sum_{k=0}^{J} \frac{y_{k,t_0+k}}{(R^f)^k}$, where $t_0 \equiv t - j$ denotes the time period in which the agent is born. $R^p_{j,t+1}$ is the agent's total return on their portfolio:

$$R^p_{j,t+1} \equiv R^f + (R_{t+1} - R^f)\alpha_{j,t} \tag{11}$$

Note that since $R^p_{j,t+1}$ is a weighted average over a constant R^f and a log-normal variable R_t, it is not itself log-normal. We follow Campbell (1993) and proceed with a log-linear approximation to the relationship between log portfolio returns and log individual-

[8] A further advantage of this framework is that, in the absence of ambiguity aversion, the model implies a constant portfolio share in risky assets for all ages (Merton, 1969). All changes we find in portfolio composition over an agent's life cycle must therefore come from ambiguity aversion.

[9] In all of the analysis here and in Section 3 we focus on the interior solution where these constraints on $\alpha_{j,t}$ do not bind. They will however become relevant in Section 5.

asset returns, taken about the point with zero excess returns:[10]

$$\log(R^p_{j,t+1}) \approx r^f + \alpha_{j,t}(r_{t+1} - r^f) + \frac{1}{2}\alpha_{j,t}(1-\alpha_{j,t})\sigma^2 \tag{12}$$

where r_{t+1} and r^f denote the log returns on the risky and safe assets respectively. With this approximation, the budget constraint (9) becomes:

$$w_{j+1,t+1} = (w_{j,t} - c_{j,t}) \exp\left[r^f + \alpha_{j,t}(r_{t+1} - r^f) + \frac{1}{2}\alpha_{j,t}(1-\alpha_{j,t})\sigma^2\right] \tag{13}$$

Solution Without Ambiguity: If there is no ambiguity, the expectations operator $E_{j,t}$ coincides with expectations formed under the objective probability distribution of returns. Proposition 1 gives the optimal consumption and portfolio allocations in this case.

Proposition 1 *Solving the household optimization (8) subject to (13) and (10), without ambiguity aversion, implies a value function of the form:*

$$V_j(w_{j,t}) = A_j \frac{w_{j,t}^{1-\gamma}}{1-\gamma} \tag{14}$$

where

$$A_j = \begin{cases} \left(\frac{1+b_{j+1}}{b_{j+1}}\right)^\gamma & \text{for } j = 0, ..., J-1 \\ 1 & \text{for } j = J \end{cases} \tag{15}$$

$$b_{j+1} = \left[\beta\phi_j A_{j+1}[1 + (1-\gamma)r^f + \frac{1}{2}(1-\gamma)\frac{(\mu - r^f)^2}{\gamma\sigma^2}]\right]^{-\frac{1}{\gamma}} \tag{16}$$

Optimal consumption and portfolio choices are given by:

$$\alpha^*_{j,t} = \frac{\mu - r^f}{\gamma\sigma^2} \tag{17}$$

$$c^*_{j,t} = \begin{cases} \frac{b_{j+1}}{1+b_{j+1}} w_{j,t} & \text{for } j < J \\ w_{j,t} & \text{for } j = J \end{cases} \tag{18}$$

Proof. Appendix A ■

As is well-known, this type of problem implies that the proportion of the agent's portfolio invested in risky assets is constant over time and age (Campbell and Viceira, 2002). To simplify notation, we remove the j, t subscripts and denote the optimal risky

[10] This approximation is exact in the limit of continuous time (Campbell and Viceira, 2002).

asset share in the absence of ambiguity as α^*. The path of optimal consumption depends on future survival probabilities, and therefore varies over the life cycle.

2.3 Ambiguity Aversion

We consider the case in which there is ambiguity over the mean of the return on the risky asset, as in Peijnenburg (2018). Formally, while equation (11) accurately reflects the return an agent would receive on a given portfolio invested in period t, the agent considers a set of models under which returns are distorted away from this by an amount $\nu_{j,t}$:

$$R^p_{j,t+1} = R^f + (R_{t+1} - R^f + \sigma_1 \nu_{j,t})\alpha_{j,t} \tag{19}$$

where σ_1 is the standard deviation of R_{t+1}:

$$\sigma_1 = \exp(\mu)\left(\exp(\sigma^2) - 1\right)^{\frac{1}{2}} \tag{20}$$

To model the agent's aversion to ambiguity, we follow Hansen and Sargent (2008) and rewrite their dynamic programming problem as:

$$V^\theta_j(w_{j,t}) = \max_{c_{j,t},\alpha_{j,t}} \min_{\nu_{j,t}} \left\{ \frac{c_{j,t}^{1-\gamma}}{1-\gamma} + \beta\phi_j \left[\frac{1}{2\theta}\nu^2_{j,t} + E_{j,t}[V^\theta_{j+1}(w_{j+1,t+1})] \right] \right\} \tag{21}$$

subject to $w_{J+1,T+1} \geq 0$, the budget constraint (9), and the distorted returns process (19).

That is, the agent makes consumption and portfolio decisions based on a distorted law of motion for their assets, in which returns on the risky asset are systematically biased towards models for the risky return which deliver low continuation values in their optimization problem. In this way they make choices which are robust to their uncertainty over the process for risky asset returns.

However, the agent does not entertain an infinite set of models. Rather, they choose the distortion in the returns process behind their consumption and portfolio choices so that it minimizes expected utility, *plus* a cost of $\frac{1}{2\theta}\nu^2_{j,t}$. Intuitively, the parameter θ controls the agent's preference for robustness: larger values of θ imply the agent entertains larger deviations from the true returns process in equation (11). Just as with the consumption and portfolio choices, the belief distortion $\nu_{j,t}$ is re-optimized every period, and the agent takes future belief distortion choices into account when making their decisions in period t. For a detailed discussion of this approach to modeling ambiguity aversion, see Hansen and Sargent (2008) and the survey in Ilut and Schneider (2022).

This formulation means that agents consider larger distortions if their value functions are more sensitive to model misspecification; in these cases the need for robustness is greater. Equation (21) shows that value functions differ by age directly through survival probabilities ϕ_j. In addition, value functions depend on wealth $w_{j,t}$, which may be correlated with age. Through both of these channels, the optimal distortions to beliefs about risky asset returns will vary across the life cycle. Intuitively, although all agents share the same preference for robustness (they have the same θ), agents of different ages have different levels of exposure to changes in the return on risky assets.

Optimal Belief Distortion: We begin by solving the inner minimization problem, in which the agent chooses how much to distort their return expectations towards the 'worst-case scenario'. In this, the following result is helpful.

Lemma 1 *Taking the same log-linear approximation approach as in (12) to the distorted returns in (19), we can write*

$$E_{j,t}[V_{j+1}^{\theta}(w_{j+1,t+1})] \approx E_{j,t}[V_{j+1}^{\theta}(w_{j+1,t+1}^{*})] + \frac{A_{j+1}}{1-\gamma}(w_{j,t}-c_{j,t})^{1-\gamma}(1-\gamma)\sigma\alpha_{j,t}\nu_{j,t} \tag{22}$$

where

$$w_{j+1,t+1}^{*} = (w_{j,t}-c_{j,t})[R^f + \alpha_{j,t}(R_{t+1}-R^f)] \tag{23}$$

is the next-period wealth the agent would achieve under the central model without return distortions.

Proof. Appendix A ■

Substituting this into the Bellman equation (21), it is then straightforward to obtain the first order condition for the inner minimization problem:

$$\nu_{j,t} = -\theta A_{j+1}(w_{j,t}-c_{j,t})^{1-\gamma}\sigma\alpha_{j,t} \tag{24}$$

Consumption and Portfolio Allocation: Substituting the optimal distortion (24) into equation (21), the household chooses consumption and the share of savings invested in risky assets to solve:

$$V_j^{\theta}(w_{j,t}) = \max_{c_{j,t},\alpha_{j,t}} \left\{ \frac{c_{j,t}^{1-\gamma}}{1-\gamma} + \beta\phi_j \left[-\frac{1}{2}\theta A_{j+1}^2 (w_{j,t}-c_{j,t})^{2-2\gamma}\sigma^2\alpha_{j,t}^2 + E_{j,t}[V_{j+1}^{\theta}(w_{j+1,t+1})] \right] \right\} \tag{25}$$

Proposition 2 characterizes the solution.

Proposition 2 *The value function takes the form:*

$$V_j^{\theta}(w_{j,t}) = A_j \frac{w_{j,t}^{1-\gamma}}{1-\gamma} + \theta B_j \frac{w_{j,t}^{2(1-\gamma)}}{2(1-\gamma)} + O(\theta^2) \tag{26}$$

where B_j is an age-dependent combination of model parameters, defined in Appendix A.

In the approximate solution where we drop terms in θ^2, optimal portfolio choice and consumption are given by:

$$\alpha_{j,t} = \alpha^* + \theta \alpha^* w_{j,t}^{1-\gamma} \Omega_{\alpha j} \tag{27}$$

$$c_{j,t} = c_{j,t}^* + \theta w_{j,t}^{2-\gamma} \Omega_{cj} \tag{28}$$

where $\alpha^*, c_{j,t}^*$ are the solutions without ambiguity defined in Proposition 1, and $\Omega_{\alpha j}, \Omega_{cj}$ are functions of $b_{j+1}, A_{j+1}, B_{j+1}$, defined in Appendix A.

Proof. Appendix A ■

With $\theta = 0$, we therefore return to the standard expected-utility solution (Proposition 1). With some ambiguity aversion ($\theta > 0$), however, both portfolio and consumption decisions shift away from this solution. The deviations from the rational-expectations solution are directly proportional to the degree of ambiguity aversion θ. Importantly, the effect of ambiguity aversion depends on both the agent's wealth and, through $\Omega_{\alpha j}$ and Ω_{cj}, their expected future lifespan. This occurs because agents of different ages are differentially exposed to the return on risky assets, and so opt for different levels of belief distortion in response to their ambiguity aversion. The resulting distortions to beliefs, consumption, and portfolio shares are generally nonlinear functions of wealth and demographics. To explore the mechanisms analytically, we therefore consider a case with a particularly simple demographic process.

3 Results with Three-Period Lifespans

We now restrict the model to $J = 2$. Agents therefore live for a maximum of three periods: at ages $j = 0, 1, 2$ we refer to them as young, middle aged, and old respectively. Furthermore, we assume that $\phi_0 = 1$, so all agents survive at least to middle age. In this simple context, population aging therefore only occurs through an increase in ϕ_1, the probability of surviving to old age.

3.1 Consumption and Portfolio Allocation: No Ambiguity Case

Consider an agent born in period t. To understand the effects of ambiguity aversion in this environment, it is helpful to first examine the forces that drive consumption and saving in the baseline model without ambiguity. In this case, the portfolio share in risky assets is constant, as in equation (17). Applying the remaining elements of Proposition 1, we obtain closed-form solutions for consumption in each period of the agent's life.

Proposition 3 *An agent with $\phi_0 = 1, \phi_2 = 0$ and initial effective wealth $w_{0,t}$ chooses consumption when young and middle-aged according to:*

$$c_{0,t} = \frac{\tilde{b}^2}{\phi_1^{\frac{1}{\gamma}} + \tilde{b}(1+\tilde{b})} w_{0,t} \tag{29}$$

$$c_{1,t+1} = \frac{\tilde{b}}{\phi_1^{\frac{1}{\gamma}} + \tilde{b}(1+\tilde{b})} R^p_{0,t+1} w_{0,t} \tag{30}$$

where $\tilde{b}$ is a strictly positive combination of age-independent parameters:

$$\tilde{b} = \left[\beta\left(1 + (1-\gamma)r^f + \frac{1}{2}\frac{(1-\gamma)}{\gamma}(\alpha^*)^2\right)\right]^{-\frac{1}{\gamma}} \tag{31}$$

Conditional on surviving to old age, they then have:

$$c_{2,t+2} = \frac{\phi_1^{\frac{1}{\gamma}}}{\phi_1^{\frac{1}{\gamma}} + \tilde{b}(1+\tilde{b})} R^p_{0,t+1} R^p_{1,t+2} w_{0,t} \tag{32}$$

Proof. Appendix A ■

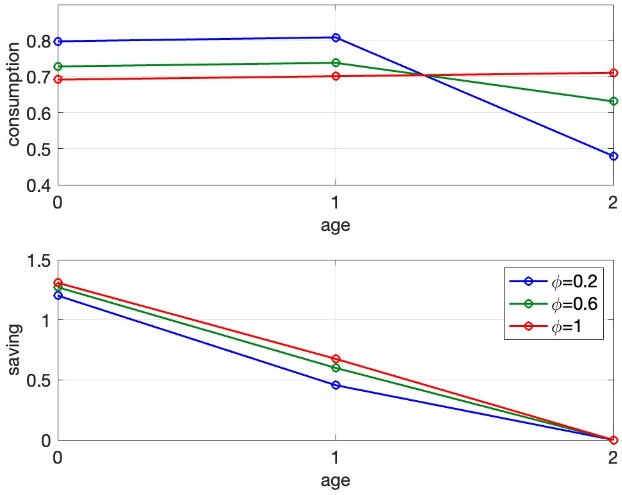
As the probability of surviving to old age (ϕ_1) increases, the incentive to save for consumption in old age rises, so agents consume less when they are young and middle-aged ($c_{0,t}$ and $c_{1,t+1}$ decrease). Savings are therefore depleted less quickly through the life cycle if life expectancy is longer.[11] For agents that do survive to old age, a greater ϕ_1 implies higher consumption $c_{2,t+2}$, due to the extra savings built up earlier in life.

These patterns are displayed in Figure 1, which plots the paths of consumption and saving over the life cycle for three different values of ϕ_1. When the probability of surviving to old age is low, agents consume a lot in their youth and middle age. If they do survive to old age, they therefore experience a large consumption drop. With a greater survival probability, young and middle-aged consumption is lower, and the subsequent

[11] This is consistent with Foltyn and Olsson (2021), who find that individuals with longer subjective life expectancies accumulate more wealth over their life cycle than those who expect to die earlier.

consumption fall in old age is lower.[12]

Figure 1: Consumption and saving paths with no ambiguity.



Note: Plots constructed using $J = 2$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 = 0.2, 0.6, 1$, $w_{0,t} = 2$, *and risky asset returns set to their expected level every period. This therefore abstracts from return shocks.*

3.2 Ambiguity Aversion

We now add ambiguity aversion back into the model. The key element of this model is how the distortion in return expectations due to ambiguity aversion varies with age, wealth, and the probability of surviving to old age. These distortions are given in Proposition 4.

Proposition 4 *The optimal distortion in beliefs about risky asset returns for an agent with* $\phi_0 = 1, \phi_2 = 0$, *wealth* $w_{j,t}$, *and age* j *is:*

$$\nu_{0,t} = -\frac{\theta\sigma\alpha^*(\phi_1^{\frac{1}{\gamma}} + \tilde{b})}{\tilde{b}^{\gamma}(\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2)^{1-\gamma}} w_{0,t}^{1-\gamma} \tag{33}$$

$$\nu_{1,t} = -\frac{\theta\sigma\alpha^*\phi_1^{\frac{1-\gamma}{\gamma}}}{(\phi_1^{\frac{1}{\gamma}} + \tilde{b})^{1-\gamma}} w_{1,t}^{1-\gamma} \tag{34}$$

$$\nu_{2,t} = 0 \tag{35}$$

Proof. Appendix A ■

To understand the implications of these distortions, we first compare agents with the same wealth but different ages, to isolate the effects of age and survival probabilities. We then go on to analyse the interactions with varying wealth.

[12]With $\phi_1 = 1$ the consumption path is slightly increasing over time here due to precautionary saving. $c_{2,t+2} > c_{1,t+1}$ in the model whenever $\phi_1 > (\tilde{b}/R^p_{1,t+2})^{\gamma}$, which is close to 1 for most calibrations.

3.2.1 Age Effects

First, Proposition 4 implies that old agents ($j = 2$) do not distort their beliefs at all (35). This is because they save nothing, and so have no exposure to asset returns. There is no need for them to make their decisions robust to doubts about average asset returns. Similarly, note that if $\phi_1 = 0$ then a middle-aged agent sets $\nu_{1,t} = 0$, for the same reason: they will die for certain at the end of the period, so they do not save and are not exposed to ambiguity. In all other cases $\nu_{j,t} < 0$, so the agents distort their beliefs towards lower returns on the risky asset.

Corollary 1 shows a further equivalence between two other extreme special cases:

Corollary 1 *Let $\nu_{j,t}(\phi)$ be the distortion chosen by an agent of age j facing a survival probability of $\phi_1 = \phi$, as well as $\phi_0 = 1, \phi_2 = 0$. Then, if $w_0 = w_1$:*

$$\nu_{0,t}(0) = \nu_{1,t}(1) \tag{36}$$

Proof. Appendix A ■

That is, a young agent who will die for certain after middle age distorts beliefs in the same way as a middle-aged agent who will survive to old age for certain. In both cases, the agent knows they have one more period of consumption, and so behavior is the same for each. This highlights that life *expectancy*, rather than age, is the critical factor in how ambiguity aversion affects beliefs in this environment.

Second, Proposition 4 also implies that changes in ϕ_1 affect the belief distortions among young and middle-aged agents away from these edge cases.

Corollary 2 *For an agent with $\phi_0 = 1, \phi_2 = 0$, as ϕ_1 changes the optimal belief distortions of young agents, holding wealth constant, are such that:*

$$\frac{\partial \nu_{0,t}}{\partial \phi_1} < 0 \tag{37}$$

Holding wealth constant, the distortions of middle-aged agents are such that:

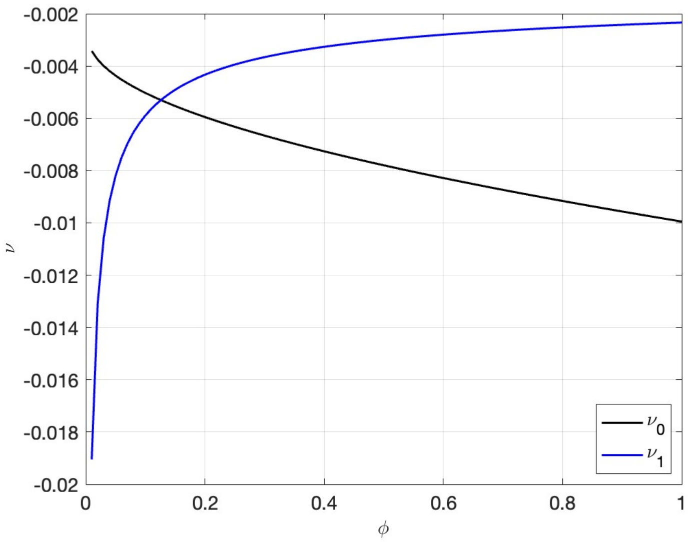
$$\frac{\partial \nu_{1,t}}{\partial \phi_1} \begin{cases} < 0 & \text{if } \gamma < 1 \\ = 0 & \text{if } \gamma = 1 \\ > 0 & \text{if } \gamma > 1 \end{cases} \tag{38}$$

Proof. Appendix A ■

As the probability of surviving to old age rises, young households distort their beliefs more strongly, becoming more pessimistic about equity returns. If the EIS is greater than 1 ($\gamma < 1$), middle-aged households do the same. However, if the EIS is less than 1 ($\gamma > 1$), they decrease the magnitude of their distortion.

This divergence is at the heart of our mechanism: in the empirically plausible case with $\gamma > 1$, as life expectancy rises the young get more pessimistic about equity returns, while older agents get more optimistic. The effect is shown in Figure 2.

Figure 2: Belief distortions as ϕ_1 varies.



Note: Plots constructed using $J = 2$, $\theta = 0.045$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 \in (0, 1]$, $w_{0,t} = w_{1,t} = 2$, and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks. With this calibration, $\sigma_1 \approx 0.01$, so (using equation (19)) a distortion of $\nu_{j,t} = -0.01$ corresponds to a reduction in expected risky returns of approximately 10 basis points.

At ϕ_1 close to 0, young households are less pessimistic than middle-aged households. As the survival probability grows, these positions reverse.

To understand the mechanisms driving the divergent responses to increasing longevity, we return to the first-order condition for the inner minimization in equation (21). The agent chooses the degree to which they distort their return expectations by balancing the marginal damage to expected continuation values with the marginal penalty to considering a larger distortion:

$$\frac{\partial}{\partial \nu_{j,t}} \left\{ \frac{1}{2\theta} \nu_{j,t}^2 + E_{j,t}[V_{j+1}^{\theta}(w_{j+1,t+1})] \right\} = 0 \tag{39}$$

Equation (24) is then simply the result of combining this with Lemma 1 and rearranging. However, we can alternatively write this condition as:

$$\nu_{j,t} = -\theta \frac{\partial E_{j,t}[V^{\theta}_{j+1}(w_{j+1,t+1})]}{\partial \nu_{j,t}} \tag{40}$$

$$\approx -\theta\sigma\alpha^{*} \underbrace{(w_{j,t} - c_{j,t})}_{\text{Wealth Channel}} \cdot \underbrace{\frac{\partial E_{j,t}[V^{\theta}_{j+1}(w_{j+1,t+1})]}{\partial w_{j+1,t+1}}}_{\text{Marginal Utility Channel}} \tag{41}$$

where the second line uses the same approximation as in Lemma 1.

The distortion is set proportional to the sensitivity of expected continuation values to asset returns. Intuitively, the more exposed the agent is to changes in risky asset returns, the more they wish to make their decisions robust to ambiguity over those returns. That sensitivity can be broken down into two channels: the wealth channel, and the marginal utility channel.

The wealth channel operates because when an agent saves more for the next period, their next-period wealth is more strongly affected by asset returns. In other words, they have more skin in the game. As discussed in Section 3.1, when ϕ_1 increases both young and middle-age agents increase their saving.[13] For both young and middle-age agents, this channel therefore implies greater belief distortions when ϕ_1 rises.

The marginal utility channel operates because a given decrease in asset returns will have a larger effect on utility for an agent with a large marginal utility of wealth in the following period. Through a standard envelope theorem, the marginal utility of wealth in period $t+1$ is equal to the marginal utility of consumption in $t+1$. Since our model features a diminishing marginal utility of consumption, this channel will be more powerful when next-period consumption is expected to be low.

This channel is what drives the divergence in beliefs across cohorts. Recall from Section 3.1 that, as ϕ_1 increases, the consumption of middle-aged agents falls, while the consumption of old agents rises. Agents who are currently young therefore expect to have a greater marginal utility of wealth in the following period, when they will be middle-aged. An increase in ϕ_1 makes them more sensitive to changes in wealth, increasing the strength of the marginal utility channel. In contrast, current middle-aged agents expect to have more wealth in future, and so a lower marginal utility, implying a smaller marginal utility channel.

For a young agent, both the wealth and marginal utility channels therefore imply that they become more pessimistic when ϕ_1 rises. For a middle-aged agent, the channels act in opposite directions. To see which dominates, note that for a middle-aged agent we

[13] The paths of consumption and saving remain qualitatively unchanged with the introduction of ambiguity aversion, as we typically consider small values of θ. Sufficient conditions for this result, and a numerical example, are provided in Appendix B.

obtain:[14]

$$\frac{\partial E_{1,t}[V_2^\theta(w_{2,t+1})]}{\partial w_{2,t+1}} \propto (w_{1,t} - c_{1,t})^{-\gamma} \tag{42}$$

Substituting this into equation (41) implies:

$$\nu_{1,t} \propto -\theta\sigma\alpha^*(w_{1,t} - c_{1,t})^{1-\gamma} \tag{43}$$

For a middle-aged agent, an increase in ϕ_1 implies a rise in $w_{j,t} - c_{j,t}$. With $\gamma < 1$, the wealth channel dominates, and middle-aged agents therefore increase the magnitude of their belief distortions when survival probabilities rise ($\nu_{j,t}$ becomes more negative). In the empirically plausible case with $\gamma > 1$, however, the marginal utility channel dominates, and middle-age agents become more optimistic about returns. With log utility ($\gamma = 1$) the effects cancel out and middle-aged agents do not adjust $\nu_{1,t}$ with ϕ_1.

Importantly, these channels depend on how consumption and saving *change* in response to an increase in longevity, not on their level. This explains why the mechanisms continue to operate in richer models with more realistic income processes, such as the one in Section 5. Quantitative models of this kind share the prediction that agents save more for old age, and consume more if they survive, when survival rates increase (Gomes, 2020).

Portfolio Allocation: Figure 3 plots the share of agent portfolios invested in the risky asset ($\alpha_{j,t}$) as ϕ_1 changes, for the same parameters as Figure 2. Both young and middle-aged agents allocate lower shares of their wealth to risky assets than they would in the absence of ambiguity, as in other models in the literature (Garlappi et al., 2007; Campanale, 2011) and consistent with empirical evidence (Dimmock et al., 2016). This is a direct consequence of Proposition 4: the belief distortions due to ambiguity aversion imply the agent acts as if the risky asset has a lower expected return than its true mean, and so invests less in that asset than they would in the absence of ambiguity. For $\phi_1 > 0.2$ in this calibration, young agents distort their return beliefs more in response to ambiguity than middle-aged agents, so their risky asset share is correspondingly lower. We find the same qualitative pattern in the quantitative illustration in Section 5.

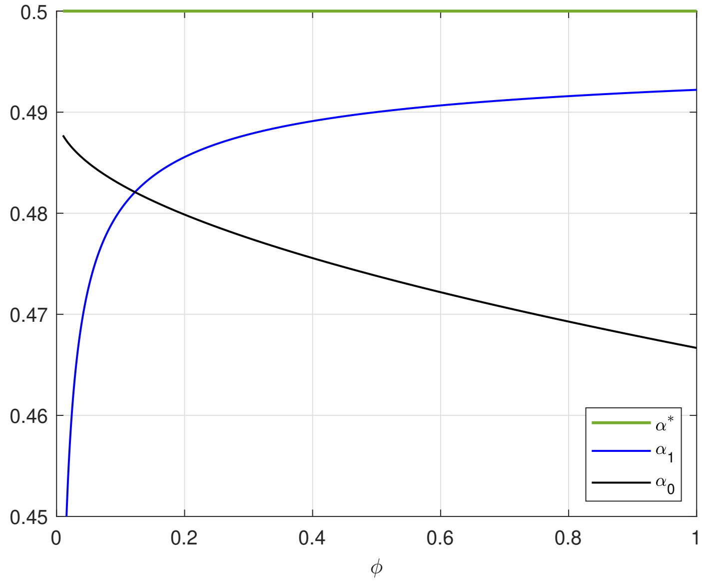
The changes in risky asset shares as the population ages (ϕ_1 increases) follow from Corollary 2. As ϕ_1 increases along the x-axis, middle-aged agents reduce the distortions in their beliefs, increasing their risky asset share towards the benchmark share without ambiguity (α^*).[15] In contrast, young households increase their distortions, and move

[14] This follows from Proposition 2, and the fact that $A_2 = 1$.

[15] Equation (34) highlights that even with $\phi_1 = 1$ the belief distortion is strictly negative, so $\alpha_{1,t}$ always remains strictly below α^*.

further from this benchmark level.

Figure 3: Risky asset shares without ambiguity (α^*), and with ambiguity for young (α_0) and middle-aged (α_1) agents, as ϕ_1 varies.



Note: Plots constructed using $J = 2$, $\theta = 0.045$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 \in (0, 1]$, $w_{0,t} = w_{1,t} = 2$, *and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks.*

Note that the models in Campanale (2011) and Peijnenburg (2018) also feature risky asset shares that increase with age ($\alpha_{1,t} > \alpha_{0,t}$). However, the mechanisms in those papers are different from ours: there agents learn over time from observed realizations of asset returns, gradually reducing the set of models they are willing to consider. In our framework, that would entail a fall in θ as agents progress from young to middle-aged, independently of survival probabilities. In contrast, we keep θ constant, but allow the optimal distortion to vary with agent exposure to asset returns. The effect of survival probabilities on the age-profile of asset shares, through the wealth and marginal utility channels, is therefore unique to our mechanism.

3.2.2 Wealth Effects

The first-order condition for belief distortions (41) highlights that, just as with age, the effects of an increase in wealth depend on the wealth and marginal utility channels.

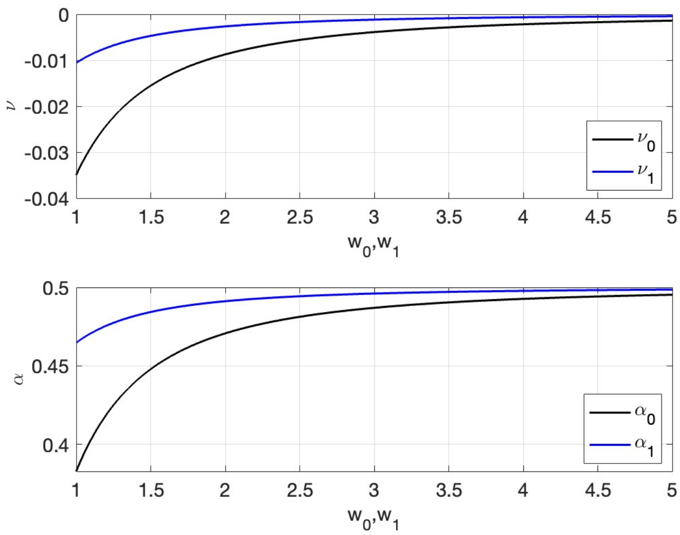
With an increase in wealth, these channels act in opposite directions. The wealth channel implies larger belief distortions for wealthier households, as they save more, so have more exposure to asset returns. The marginal utility channel implies the opposite: wealthier households have smaller belief distortions, because their continuation values are less sensitive to marginal changes in future wealth. As with the effect of survival probabilities on middle-aged agents, which channel dominates depends on whether the EIS is greater or less than 1.

Corollary 3 *For agents with $\phi_0 = 1, \phi_2 = 0$, $w_{j,t}$ affects optimal belief distortions such that for $j = 0, 1$:*

$$\frac{\partial \nu_{j,t}}{\partial w_{j,t}} \begin{cases} < 0 & \text{if } \gamma < 1 \\ = 0 & \text{if } \gamma = 1 \\ > 0 & \text{if } \gamma > 1 \end{cases} \tag{44}$$

Proof. Appendix A ■

Figure 4: Belief distortions and risky asset shares vary with wealth.



Note: Plots constructed using $J = 2$, $\theta = 0.045$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 = 0.7$, $w_{0,t}$ and $w_{1,t}$ vary from 1 to 5, and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks.

Under our preferred calibrations ($\gamma > 1$), being wealthier causes agents to become more optimistic about asset returns. As a result, wealthier agents invest a greater share of their wealth in the risky asset. These patterns are shown in Figure 4. Although this analytic model does not feature non-participation, the direction of this effect is consistent with evidence in Briggs et al. (2020) that exogenous increases in wealth increase the probability that households invest in equities.

Interactions with Age Effects: As well as directly affecting belief distortions as in Corollary 3, an agent's wealth can affect the strength of the age effects on beliefs studied in

Section 3.2.1. In our preferred parameter region of $\gamma > 1$, when wealth is higher, age effects are smaller in magnitude, for agents of all ages.

Corollary 4 *For agents with $\phi_0 = 1, \phi_2 = 0$, the age effects on optimal belief distortions change with wealth such that for $j = 0, 1$:*

$$sign\left\{\frac{\partial}{\partial w_{j,t}}\left(\frac{\partial \nu_{j,t}}{\partial \phi_1}\right)\right\} = \begin{cases} sign\left\{\frac{\partial \nu_{j,t}}{\partial \phi_1}\right\} & if\, \gamma < 1 \\ 0 & if\, \gamma = 1 \\ -sign\left\{\frac{\partial \nu_{j,t}}{\partial \phi_1}\right\} & if\, \gamma > 1 \end{cases} \tag{45}$$

This implies that if $\gamma > 1$, the effect of ϕ_1 on optimal belief distortions decreases in magnitude as $w_{j,t}$ increases.

Proof. Appendix A ∎

Intuitively, at high levels of wealth, future marginal utility is less sensitive to changes in returns, and so the marginal utility channel is weakened. When $\gamma > 1$, the marginal utility channel is the dominant channel determining how belief distortions $\nu_{j,t}$ change with ϕ_1 at all ages. Weakening that channel therefore weakens the effects of ϕ_1 on beliefs.

Throughout this analysis we maintain the simplifying assumption that any wealth left un-consumed by agents who die before reaching their maximum lifespan is destroyed. One consequence of Corollary 4 is that this assumption, if anything, implies we will understate the magnitude of the mechanisms we study.

Specifically, a common alternative assumption in this literature is that any such un-consumed wealth is left as an 'accidental bequest', and so is distributed between surviving agents in the following period (e.g. Gagnon et al., 2021). If we assumed this, then a rise in the survival probability ϕ_1 would imply fewer agents die before old age, which in turn means fewer agents leave accidental bequests, which reduces the wealth of all agents.[16] With that lower wealth, Corollary 4 implies the direct effects of ϕ_1 on beliefs become stronger. The differences between young and middle-aged agents widen. To keep the exposition of the main channels as clear as possible, we continue to abstract from this endowment effect in the remainder of the paper.

[16] There would also be an offsetting effect, that greater ϕ_1 implies agents save more for old age, so conditional on dying before old age an agent leaves a larger accidental bequest. Since old-age consumption (and thus wealth) are concave in ϕ_1 for our preferred parameter range of $\gamma > 1$ (32), this effect is dominated by the simple channel of fewer early deaths as long as ϕ_1 is not too small. In the calibration used for the figures in this section, accidental bequests decline with ϕ_1 for all $\phi_1 > 0.032$.

3.3 Intergenerational Inequality

We now use the results developed above to analyze the impacts of increased longevity through ambiguity aversion. The first implication we study concerns how wealth evolves over an agent's life-cycle. If realizations of the risky asset return are close to its mean, this is also informative about intergenerational wealth inequality, as there are no other shocks that change from one cohort to the next.

Using equation (13), we can express the ratio between wealth at ages $j+1$ and j as:

$$\frac{w_{j+1,t+1}}{w_{j,t}} = \left(1 - \frac{c_{j,t}}{w_{j,t}}\right) \exp\left(r^f + \alpha_{j,t}(r_{t+1} - r^f) + \frac{1}{2}\sigma^2 \alpha_{j,t}(1-\alpha_{j,t})\right) \tag{46}$$

Expanding out $c_{j,t}$ and $\alpha_{j,t}$ using Proposition 2, this becomes:

$$\frac{w_{j+1,t+1}}{w_{j,t}} = \left(\frac{1}{1+b_{j+1}} - \theta w_{j,t}^{1-\gamma}\Omega_{c,j}\right) \exp\Bigg(r^f + \alpha^*(r_{t+1} - r^f) + \frac{1}{2}\sigma^2\alpha^*(1-\alpha^*)$$
$$+ \theta\alpha^* w_{j,t}^{1-\gamma}\Omega_{\alpha,j}\left[r_{t+1} - r^f + \frac{1}{2}\sigma^2(1 - 2\alpha^* - \theta\alpha^* w_{j,t}^{1-\gamma}\Omega_{\alpha,j})\right]\Bigg) \tag{47}$$

Finally, using the definition of b_{j+1} for $j = \{0, 1\}$ (Appendix A) note that:

$$\frac{1}{1+b_1} = \frac{\phi_1^{\frac{1}{\gamma}} + \tilde{b}}{\phi_1^{\frac{1}{\gamma}} + 2\tilde{b}} \tag{48}$$

$$\frac{1}{1+b_2} = \frac{\phi_1^{\frac{1}{\gamma}}}{\phi_1^{\frac{1}{\gamma}} + \tilde{b}} \tag{49}$$

both of which are strictly increasing in ϕ_1.

In the absence of ambiguity ($\theta = 0$), only the first term in equation (47) changes with ϕ_1. As the survival probability rises, agents save more for the future, so middle-aged agents become wealthier relative to young agents, and similarly old agents become wealthier relative to middle-aged agents.

Ambiguity adds two further channels to this change in wealth across the life-cycle. First, a rise in ϕ_1 affects agent portfolio choices, and so affects average returns. This generates the terms in square brackets in equation (47). In Section 3.2.1 we showed that rising ϕ_1 has opposing effects on the risky asset shares of young and middle-aged agents in the empirically plausible case of $\gamma > 1$. Young agents reduce $\alpha_{0,t}$, which reduces the wealth of the middle-aged relative to the young. Middle-aged agents increase $\alpha_{1,t+1}$, increasing the relative wealth of the old. Through this channel, an aging population leads to a greater wealth concentration among older households.

Second, ambiguity also affects the amount saved by each agent, through the term $-\theta w_{j,t}^{1-\gamma}\Omega_{c,j}$. For both young and middle-aged agents, $\Omega_{c,j}$ varies with ϕ_1, and for middle-aged agents so does $w_{j,t}$. In Appendix A (equation (82)) we show that this term is potentially non-monotonic in ϕ_1, so it has an ambiguous effect on wealth inequality. However, in the simple calibration used throughout this section this effect is negligible relative to the other channels.[17]

These effects concern how changes in life expectancy affect the relative wealth between generations. In this simple model, wealth typically declines with age, because we lack many of the features commonly added to quantitative life-cycle models.[18] The mechanisms identified in this section imply that a rise in ϕ_1 causes a greater concentration of wealth among older generations relative to this baseline. When we extend the model for our quantitative exercises in Section 5, the life-cycle of wealth in the model matches the hump-shaped pattern observed in the data, but the mechanisms identified here continue to operate.

3.4 Aggregation

Next, we study how aging affects the composition of aggregate asset demand.

Recall that each period a new cohort of agents is born with measure 1, so there are 3 overlapping generations alive in each period. All agents within a cohort are identical. The aggregate demand for safe and risky assets is therefore given by:

$$AD_t(safe) = (1-\alpha_{0,t})(w_{0,t}-c_{0,t}) + (1-\alpha_{1,t})(w_{1,t}-c_{1,t}) \tag{50}$$

$$AD_t(risky) = \alpha_{0,t}(w_{0,t}-c_{0,t}) + \alpha_{1,t}(w_{1,t}-c_{1,t}) \tag{51}$$

where we have used the result that old agents do not save in either asset, and that all agents survive to middle age. Note that this implies the composition effects of aging, as studied in e.g. Auclert et al. (2021), are absent here: the age-composition of *asset market participants* is constant as ϕ_1 rises. This simplification allows us to focus on the novel channels introduced by ambiguity aversion here.

We showed previously that if there is no ambiguity aversion ($\theta = 0$), risky asset shares α_j are constant, and both young and middle-aged agents cut consumption when ϕ_1 rises (Proposition 3). As a result, the aggregate demand for both types of asset rises, as longer

[17]Specifically, with the calibration used in Figure 2, an increase in ϕ_1 from 0.2 to 0.8 implies that $w_{2,t+2}/w_{1,t+1}$ and $w_{1,t+1}/w_{0,t}$ increase by 31.02% and 7.48% respectively. The change in consumption due to ambiguity accounts for 0.014% and 0.019% of those changes.

[18]We use the word 'typically' here because if there is a very large realization of the risky asset return, it is possible for wealth to increase from age j to $j+1$.

life expectancy encourages greater saving for old age.

In the case with ambiguity aversion, the deviation of $\alpha_{j,t}$ and $c_{j,t}$ from the no-ambiguity benchmark is proportional to the degree of ambiguity aversion θ (Proposition 2). For small θ, we therefore maintain the result that $AD_t(safe)$ and $AD_t(risky)$ rise with ϕ_1, as in the no-ambiguity benchmark.

Ambiguity aversion does, however, affect the speed at which each aggregate asset demand rises, which therefore affects the *composition* of asset demand as the population ages. This is displayed in equation (52), which gives the population analogue of the individual-level risky share $\alpha_{j,t}$.

$$\frac{AD_t(risky)}{AD_t(safe) + AD_t(risky)} = \frac{\alpha_{0,t}(w_{0,t} - c_{0,t}) + \alpha_{1,t}(w_{1,t} - c_{1,t})}{w_{0,t} - c_{0,t} + w_{1,t} - c_{1,t}} \tag{52}$$

Substituting out for the individual risky asset shares $\alpha_{j,t}$ using equation (27), this becomes:

$$\frac{AD_t(risky)}{AD_t(safe) + AD_t(risky)} = \alpha^* + \theta\alpha^* \left(\frac{\Omega_{\alpha,0} w_{0,t}^{1-\gamma}(w_{0,t} - c_{0,t}) + \Omega_{\alpha,1} w_{1,t}^{1-\gamma}(w_{1,t} - c_{1,t})}{w_{0,t} - c_{0,t} + w_{1,t} - c_{1,t}} \right) \tag{53}$$

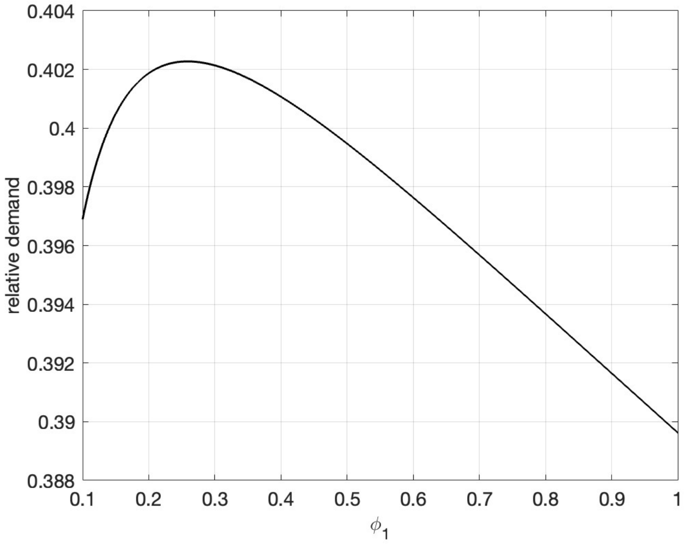
In the absence of ambiguity, the relative demand for risky assets is a constant (α^*). However, with ambiguity ($\theta > 0$), demand for safe and risky assets are no longer in fixed proportions, and the relative demand for each asset changes with the survival probability.

These relative demand changes are shown in Figure 5. The no-ambiguity relative demand is constant at α^*, which with this calibration is equal to 0.5. As in e.g. Garlappi et al. (2007), ambiguity aversion reduces the relative demand for risky assets below this level. The contribution of our model is that we can ask how that relative demand changes with survival rates. As ϕ_1 rises, the demand for risky assets relative to safe assets follows a hump-shape: it rises, reaches a peak, then falls.

This hump-shape in relative risky asset demand occurs because young and middle-aged agents shift their belief distortions in different directions as ϕ_1 increases, as shown in Section 3.2.1. As the survival probability rises, the young become more pessimistic about asset returns, while the middle-aged become more optimistic (Corollary 2). As a result, young agents decrease their relative demand for risky assets, while middle-aged agents increase their relative demand.

When the probability of surviving to old age is low, young households save only a small fraction of their endowment (Proposition 3, extended to the ambiguity case in Appendix B). As a result, when they become middle-aged, they have little wealth: $w_{1,t}$ is

Figure 5: Relative asset demand $\frac{AD_t(risky)}{AD_t(safe)+AD_t(risky)}$ varies with ϕ_1.



Note: Plots constructed using $J = 2$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\theta = 0.045$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 \in (0, 1]$, $w_{0,t} = 1$, *and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks.*

low relative to $w_{0,t}$. Corollary 4 then implies that any small increase in ϕ_1 has a stronger effect on the beliefs of middle-aged than young agents. Initially, the middle-aged agents react most strongly to ϕ_1, and relative risky asset demand rises.

However, as ϕ_1 rises, young agents increase their saving, and these mechanisms work in reverse. Wealth $w_{1,t}$ rises, and so age effects become weaker for middle-aged agents, ultimately becoming smaller than the effects on young agents. At high ϕ_1, further aging of the population therefore implies a fall in relative risky asset demand.

Finally, note that relative risky asset demand is also affected by a composition channel. As ϕ_1 rises, $w_{1,t}$ rises, which means that middle-aged agents account for a greater share of aggregate saving. Since for most values of ϕ_1 middle-aged agents are more optimistic than young agents (Figure 2), this also causes the aggregate relative risky asset demand to rise with ϕ_1, shifting the peak in Figure 5 to the right.

3.5 Endogenous Equity Premium

So far, we have considered a small open economy aging alone. In that case, the variation in relative demand for safe and risky assets shown in Figure 5 does not affect returns or asset prices. However, in a closed economy, or indeed in a world where all countries are aging simultaneously, this will no longer be true.

We therefore extend the model here, and instead assume that the relative supply of safe and risky assets is fixed. This allows us to study the effects of aging on the equity premium $\mu - r^f$, as this must adjust to ensure that asset markets clear. The equity premium is particularly of interest because it controls how much heterogeneity there is between the wealth accumulation of agents with different beliefs. It is therefore central to how our mechanisms will affect intergenerational inequality.

Specifically, let $S_t(safe)$ and $S_t(risky)$ denote the supply of safe and risky assets in period t. The relative supply of risky assets is assumed to be fixed at $\bar{S}$:

$$\frac{S_t(risky)}{S_t(safe) + S_t(risky)} = \bar{S} \tag{54}$$

For asset markets to clear, we therefore require:[19]

$$\frac{AD_t(risky)}{AD_t(safe) + AD_t(risky)} = \bar{S} \tag{55}$$

This particular assumption on asset supply is useful here, because it implies that if there is no ambiguity aversion, the solution is trivial. From equation (53), if $\theta = 0$ then the relative demand for risky assets is constant at α^*, which itself is directly proportional to the equity premium (equation (17)). In this case, the equilibrium equity premium is therefore a constant, unaffected by changes in survival probabilities:

$$(\mu - r^f | \theta = 0) = \gamma \sigma^2 \bar{S} \tag{56}$$

As a result, any dependence of the equity premium on ϕ_1 must come through ambiguity aversion. In this way, our equity premium analysis is similar in spirit to our analysis of the small open economy above, in which individual portfolio choices are independent of ϕ_1 unless there is some ambiguity over risky returns.

To analyze the equity premium in the case with ambiguity, it is useful to first note that relative aggregate demand increases monotonically in the equity premium.

Lemma 2 *In the model with $\phi_0 = 1, \phi_2 = 0$, for any $\theta < \theta^*$:*

$$\frac{\partial}{\partial \mu} \left(\frac{AD_t(risky)}{AD_t(safe) + AD_t(risky)} \right) > 0 \tag{57}$$

$$\frac{\partial}{\partial r^f} \left(\frac{AD_t(risky)}{AD_t(safe) + AD_t(risky)} \right) < 0 \tag{58}$$

where $\theta^* > 0$ is a threshold defined in Appendix A.

[19] This condition would still be necessary, though not sufficient, for asset market clearing in a model with fixed supplies of both assets individually.

Proof. Appendix A ■

This is intuitive: as in the case without ambiguity, if the equity premium rises, then the expected return on risky assets rises relative to safe assets, rendering them more attractive to investors. Whether the equity premium rises because μ rises or r^f falls, the relative demand for risky assets therefore rises.

From this we arrive at two implications. First, Proposition 4 implies that for all values of ϕ_1, young and middle-aged agents distort their beliefs towards (weakly) lower risky asset returns. This is why ambiguity reduces the relative demand for risky assets, as shown in Figure 5. To offset this and ensure market clearing, the equity premium must therefore be higher than if there was no ambiguity. Ambiguity aversion can therefore help explain the equity premium puzzle, as in other models in this literature (e.g. Dimmock et al., 2016).

Second, the analysis in the previous sections highlights that individual and aggregate portfolio choices change with ϕ_1, implying that the equity premium will change as survival probabilities rise. Specifically, the equity premium is U-shaped in ϕ_1.

The intuition for this result follows directly from the discussion in Section 3.4. As ϕ_1 rises from a low level, then the relative aggregate demand for risky assets rises, as middle-aged agents become more optimistic about risky returns. This pushes the equity premium down, to clear asset markets. As ϕ_1 continues to rise, the relative aggregate demand for risky assets begins to fall, as increasing pessimism from young agents dominates the optimism from the middle-aged. That in turn implies the equity premium rises.

Interestingly, although the model is extremely simple, this is consistent with qualitative patterns in equity premia in the last 75 years. Since 1950, developed economies have experienced substantial rises in life expectancy. Over the same period, Kuvshinov and Zimmermann (2020) document that equity premia in developed economies have followed a U-shape, first falling, and then rising again after 1990. However, note that in Appendix D.7 we find that in our quantitative model the effects of life expectancy on equity premia are rather small, so the channel discussed here is unlikely to explain all of the historical equity premium experience.

4 Testing the Mechanism

The key novel prediction of our model is that *life expectancy*, not just the current age, affects beliefs and portfolio decisions. In this section, we test this prediction using survey data from the US. We find evidence in favor of the mechanisms outlined in Section 3: for young households, a longer subjective life expectancy is associated with smaller portfolio shares invested in equities. For older households, that relationship is reversed. This qualitative pattern is produced by our model for $\gamma > 1$, as shown in Figure 3.

4.1 Data

We use the Household Finance Module included in the August waves of the Federal Reserve Bank of New York Survey of Consumer Expectations (SCE) between 2014-2019.[20] In each of the six waves, approximately 1,100 households are asked detailed questions about their portfolio choices and, importantly for our purposes, their subjective life expectancy. The survey also collects rich demographic information, including age, gender, race, and education. The sample is designed to be representative of the US population.

Household Portfolios: The first question we use asks about the share of the respondent's portfolio invested in equities:

> "What proportion of the money in your (and your spouse's/partner's) saving and investment accounts (excluding funds in retirement accounts) is invested in stocks/stock mutual funds?"

This question is not conditional on asset market participation: people who do not invest in equities give an answer of 0.

Note that this question is similar, but not quite identical, to $\alpha_{j,t}$ in the simple model. The question elicits the share of financial wealth invested in stocks, and the model concept is the share of total (effective) wealth invested. We focus on the untransformed data in our main analysis for clarity. In Appendix C, we show that our results are robust to transforming this variable to measure the fraction of total wealth invested in stocks.

As well as portfolio shares, we also use the information recorded in the survey on household income. We use the many questions on assets and liabilities to construct a measure of the household's net wealth following Armantier et al. (2016).

Subjective Life Expectancy: The key reason for using the SCE here, rather than larger household finance surveys such as the Survey of Consumer Finances (SCF), is that households are also asked for their beliefs about their own longevity. Specifically, survey participants under the age of 65 are asked:

> "What do you think is the percent chance that you will live to age 85?"

This measures subjective survival beliefs. We do not take a stand on whether these beliefs are accurate or not. In our model we assumed that agents are correct about

[20] For a detailed overview of the SCE see Armantier et al. (2017).

their survival probabilities, but the mechanisms are exactly the same if they are not: what matters for ambiguity-driven belief distortions and portfolio decisions is the agent's belief about their survival probability.

Alongside this question, survey participants are also asked for the probability that they will live to the age of 65 and 75. We focus on the longest-horizon question, as this contains the most cross-sectional variation, giving us the most power to detect our mechanism.[21] In Appendix C we repeat the regressions of Section 4.2 using these shorter-horizon questions, and find our results are qualitatively robust, though imprecisely estimated due to the smaller variation.

Summary Statistics: Table 1 displays summary statistics for the key variables in our analysis. In all cases, the statistics displayed concern the population who answered the subjective survival probability questions (i.e., those under 65).

Table 1: Summary statistics for key variables in the SCE Household Finance module.

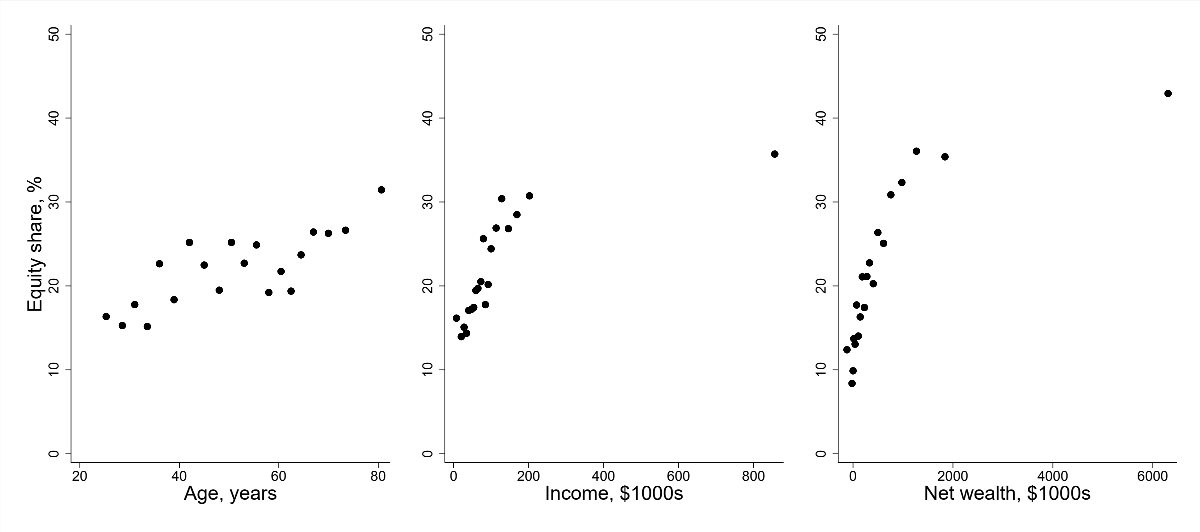
	Mean	Median	Lower Quartile	Upper Quartile
Equity share (%)	20	0	0	35
Equity share, conditional (%)	45	40	20	70
Net wealth ($1000s)	503	139	6	460
Income ($1000s)	105	70	38	110
Subjective survival prob. (%)	52	50	30	75
Age (years)	45	46	35	56

Note: Summary statistics are computed for the sample of survey respondents with non-missing values for the subjective survival probability to age 85, which therefore only includes households with age < 65. *"Equity share, conditional" refers to the equity share among the sample with non-zero shares of wealth invested in equity. Net wealth consists of the value of financial assets, housing, retirement accounts (IRAs and 401ks), other land, vehicles, and other assets, minus the value of mortgages, other home equity lines of credit, and non-housing presonal debt (including credit card debt and student debt). Units for each variable are in brackets after the variable name. Sample period: 2014-2019. Source: Survey of Consumer Expectations, core survey and household finance module.*

Previous literature has highlighted in more detailed data that risky asset portfolio shares are typically increasing in age, income, and net worth. The binned scatter plots in Figure 6 show that this is also the case in the SCE, even though this data does not (for example) include indirect equity holdings, which are typically included in portfolio data from the SCF.

[21] This is simply because almost all households in the sample expect to reach the age of 65 with a very high probability: the mean subjective survival probability to 65 in our sample is 82%. In contrast, the mean subjective survival probability to 85 is 53%. The variance in the answers to the 'age 85' question is more than double that of the 'age 65' question.

Figure 6: Share of financial assets invested in equities varies with age, income, and net wealth.



Note: Plots show binned scatter plots of the share of financial assets invested in stocks against age, income, and net wealth respectively. In each panel, each point represents the mean equity share and the mean x-axis realization in a given ventile of the x-axis distribution. Sample period: 2014-2019. Source: Survey of Consumer Expectations, core survey and household finance module.

4.2 Analysis

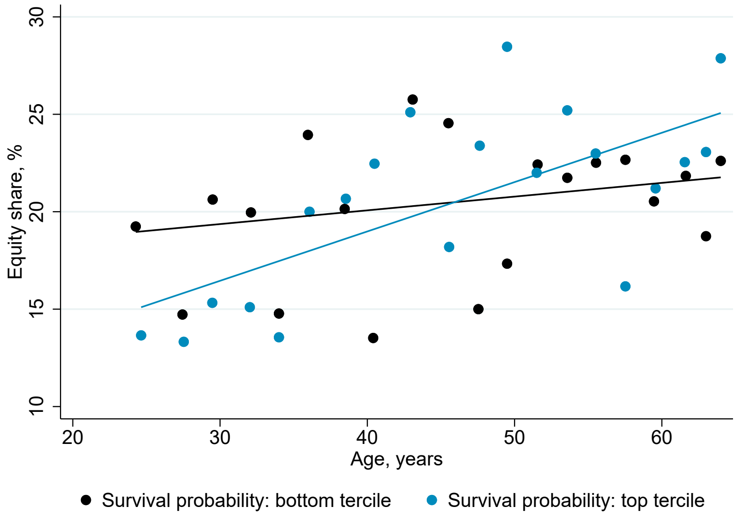
Corollary 2 predicts that when life expectancy increases, younger households become more pessimistic about risky asset returns, while older households become more optimistic – assuming a standard calibration in which the EIS is less than 1. This is reflected in portfolio choices (Figure 3): increasing survival probabilities imply young households reduce the share of their wealth invested in risky assets, while middle-aged households increase their risky share. The age-gradient of risky asset shares therefore increases with survival beliefs.

This is the prediction that we test in the data. First, we simply plot the age-gradient of risky asset shares among those with high and low subjective survival probabilities. Figure 7 plots the binned scatter plot of risky asset shares against age for those in the bottom tercile of the subjective survival probability distribution (black) and those in the top tercile (blue). Consistent with the model, a greater subjective survival probability is associated with lower risky asset shares among young households, but a steeper age gradient, such that greater longevity is associated with greater risky asset shares for older households.

Next, we test for this result in a more systematic way, and examine its statistical significance. For that we estimate the following equation via OLS:

$$\tilde{\alpha}_{it} = \beta_0 + \beta_1 age_{it} + \beta_2 \tilde{\phi}_{it} + \beta_3 age_{it} \cdot \tilde{\phi}_{it} + \delta_t + \Gamma' X_{it} + \varepsilon_{it} \tag{59}$$

Figure 7: Age-profile of portfolio share in equity, split by subjective survival probability.



Note: Plot shows the binned scatter plot of the share of financial assets invested in stocks against age, among those in the bottom ($\tilde{\phi}_{it} \leq 40\%$, black) and top ($\tilde{\phi}_{it} \geq 67\%$, blue) terciles of the subjective survival probability distribution. In each tercile, each point represents the mean equity share and the mean age in a given ventile of the age distribution. The solid lines give the predicted values from a linear regression of the equity share on age in each tercile, with equations $\tilde{\alpha}_{it} = 17.251 + 0.070 age_{it}$ and $\tilde{\alpha}_{it} = 8.857 + 0.253 age_{it}$ respectively. The slope coefficient is not significantly different from zero for the bottom tercile ($p = 0.317$) but is for the top tercile ($p = 0.00009$). Sample period: 2014-2019. Source: Survey of Consumer Expectations, core survey and household finance module.

where $\tilde{\alpha}_{it}$ is the share of household i's financial wealth invested in stocks in period t, age_{it} is their age in years, $\tilde{\phi}_{it}$ is their subjective probability of surviving to age 85,[22] δ_t are period fixed-effects, X_{it} is a vector of controls which we vary to check the robustness of the results, and ε_{it} is an error term.

The coefficients of interest are β_2 and β_3. Our model predicts that $\beta_2 < 0$ and $\beta_3 > 0$. The first of these inequalities implies young households invest less in equities when their life expectancy is longer, and the second implies that the age-gradient of equity shares is greater when life expectancy is longer.

Table 2 shows the results. In column 1, the only controls are period fixed effects. Column 2 adds income and net wealth as extra controls, since the analysis in Corollary 2 and Figure 3 holds wealth (financial and human) constant. Column 3 adds a further range of demographic controls (details in table note). In all cases, the estimated coefficient on subjective survival probability (β_2) is negative, and the estimated interaction between subjective survival probability and age (β_3) is positive. Both are significantly different from zero.

The signs of the coefficients β_2 and β_3 are therefore consistent with the model. The

[22] We use tildes here to distinguish these concepts from their model counterparts α_{it} and ϕ_{age}.

Table 2: Regressions on share of financial assets invested in equity.

	(1) Equity share	(2) Equity share	(3) Equity share
Age	-0.0919	-0.1189	-0.0745
	(0.0991)	(0.0989)	(0.0995)
Subjective survival prob.	-0.1659**	-0.1626**	-0.1467**
	(0.0745)	(0.0746)	(0.0718)
Age × Subjective survival prob.	0.0040**	0.0039**	0.0035**
	(0.0017)	(0.0017)	(0.0016)
Net wealth ($1000s)		0.0007*	0.0004
		(0.0004)	(0.0003)
Income ($1000s)		0.0023	0.0018
		(0.0017)	(0.0013)
Period FE	Yes	Yes	Yes
Demographic controls	No	No	Yes
Observations	3466	3446	3438
R^2	0.0054	0.0110	0.0724

Note: Robust standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. *All regressions are weighted using the SCE-provided survey weights. The demographic controls in column 3 consist of dummy variables for the state in which the respondent lives, gender, home ownership status, marriage status, race, and education. Sample period: 2014-2019. Source: Survey of Consumer Expectations, core survey and household finance module.*

magnitudes are also substantial. The 25th percentile of subjective survival probability is a household who believes they have a 30% chance of living to the age of 85. At this subjective life expectancy, using the estimates from the most demanding specification (Table 2 column 3) one extra year of age is associated with a 3 basis point increase in the unconditional risky asset share. At the 75th percentile of subjective survival probabilities (75% chance of surviving to 85), that gradient is more than 6 times steeper: an extra year of age is associated with a 19 basis point increase in the unconditional risky asset share.

Table 3 shows the results from the same analysis, estimated in the restricted sample for whom $\tilde{\alpha}_{it} > 0$. The dependent variable here is therefore the conditional risky asset share. Qualitatively, the results are the same as in Table 2.

These results are robust to a number of specification changes and checks, documented in the Appendix C. In particular, columns 2 and 3 of Tables 2 and 3 use income and net wealth in levels, ensuring that we do not exclude households with zero or (for net wealth) negative values in these variables. However, restricting the sample to those with

Table 3: Regressions on share of financial assets invested in equity, conditional on equity market participation.

	(1) Equity share	(2) Equity share	(3) Equity share
Age	-0.1935	-0.2115	-0.1112
	(0.1685)	(0.1688)	(0.1713)
Subjective survival prob.	-0.3444***	-0.3360**	-0.2625**
	(0.1323)	(0.1327)	(0.1323)
Age × Subjective survival prob.	0.0076***	0.0074***	0.0055*
	(0.0028)	(0.0029)	(0.0028)
Net wealth ($1000s)		0.0005*	0.0004*
		(0.0003)	(0.0002)
Income ($1000s)		0.0011	0.0011
		(0.0013)	(0.0010)
Period FE	Yes	Yes	Yes
Demographic controls	No	No	Yes
Observations	1579	1568	1562
R^2	0.0166	0.0181	0.0735

Note: Robust standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. *For other regression details see the note to Table 2.*

strictly positive income and net wealth and including these variables in logs does not meaningfully alter the coefficients of interest.

Importantly, this pattern is not predicted by any existing mechanism proposed in the literature. In particular, models in which beliefs depend only on age predict a constant *age profile* of return expectations (Campanale, 2011; Peijnenburg, 2018). In such models, it does not matter whether a 30-year old believes they will live a long life or not: their beliefs depend only on the fact that they are 30. In a Merton-style model such as the one we develop above, this would imply that risky asset shares conditional on age should be unaffected by life expectancy.

In fact, in richer models pure age-based mechanisms are even more at odds with the data. Households with longer life expectancy should invest more in equities, as their planning horizon is longer and so they can absorb more short-term risk (e.g. Yogo, 2016). Younger households should therefore have larger risky asset shares when their subjective life expectancy is high, which is the opposite of the patterns we observe in the data. Our model can therefore explain a fact in the data which would otherwise be extremely puzzling.

5 Quantitative Analysis

We now return to the model with J-period maximum lifespans from Section 2, and add the features now standard in quantitative life-cycle portfolio-choice models: risky age-dependent labor income, Epstein-Zin preferences, and equity market participation costs.[23] The resulting model is similar to that in Gomes and Michaelides (2005), with the addition of ambiguity aversion as specified in Section 2. As these quantitative model features are standard in this literature, we leave the formal details to Appendix D.1.

We calibrate the model demographics to survival probabilities in the US in 2019. We then use the model to examine the effect of demographic change on portfolios in the coming decades. To do this, we compare the 2019 calibration with a counterfactual using projected survival probabilities for 2100. Even holding the distribution of labor income fixed, changes in life expectancy imply non-trivial changes in the age profile of portfolio composition.

5.1 Calibration

We calibrate the model such that one period is one year. First, we set the majority of the parameters to typical values in the literature, as discussed in Gomes (2020) (see Appendix D.2 for details). We set the initial age to $j = 20$, and the maximum possible lifespan to $J = 109$: this is the first age in the 2019 mortality data discussed below at which the annual mortality rate exceeded 50%. The remaining parameters are the survival probabilities, and the two parameters not present in the model reviewed in Gomes (2020): the equity market participation cost, and the degree of ambiguity aversion (θ).

Survival probabilities: We first split households into three groups. For the first group, we calibrate their survival probabilities using age-specific mortality rates in the US in 2019 reported by the US Office of the Chief Actuary at the Social Security Administration. This mortality data is representative for the US population. The survival probability ϕ_j is one minus the empirical mortality rate for people of age j. The second group (high perceived mortality) have perceived mortality rates equal to $(1 - \phi_j)(1 + \zeta)$, while the third group (low perceived mortality) have perceived mortality rates equal to $(1 - \phi_j)(1 - \zeta)$.[24] The perceived survival rates for each group are one minus the relevant perceived mortality

[23] As in Cocco et al. (2005) and others, we do not include a bequest motive in our baseline exercises. However, in Appendix D.5, we extend the model to include a bequest motive, and find that our results are qualitatively unchanged.

[24] This heterogeneity is in perceived mortality, not actual mortality. All agents' true mortality rates are the ones assigned to the first group, i.e., those taken from US data. All results except those in Appendix D.7 concern portfolio decisions conditional on age, so this distinction is largely not relevant, as portfolio decisions at each age depend on perceived and not actual mortality rates.

rates.

This heterogeneity in perceived survival probabilities is a robust fact of our survey data (see Table 1). We therefore calibrate the scaling factor ζ to match that data. For each perceived-mortality group we calculate the implied perceived probability of surviving to age 85 at each age. We choose ζ so that the standard deviation of this perceived survival probability across agents below the age of 65 is equal to that in our SCE sample (28%).[25] All resulting mortality rates for all groups remain in $(0, 1)$.

The key reason for allowing heterogeneous perceived mortality rates is to provide a disciplined way to calibrate the degree of ambiguity aversion, as explained below. However, it is not crucial for our qualitative results. Appendix D.6 presents the key quantitative exercise of this section without any heterogeneity in perceived mortality rates, and our conclusions are robust. Throughout the rest of this section, we will compare the model with ambiguity to one without, but which is otherwise identical. That 'no ambiguity benchmark' replicates the standard results from other quantitative portfolio choice models, showing further than mortality rate heterogeneity is not driving any of our results.

Participation costs and ambiguity aversion: We calibrate the equity market participation cost to match the average participation rate in the 2019 Survey of Consumer Finances (53%). Finally, we calibrate the degree of ambiguity aversion to target the results of the empirical test of our mechanism in Section 4. Specifically, we target β_2 and β_3 in estimates of equation (59), as these coefficients form the key test of our ambiguity-driven mechanism.

Matching the regression coefficients: To target the results of Section 4, we estimate equation (59) on simulated data from the model.

Solving the model yields decision rules for all three mortality-rate groups. We simulate 10000 agents from each group, to give a simulation sample of 30000 agents, who differ by age, subjective probability of surviving to age 85, income, wealth, and portfolio choices. With this sample, we then estimate equation (59) on the simulated data for all agents below age 65 (to match the SCE sample). We use the conditional risky asset share as the dependent variable, because the unconditional share is more strongly affected by the participation cost, which hinders our ability to separately identify θ from that cost.

This means that we replicate Table 3 in the model. We choose θ to target the key coefficients (β_2 and β_3) in the most demanding specification (column 3). With a plausible degree of ambiguity aversion (see Appendix D.3), we obtain $\beta_2 = -0.3073$, $\beta_3 = 0.0047$.[26]

[25] We only consider agents below age 65 to match the SCE sample described in Section 4.1.

[26] The match is not exact because we have one parameter and two targets. Increasing θ further increases β_2 (closer to target) but decreases β_3 (further from target). Our choice balances the two deviations in

This slightly overstates the level effect of greater survival probability (β_2), and understates the interaction effect between age and subjective survival probability in particular (β_3), relative to the data. Since our main exercise below concerns the effect of increased longevity on the gradient of the age-profile of risky asset shares (Section 5.3), understating the interaction effect implies that our results in that projection are if anything somewhat conservative.

Note that this calibration approach is only possible because we have allowed for heterogeneous perceived mortality rates. Without that (i.e., if $\zeta = 0$), there would be no heterogeneity in subjective survival probabilities once we condition on an agent's age. In that case, we could not run the regression.

5.2 Age Profiles of Portfolio Allocations

Figure 8 plots the conditional risky asset share for agents of different ages in the calibrated baseline model, alongside the profile from an otherwise-identical model without ambiguity, and the empirical age profile from the SCE.[27] As in Section 4, we use $\tilde{\alpha}_{j,t}$ to refer to the share of risky assets in financial wealth, with the tilde to distinguish this from the share of risky assets in total wealth analyzed in Sections 2 and 3.

The calibrated model generates similar levels of conditional risky shares as seen in the data, and replicates the generally increasing age profile, despite neither of these being targeted in the calibration. Note that while the data displays a slight downward slope from ages 40-60, this is not systematically present in other datasets (see e.g. Chang et al., 2018; Catherine, 2022). The life-cycles of the participation rate and the unconditional risky asset share are in Appendix D.4.

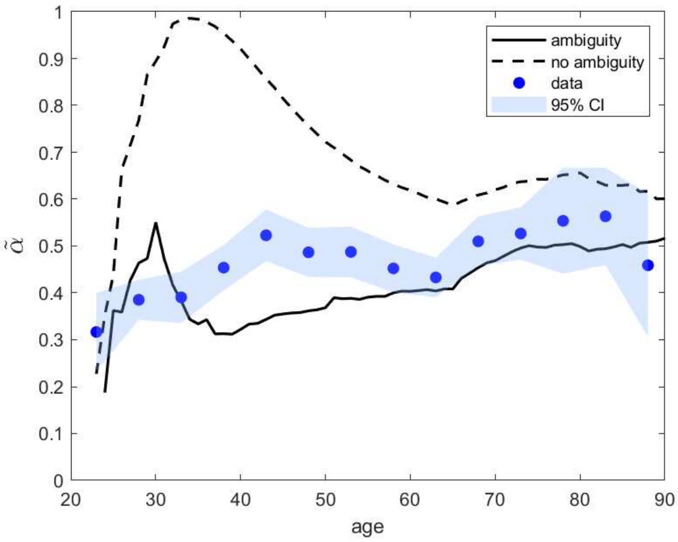
Of course, the empirical age-profile shown is a snapshot of a particular point in time, and so conflates age, period, and cohort effects. A large literature attempts to disentangle the pure age effect using estimated models and a range of identifying assumptions (see Gomes and Smirnova, 2021, for a recent example). However, this is not our focus. A key part of our mechanism is that changes in life expectancy affect portfolio decisions, through changes in ambiguity-driven belief distortions. Since life expectancy changes over time and by cohort, we do not want to strip out these effects. Indeed, exploring how this profile might change over time is the purpose of Section 5.3.

To understand the mechanisms at work in this result, we begin by discussing the case without ambiguity, which is similar to many other standard models in this literature. In

percentage terms.

[27]We use the SCE as this was used in the calibration step, but note that a similar age-profile has been observed in the Survey of Consumer Finances (Chang et al., 2018). The SCE data is pooled across 2014-2019 to obtain reasonable sample sizes in each age bin.

Figure 8: Model-generated vs. empirical conditional risky asset shares, 2019 calibration.



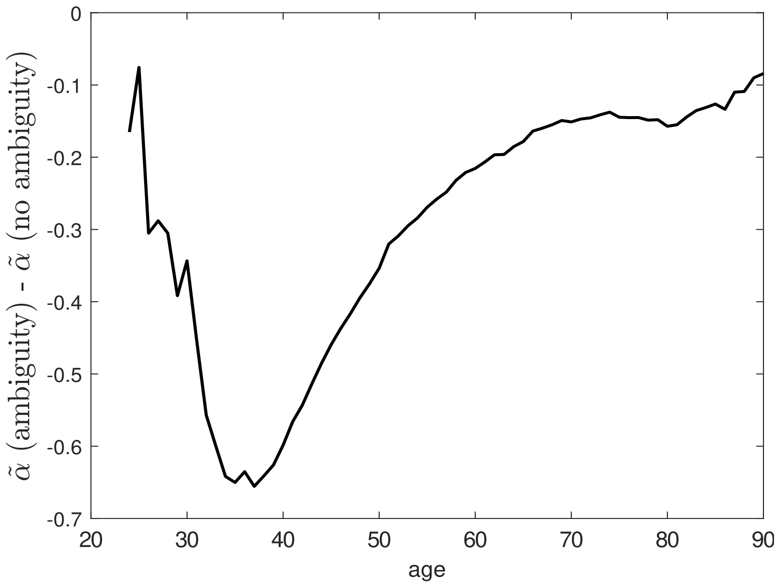
Note: The solid line is constructed using the model, with calibration described in Section 5.1 and Appendix D.2. The dashed line is constructed from the same model, with the preference for ambiguity set to $\theta = 0$. The circles and shaded areas are from the August waves of SCE from 2014-2019, as described in Section 4. Each circle is the mid-point of a five-year age range, and denotes the mean conditional risky asset share in that range. The shaded area is the 95% confidence interval around these means.

this benchmark, the risky asset share rises rapidly early in life. This is a well-known effect of participation costs (see e.g. the discussion in Gomes and Michaelides, 2005): as wealth rises rapidly early in life more agents find it optimal to pay the participation cost. After this point, the risky asset share decreases, then after retirement at age 65 it is mildly increasing. This path reflects the changing ratio of financial assets to 'human wealth' (the present value of future income). Since labor and retirement income are uncorrelated with risky asset returns, any expected future income acts like an extra endowment of risk-free bonds in the agent's portfolio problem. As the agent moves through their working life, their financial assets increase relative to the remaining expected future labor income. To maintain a constant share of risky assets out of total wealth, this shift to more financial wealth implies the agent must decrease the share of their financial assets invested in the risky asset. After retirement, wealth is gradually spent, implying that the financial wealth to income ratio decreases, reversing this trend. For a complete discussion of these well-known effects, see Gomes (2020) and the references therein.

The model with ambiguity features a similar path of risky asset shares at very young ages. However, the initial rise in α is much smaller, and the subsequent fall lasts for less time. After age 35, the risky asset share steadily increases, gradually converging back to the no-ambiguity benchmark.

To express this another way, Figure 9 plots the gap between the average conditional risky asset share in the benchmark no-ambiguity model and the model with ambiguity.

Figure 9: The effects of ambiguity on conditional risky asset shares.



Note: The solid line is the difference between the 'ambiguity' and 'no ambiguity' lines in Figure 8. See the note to Figure 8 for details.

At very young ages, ambiguity has little effect. This is because the vast majority of very young agents' total wealth is contained in the present value of future labor income, not in financial wealth. Even though they have a high marginal utility of income, they are not therefore particularly exposed to fluctuations in financial portfolio returns.

However, outside of these very early years, the optimal belief distortions are similar to those implied by the simple model. In Figure 2, for the majority of the parameter space we found that young agents distorted their beliefs more than old agents. In the language of Section 3.2.1, the marginal utility channel dominates the wealth channel. As agents age from 35 to 65, their consumption rises (see Appendix D.4), and so their marginal utility of income falls, implying a smaller exposure to ambiguity in risky asset returns. The optimal belief distortion shrinks over these periods, reducing the effect of ambiguity. After retirement at 65, consumption does begin to fall, but more slowly than wealth. In this region, the wealth channel therefore dominates, leading to further declines in belief distortions.

5.3 Projecting Asset Demand

We showed in Section 3 that life expectancy is a key driver of the age profile of portfolio choices. As populations age, survival probabilities increase particularly for older cohorts.

In our model, this leads to differential changes in the portfolio choices of different age groups, with implications for inequality within and between cohorts.

To explore this dependence on demographics, we take the calibrated model and replace the 2019 survival rates with demographic projections for the year 2100 in the US.[28] As well as ϕ_j, we also recalibrate the maximum lifespan J using the same approach as before: we find the first age at which the annual mortality rate is expected to exceed 50%, which for 2100 implies $J = 117$. These projections come from the US Office of the Chief Actuary, who predict survival probabilities rising substantially, particularly for the oldest age groups.[29] This exercise therefore gives a projection of the direct effect of extending life expectancy on portfolio choices, holding everything else fixed.

Figure 10a shows the results, plotting the model-implied age profile of conditional risky asset shares in 2019 and 2100. Solid lines plot $\tilde{\alpha}_{j,2019}$ and dashed lines plot the equivalent $\tilde{\alpha}_{j,2100}$. The increase in life expectancy by 2100 causes younger middle-aged households to invest less in risky assets, as they distort their beliefs more strongly towards low risky returns. For older households, return expectations decline by less, so they become more optimistic relative to the young. To make these effects clearer, Figure 10b plots the difference between 2100 and 2019 for each age, alongside the equivalent "longevity effect" in the model without ambiguity.

Beyond very young agents, who as highlighted above are not very susceptible to this form of ambiguity, the age profile of risky asset shares therefore becomes steeper, in line with the results with $J = 2$ (Corollary 2).[30] Quantitatively, the gap between the risky asset shares of those aged 80 and 35 rises from 16.6 p.p. to 20.9 p.p., an increase of 26%. There is no such effect in the model without ambiguity. If anything, older households decrease their risky asset shares slightly relative to younger households when facing increased longevity, which is counter to the cross-sectional evidence in Section 4.

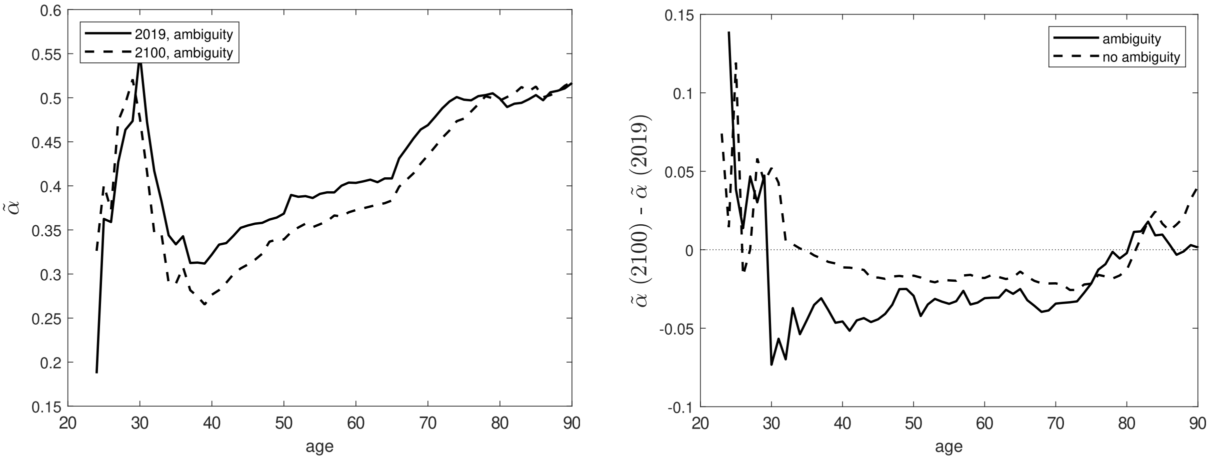
Plausible increases in survival probabilities therefore cause substantial changes in average portfolio decisions. Importantly, this projection only captures changes directly

[28] A related exercise is performed in Auclert et al. (2021), which takes an OLG model and forecasts future wealth-to-income ratios by holding fixed the average wealth and income of each age group, but varying the proportions of households of each age according to UN projections. When we change ϕ_j, we keep the parameter governing heterogeneity in perceived mortality (ζ) constant.

[29] For example, in 2019 the death rate among 70-year-olds in the US was 1.9%. In 2100 that is projected to fall to 1.0%. These changes in death rates mean that life expectancy is projected to rise substantially. Life expectancy conditional on reaching age 30, for example, is projected to rise by more than 6 years, from 79.5 to 85.7. Conditional on reaching age 70, the projected rise is from 85.2 to 89.1. The data is available at https://www.ssa.gov/oact/HistEst/Death/2023/DeathProbabilities2023.html. Note that these projections contain substantial uncertainty, which we are abstracting from here. If life expectancy instead declines over this period, then the results would be the opposite of those we plot below, as the mechanisms outlined in Section 3 would operate in reverse.

[30] Note the participation rate is small for young agents in this model, so the simulated conditional risky share for those below the age of 30 is based on a small number of agents. This explains the noisy behavior of Figure 10b below age 30.

Figure 10: Model-implied conditional risky asset shares, 2019 and 2100.



(a) Levels, 2019 and 2100 **(b)** Difference between 2100 and 2019

Note: Plots constructed using the calibration and data described in Section 5.1 and Appendix D.2. In panel (b), each line is the average conditional risky asset share at age j in the 2100 calibration, minus the equivalent average at that age in the 2019 calibration.

due to the age effect on ambiguity aversion and saving. If the income distribution or asset returns change over time, they would further alter these results. In Appendix D.7, for example, we explore one possibility, endogenizing the equity premium as in Section 3.5. In this case, the increase in life expectancy to 2100 causes the equity premium to rise, continuing its recent rising trend. However, we find that quantitatively the effect is modest, with the equity premium rising by 7 basis points, implying this particular channel does not substantially alter the conclusions presented here.

6 Conclusion

We develop a model in which investors face ambiguity over expected returns on risky assets. In contrast to previous literature, we allow agents to choose the degree to which they respond to this ambiguity optimally. With the same preferences, ambiguity aversion causes stronger distortions to return expectations among agents whose utility is very sensitive to risky asset returns. This implies differential effects of ambiguity on expected returns and portfolio choices for agents with different levels of wealth, and different life expectancies.

In particular, as life expectancy rises, younger investors become more sensitive to rates of return, which means they distort their beliefs to be more pessimistic about the returns to risky assets, and they invest less in those assets. In contrast, in the empirically reasonable case where the elasticity of intertemporal substitution is less than 1, older agents become more optimistic about risky asset returns, and allocate a greater share of their savings to them. In this case wealthier households are also endogenously more optimistic about risky asset returns, fueling greater savings rates and risky asset shares, as documented in Straub (2019), Briggs et al. (2020) and others.

The prediction that greater life expectancy causes differential effects on the portfolio decisions of young and old households is novel to this model. We test it in survey data on US households, and find strong support: the age-profile of equity shares in portfolios is substantially steeper among households with a longer subjective life expectancy. This is difficult to explain without the ambiguity aversion we study.

In a quantitative extension of the model, we generate empirically plausible age profiles of risky asset shares in the US. We then use this quantitative model to project asset demand forward to 2100, to uncover the likely effect of demographic changes on savings behavior. As the population ages, and life expectancies increase, older households increase the share of their wealth invested in risky assets relative to the young.

To obtain these results, we have focused on a particular dimension of ambiguity that has received extensive empirical support (Dimmock et al., 2016). However, there are a range of related areas in which it is also plausible that agents face substantial ambiguity, which may themselves interact with life expectancy, and which could further alter financial decision-making. For example, agents may face ambiguity about their future income, or their own survival probabilities,[31] in addition to the ambiguity about their asset returns. Future research could profitably explore the interactions of these multiple dimensions of ambiguity, and how they are jointly affected by demographic change.

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[31] Caliendo et al. (2020) discuss the plausibility of survival ambiguity in detail.

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A Proofs

Proposition 1: Since $\phi_J = 0$ and there are no bequests, there is no incentive to save at age J. As we also impose that $w_{J+1,T+1} \geq 0$ (10), all agents at age J choose to consume all of their wealth. With no continuation value, the value function of these agents is:

$$V_J(w_{j,t}) = \frac{w_{j,t}^{1-\gamma}}{1-\gamma} \tag{60}$$

For all $j < J$, we now conjecture that the value function takes the same functional form:

$$V_j(w_{j,t}) = A_j \frac{w_{j,t}^{1-\gamma}}{1-\gamma} \tag{61}$$

for some age-dependent constant A_j. The expectation of the next-period value function becomes:

$$E_{j,t}[V_{j+1}(w_{j+1,t+1})] = E_{j,t}[A_{j+1}\frac{w_{j+1}^{1-\gamma}}{1-\gamma}] \tag{62}$$

$$\approx \frac{A_{j+1}}{1-\gamma}(w_{j,t} - c_{j,t})^{1-\gamma} E_{j,t}\left\{\exp\left[r^f + \alpha_{j,t}(r_{t+1} - r^f) + \frac{1}{2}\alpha_{j,t}(1-\alpha_{j,t})\sigma^2\right]\right\}^{1-\gamma} \tag{63}$$

where equation (63) uses the log-linear approximation to the one-period excess return around zero in equation (12) and Campbell (1993). Since r_{t+1} is normally distributed, we can further write:

$$E_{j,t}[V_{j+1}(w_{j+1,t+1})] = \frac{A_{j+1}}{1-\gamma}(w_{j,t}-c_{j,t})^{1-\gamma} \exp\left\{(1-\gamma)\left[r^f + \alpha_{j,t}(\widetilde{\mu} - r^f) + \frac{1}{2}\alpha_{j,t}(1-\alpha_{j,t})\sigma^2\right] + \frac{1}{2}(1-\gamma)^2\alpha_{j,t}^2\sigma^2\right\} \quad (64)$$

Using this in equation (8) and taking first order conditions with respect to $\alpha_{j,t}, c_{j,t}$ gives equations (17) and (18), with b_{j+1} as in equation (16) (noting that $\mu \equiv \tilde{\mu} + \sigma^2/2$).

Finally, we verify the conjecture in equation (61) for $j = 0, ..., J-1$. Substituting equation (64) into the value function (8) we have:

$$A_j \frac{w_{j,t}^{1-\gamma}}{1-\gamma} = \frac{(c_{j,t}^*)^{1-\gamma}}{1-\gamma} + \beta\phi_j \frac{A_{j+1}}{1-\gamma}(w_{j,t} - c_{j,t}^*)^{1-\gamma}\left\{1 + (1-\gamma)[r^f + (\mu - r^f)\alpha_{j,t}^*] - \frac{1}{2}\gamma(1-\gamma)\sigma^2(\alpha^*)^2\right\} \quad (65)$$

where $\alpha^*, c_{j,t}^*$ denote the optimal risky share and consumption from equations (17) and (18) respectively. Since α^* is a constant, and $c_{j,t}^*$ is proportional to $w_{j,t}$, this verifies the conjectured functional form for $V_j(w_{j,t})$ (61). Matching coefficients, we obtain equation (15).

Lemma 1: First conjecture that the value function takes the form:

$$V_j^\theta(w_{j,t}) = A_j \frac{w_{j,t}^{1-\gamma}}{1-\gamma} + \theta B_j \frac{w_{j,t}^{2(1-\gamma)}}{2(1-\gamma)} + O(\theta^2) \quad (66)$$

Taking the same log-linear approximation approach as in (12) to the distorted returns in (19), we can write

$$\begin{aligned} E_{j,t}[V_{j+1}^\theta(w_{j+1,t+1})] \approx\ & \frac{A_{j+1}}{1-\gamma}(w_{j,t} - c_{j,t})^{1-\gamma}\{1 + (1-\gamma)[r^f + (\widetilde{\mu} - r^f + \sigma\nu_{j,t})\alpha_{j,t}] \\ & + \frac{1}{2}(1-\gamma)\sigma^2(\alpha_{j,t} - \gamma\alpha_{j,t}^2)\} \\ & + \theta\frac{B_{j+1}}{2(1-\gamma)}(w_{j,t} - c_{j,t})^{2(1-\gamma)}\{1 + 2(1-\gamma)[r^f + (\widetilde{\mu} - r^f + \sigma\nu_{j,t})\alpha_{j,t}] \\ & + (1-\gamma)\sigma^2(\alpha_{j,t} + \alpha_{j,t}^2 - 2\gamma\alpha_{j,t}^2)\} \\ =\ & E[V_{j+1}(w_{j+1,t+1}^*)] + \frac{A_{j+1}}{1-\gamma}(w_{j,t} - c_{j,t})^{1-\gamma}(1-\gamma)\sigma\alpha_{j,t}\nu_{j,t} \\ & + \theta\frac{B_{j+1}}{2(1-\gamma)}(w_{j,t} - c_{j,t})^{2(1-\gamma)}2(1-\gamma)\sigma\alpha_{j,t}\nu_{j,t} \end{aligned} \quad (67)$$

where in the first approximation we drop the term with θ^2 and higher orders, and use the approximation:

$$R_{t+1} + \sigma_1 \nu_{j,t} \approx \exp(r_{t+1} + \sigma \nu_{j,t}) \tag{68}$$

In equation (68), σ_1 is the standard deviation of R_{t+1}, which equals to $[\exp(\sigma^2) - 1]^{\frac{1}{2}}[\exp(2\widetilde{\mu}+\sigma^2)]^{\frac{1}{2}}$. Note that $\theta\nu_{j,t} << \nu_{j,t}$, we can further approximate $E_{j,t}[V^{\theta}_{j+1}(w_{j+1,t+1})]$ as:

$$E_{j,t}[V^{\theta}_{j+1}(w_{j+1,t+1})] \approx E_{j,t}[V_{j+1}(w^*_{j+1,t+1})] + \frac{A_{j+1}}{1-\gamma}(w_{j,t} - c_{j,t})^{1-\gamma}(1-\gamma)\sigma\alpha_{j,t}\nu_{j,t} \tag{69}$$

Proposition 2: Continue with the guess of the value function's form:

$$\begin{aligned} V^{\theta}_j(w_{j,t}) &= A_j \frac{w_{j,t}^{1-\gamma}}{1-\gamma} + \theta B_j \frac{w_{j,t}^{2(1-\gamma)}}{2(1-\gamma)} + O(\theta^2) && (70) \\ &\approx V^0_j(w_{j,t}) + \theta V^1_j(w_{j,t}) && (71) \end{aligned}$$

Note that A_j is the same as in the benchmark model. When there is no ambiguity aversion (i.e. $\theta = 0$), the value function degenerates into that of the benchmark model.

It is intuitive to conjecture that the optimal portfolio choice and consumption take the following forms:

$$\begin{aligned} \alpha_{j,t} &= \alpha^* + \theta\alpha'_{j,t} && (72) \\ c_{j,t} &= c^*_{j,t} + \theta c'_{j,t} && (73) \end{aligned}$$

where $\alpha^*, c^*_{j,t}$ are the solutions without ambiguity defined in Proposition 1.

Then we can approximate the value function by dropping the term with θ^2 and higher orders:

$$\begin{aligned} V^{\theta}_j(w_{j,t}) &= \max_{c,\alpha}\{\frac{c_{j,t}^{1-\gamma}}{1-\gamma} + \beta\phi_j[-\frac{1}{2}\theta A^2_{j+1}(w_{j,t} - c_{j,t})^{2-2\gamma}\sigma^2\alpha^2_{j,t} + E_{j,t}[V^{\theta}_{j+1}(w^*_{j+1,t+1})]]\} \\ &\approx \max_{c,\alpha}\{\frac{c_{j,t}^{1-\gamma}}{1-\gamma} + \beta\phi_j[-\frac{1}{2}\theta A^2_{j+1}(w_{j,t} - c_{j,t})^{2-2\gamma}\sigma^2\alpha^2_{j,t} + E_{j,t}[V^0_{j+1}(w^*_{j+1,t+1})] \\ &\quad +\theta E_{j,t}[V^1_{j+1}(w^*_{j+1,t+1})]]\} \end{aligned}$$

where using log linearization we have:

$$\begin{aligned}
E_{j,t}[V^0_{j+1}(w^*_{j+1,t+1})] &\approx \frac{A_{j+1}}{1-\gamma}(w_{j,t}-c_{j,t})^{1-\gamma}\{1+(1-\gamma)[r^f+(\widetilde{\mu}-r^f)\alpha_{j,t}] \\
&+\frac{1}{2}(1-\gamma)\sigma^2(\alpha_{j,t}-\gamma\alpha^2_{j,t})\} \\
E_{j,t}[V^1_{j+1}(w^*_{j+1,t+1})] &\approx \frac{B_{j+1}}{2(1-\gamma)}(w_{j,t}-c_{j,t})^{2(1-\gamma)}\{1+2(1-\gamma)[r^f+(\widetilde{\mu}-r^f)\alpha_{j,t}] \\
&+(1-\gamma)\sigma^2(\alpha_{j,t}+\alpha^2_{j,t}-2\gamma\alpha^2_{j,t})\}
\end{aligned}$$

The FOC w.r.t. $\alpha_{j,t}$ gives:

$$\begin{aligned}
0 &= -\theta A^2_{j+1}(w_{j,t}-c_{j,t})^{1-\gamma}\sigma^2\alpha_{j,t}+A_{j+1}\{(\widetilde{\mu}-r^f)+\frac{1}{2}\sigma^2-\gamma\sigma^2\alpha_{j,t}\} \\
&+\theta B_{j+1}(w_{j,t}-c_{j,t})^{1-\gamma}\{(\widetilde{\mu}-r^f)+\frac{1}{2}\sigma^2+(1-2\gamma)\sigma^2\alpha_{j,t}\}
\end{aligned}$$

Substituting $\mu=\widetilde{\mu}+\frac{1}{2}\sigma^2$ into the equation gives:

$$\begin{aligned}
0 &= -\theta A^2_{j+1}(w_{j,t}-c_{j,t})^{1-\gamma}\sigma^2\alpha_{j,t}+A_{j+1}\{(\mu-r^f)-\gamma\sigma^2\alpha_{j,t}\} \\
&+\theta B_{j+1}(w_{j,t}-c_{j,t})^{1-\gamma}\{(\mu-r^f)+(1-2\gamma)\sigma^2\alpha_{j,t}\}
\end{aligned}$$

By plugging $\alpha_{j,t}=\alpha^*_{j,t}+\theta\alpha'_{j,t}=\frac{\mu-r}{\gamma\sigma^2}+\theta\alpha'_{j,t}$, we get:

$$\begin{aligned}
0 &= -\theta A^2_{j+1}(w_{j,t}-c_{j,t})^{1-\gamma}\sigma^2(\alpha^*_{j,t}+\theta\alpha'_{j,t})-\theta A_{j+1}\gamma\sigma^2\alpha'_{j,t} \\
&+\theta B_{j+1}(w_{j,t}-c_{j,t})^{1-\gamma}\{(\mu-r^f)+(1-2\gamma)\sigma^2(\frac{\mu-r^f}{\gamma\sigma^2}+\theta\alpha'_{j,t})\}
\end{aligned}$$

Further approximation by dropping terms with θ^2 and simplification give:

$$\begin{aligned}
0 &\approx -A^2_{j+1}(w_{j,t}-c_{j,t})^{1-\gamma}\sigma^2\alpha^*_{j,t}-A_{j+1}\gamma\sigma^2\alpha'_{j,t} \\
&+B_{j+1}(w_{j,t}-c_{j,t})^{1-\gamma}(\mu-r^f)(\frac{1}{\gamma}-1) \\
\alpha'_{j,t} &= -\frac{A^2_{j+1}+B_{j+1}(\gamma-1)}{A_{j+1}\gamma}\alpha^*_{j,t}(w_{j,t}-c^*_{j,t})^{1-\gamma} \\
&= -\frac{A^2_{j+1}+B_{j+1}(\gamma-1)}{A_{j+1}\gamma}\alpha^*_{j,t}(\frac{w_{j,t}}{1+b_{j+1}})^{1-\gamma} \qquad (74)
\end{aligned}$$

Then we can simplify $E_{j,t}[V^0_{j+1}(w^*_{j+1,t+1})]$ and $E_{j,t}[V^1_{j+1}(w^*_{j+1,t+1})]$:

$$\begin{aligned} E_{j,t}[V^0_{j+1}(w^*_{j+1,t+1})] &\approx \frac{A_{j+1}}{1-\gamma}(w_{j,t}-c_{j,t})^{1-\gamma}\{1+(1-\gamma)r^f+\frac{1}{2}\gamma(1-\gamma)\sigma^2(\alpha^*_{j,t})^2\} \\ E_{j,t}[V^1_{j+1}(w^*_{j+1,t+1})] &\approx \frac{B_{j+1}}{2(1-\gamma)}(w_{j,t}-c_{j,t})^{2(1-\gamma)}\{1+2(1-\gamma)r^f \\ &\quad +(1-\gamma)\sigma^2(\alpha^*_{j,t})^2\} \end{aligned}$$

We move onto the RHS of the Bellman equation:

$$RHS = \frac{c^{1-\gamma}_{j,t}}{1-\gamma}+\beta\phi_j[-\frac{1}{2}\theta A^2_{j+1}(w_{j,t}-c_{j,t})^{2-2\gamma}\sigma^2(\alpha^*_{j,t})^2+E_t[V^0_{j+1}(w^*_{j+1,t+1})]+\theta E_{j,t}[V^1_{j+1}(w^*_{j+1,t+1})]]$$

The FOC w.r.t $c_{j,t}$ gives:

$$\begin{aligned} 0 &= c^{-\gamma}_{j,t}-(w_{j,t}-c_{j,t})^{-\gamma}b^{-\gamma}_{j+1} \\ &\quad +\theta\beta\phi_j(w_{j,t}-c_{j,t})^{1-2\gamma}\{(A^2_{j+1}-B_{j+1})(1-\gamma)\sigma^2(\alpha^*_{j,t})^2-B_{j+1}[1+2(1-\gamma)r^f]\} \end{aligned}$$

By plugging $c_{j,t}=c^*_{j,t}+\theta c'_{j,t}$, we get:

$$\gamma b^{-\gamma-1}_{j+1}c'_{j,t}+\gamma b^{-\gamma}_{j+1}c'_{j,t} = \beta\phi_j(w_{j,t}-c^*_{j,t})^{2-\gamma}[(A^2_{j+1}-B_{j+1})(1-\gamma)\sigma^2(\alpha^*_{j,t})^2-B_{j+1}[1+2(1-\gamma)r^f]]$$

Therefore:

$$\begin{aligned} c'_{j,t} &= \frac{\beta\phi_j\{(A^2_{j+1}-B_{j+1})(1-\gamma)\sigma^2(\alpha^*_{j,t})^2-B_{j+1}[1+2(1-\gamma)r^f]\}}{\gamma b^{-\gamma}_{j+1}(1+b^{-1}_{j+1})}(w_{j,t}-c^*_{j,t})^{2-\gamma} \\ &= \frac{\beta\phi_j\{(A^2_{j+1}-B_{j+1})(1-\gamma)\sigma^2(\alpha^*_{j,t})^2-B_{j+1}[1+2(1-\gamma)r^f]\}}{\gamma b^{-\gamma}_{j+1}(1+b^{-1}_{j+1})}(\frac{w_{j,t}}{1+b_{j+1}})^{2-\gamma} \end{aligned} \quad (75)$$

where we use first-order Taylor approximation:

$$\begin{aligned} c^{-\gamma}_{j,t} &\approx (c^*_{j,t})^{-\gamma}-\gamma(c^*_{j,t})^{-\gamma-1}\theta c'_{j,t} \\ (w_{j,t}-c_{j,t})^{-\gamma} &\approx (w_{j,t}-c^*_{j,t})^{-\gamma}+\gamma(w_{j,t}-c^*_{j,t})^{-\gamma-1}\theta c'_{j,t} \end{aligned}$$

Plugging the above results into the Bellman equation gives:

$$\begin{aligned} B_j\frac{w^{2(1-\gamma)}_{j,t}}{2(1-\gamma)} &= [\frac{1}{\gamma(1+b^{-1}_{j+1})}-\frac{1}{2(1-\gamma)}]\beta\phi_j\{(A^2_{j+1}-B_{j+1})(1-\gamma)\sigma^2(\alpha^*_{j,t})^2 \\ &\quad -B_{j+1}[1+2(1-\gamma)r^f]\}(w_{j,t}-c^*_{j,t})^{2-2\gamma} \end{aligned}$$

where we use first-order Taylor approximation again:

$$c_{j,t}^{1-\gamma} \approx (c_{j,t}^*)^{1-\gamma} - (\gamma-1)(c_{j,t}^*)^{-\gamma}\theta c'_{j,t}$$

Matching coefficients w.r.t. $w_{j,t}$ gives:

$$B_j = \beta\phi_j[\frac{2(1-\gamma)}{\gamma(1+b_{j+1}^{-1})} - 1]\{(A_{j+1}^2 - B_{j+1})(1-\gamma)\sigma^2(\alpha_{j,t}^*)^2 - B_{j+1}[1+2(1-\gamma)r^f]\}(\frac{1}{1+b_{j+1}})^{2-2\gamma} \tag{76}$$

with $B_J = 0$ because the agents in the last period do not invest and hence are not exposed to uncertainty. The system can be solved backward.

We can express $\alpha_{j,t}$ and $c_{j,t}$ in another way:

$$\alpha_{j,t} = \alpha^* + \theta\alpha^* w_{j,t}^{1-\gamma}\Omega_{\alpha j} \tag{77}$$
$$c_{j,t} = c_{j,t}^* + \theta w_{j,t}^{2-\gamma}\Omega_{cj} \tag{78}$$

where

$$\Omega_{\alpha j} = -\frac{A_{j+1}^2 + B_{j+1}(\gamma-1)}{\gamma A_{j+1}(1+b_{j+1})^{\gamma-1}} \tag{79}$$
$$\Omega_{cj} = \frac{\beta\phi_j\{(A_{j+1}^2 - B_{j+1})(1-\gamma)\sigma^2(\alpha^*)^2 - B_{j+1}[1+2(1-\gamma)r^f]\}}{\gamma b_{j+1}^{-\gamma-1}(1+b_{j+1})^{3-\gamma}} \tag{80}$$

For Section 3.3, we note that Ω_{cj} is potentially non-monotonic in ϕ_1, depending on parameters:

$$sgn(\frac{\partial\Omega_{c1}}{\partial\phi_1}) = sgn((1-\gamma)\frac{\partial\phi_1^{-\frac{1}{\gamma}}(1+\tilde{b}\phi_1^{-\frac{1}{\gamma}})^{\gamma-3}}{\partial\phi_1}) \tag{81}$$
$$= sgn((\gamma-1)(1+(\tilde{b}+\gamma-3)\phi_1^{-\frac{1}{\gamma}})) \tag{82}$$

Proposition 3: The result follows largely from applying Proposition 1.

Since $J = 2$, we have that $A_2 = 1$, and so:

$$b_2 = \left[\beta\phi_1[1+(1-\gamma)r^f + \frac{1}{2}(1-\gamma)\frac{(\mu-r^f)^2}{\gamma\sigma^2}]\right]^{-\frac{1}{\gamma}} \tag{83}$$
$$= \phi_1^{-\frac{1}{\gamma}}\tilde{b} \tag{84}$$

where $\tilde{b}$ is defined in (31).

Applying the definitions in Proposition 1, we further obtain:

$$A_1 = \left(\frac{\phi_1^{\frac{1}{\gamma}} + \tilde{b}}{\tilde{b}}\right)^{\gamma} \quad (85)$$

$$b_1 = \frac{\tilde{b}}{\phi_1^{\frac{1}{\gamma}} + \tilde{b}} \quad (86)$$

Substituting these into equation (18) gives equations (29) and (30). Finally, note using equations (10) and (9) that:

$$c_{2,t+2} = w_{2,t+2} = w_{0,t}R^p_{0,t+1}R^p_{1,t+2} - c_{0,t}R^p_{0,t+1}R^p_{1,t+2} - c_{1,t+1}R^p_{1,t+2} \quad (87)$$

Substituting in equations (29) and (30) implies equation (32).

Proposition 4: Since in a simple three-period model, the old do not invest in the stock market, we have $\nu_{2,t} = 0$. Substituting equations (27) and (28) into equation (24), and using the expressions for A_1, b_1 and b_2 gives us the optimal distortions for the young and the middle-aged.

Corollary 1: Using Proposition 4 we have that:

$$\nu_{1,t}(1) = -\frac{\theta\sigma\alpha^*}{(1+\tilde{b})^{1-\gamma}} w_{1,t}^{1-\gamma} \quad (88)$$

And:

$$\nu_{0,t}(0) = -\frac{\theta\sigma\alpha^*}{\tilde{b}^{\gamma-1}(\tilde{b}+\tilde{b}^2)^{1-\gamma}} w_{0,t}^{1-\gamma} = -\frac{\theta\sigma\alpha^*}{(1+\tilde{b})^{1-\gamma}} w_{0,t}^{1-\gamma} \quad (89)$$

Therefore $\nu_{0,t}(0) = \nu_{1,t}(1)$

Corollary 2: For the middle-aged:

$$\begin{aligned}\frac{\partial \nu_{1,t}}{\partial \phi_1} &= -\theta\sigma\alpha^*_{1,t}(w_{1,t})^{1-\gamma}\frac{\partial(1+\phi_1^{-\frac{1}{\gamma}}\tilde{b})^{\gamma-1}}{\partial\phi} \\ &= \frac{\gamma-1}{\gamma}\theta\sigma\alpha^*_1(w_{1,t})^{1-\gamma}(1+\phi_1^{-\frac{1}{\gamma}}\tilde{b})^{\gamma-2}\phi_1^{-\frac{1}{\gamma}-1}\tilde{b}\begin{cases} < 0 & \text{if } \gamma < 1 \\ = 0 & \text{if } \gamma = 1 \\ > 0 & \text{if } \gamma > 1 \end{cases}\end{aligned} \quad (90)$$

For the young:

$$\frac{\partial \nu_{0,t}}{\partial \phi_1} = -\theta\sigma\alpha^*_{0,t}(w_{0,t})^{1-\gamma}\tilde{b}^{-\gamma}(\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2)^{-2+\gamma}\frac{1}{\gamma}\phi_1^{\frac{1}{\gamma}-1}(\tilde{b}^2 + \gamma\phi_1^{\frac{1}{\gamma}} + \tilde{b}\gamma) < 0 \qquad (91)$$

Corollary 3: For the middle-aged:

$$\frac{\partial \nu_{1,t}}{\partial w_{1,t}} = -\frac{\theta\sigma\alpha^*\phi_1^{\frac{1-\gamma}{\gamma}}}{(\phi_1^{\frac{1}{\gamma}} + \tilde{b})^{1-\gamma}} w_{1,t}^{-\gamma}(1-\gamma) \begin{cases} < 0 & \text{if } \gamma < 1 \\ = 0 & \text{if } \gamma = 1 \\ > 0 & \text{if } \gamma > 1 \end{cases}$$

For the young:

$$\frac{\partial \nu_{0,t}}{\partial w_{0,t}} = -\frac{\theta\sigma\alpha^*(\phi_1^{\frac{1}{\gamma}} + \tilde{b})}{\tilde{b}^{\gamma}(\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2)^{1-\gamma}} w_{0,t}^{-\gamma}(1-\gamma) \begin{cases} < 0 & \text{if } \gamma < 1 \\ = 0 & \text{if } \gamma = 1 \\ > 0 & \text{if } \gamma > 1 \end{cases}$$

Corollary 4: Differentiating equations (90) and (91) with respect to w_0, w_1 respectively:

$$\begin{aligned} \frac{\partial}{\partial w_{0,t}}\left(\frac{\partial \nu_{0,t}}{\partial \phi_1}\right) &= -\frac{(1-\gamma)}{\gamma}\theta\sigma\alpha^*_{0,t}(w_{0,t})^{-\gamma}\tilde{b}^{-\gamma}(\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2)^{-2+\gamma}\phi_1^{\frac{1}{\gamma}-1}(\tilde{b}^2 + \gamma\phi_1^{\frac{1}{\gamma}} + \tilde{b}\gamma) \\ \frac{\partial}{\partial w_{1,t}}\left(\frac{d\nu_{1,t}}{d\phi_1}\right) &= -\frac{(1-\gamma)^2}{\gamma}\theta\sigma\alpha^*_1(w_{1,t})^{-\gamma}(1 + \phi_1^{-\frac{1}{\gamma}}\tilde{b})^{\gamma-2}\phi_1^{-\frac{1}{\gamma}-1}\tilde{b} \end{aligned}$$

If $\gamma = 1$, both differentials equal 0. If $\gamma \neq 0$, then:

$$\begin{aligned} \frac{\partial}{\partial w_{0,t}}\left(\frac{\partial \nu_{0,t}}{\partial \phi_1}\right) &\propto -(1-\gamma) \\ \frac{\partial}{\partial w_{1,t}}\left(\frac{\partial \nu_{1,t}}{\partial \phi_1}\right) &\propto -(1-\gamma)^2 \end{aligned}$$

Combined with the signs derived in Corollary 2 this delivers Corollary 4.

Lemma 2: To reduce notation, in this proof we use $AD_{r,t}$ to denote the relative aggregate demand for risky assets:

$$AD_{r,t} \equiv \frac{AD_t(risky)}{AD_t(safe) + AD_t(risky)} \tag{92}$$

Using equation (17) to substitute out for α^*, equation (53) can be rewritten:

$$AD_{r,t} = \frac{\mu - r^f}{\gamma\sigma^2}\left(1 + \theta(\Gamma_{0,t} + \Gamma_{1,t})\right) \tag{93}$$

where:

$$\Gamma_{j,t} = \frac{\Omega_{\alpha,j} w_{j,t}^{1-\gamma}(w_{j,t} - c_{j,t})}{w_{0,t} - c_{0,t} + w_{1,t} - c_{1,t}} \tag{94}$$

for $j \in \{0, 1\}$.

From this we have:

$$\frac{\partial AD_{r,t}}{\partial \mu} = \frac{1}{\gamma\sigma^2}\left(1 + \theta(\Gamma_{0,t} + \Gamma_{1,t})\right) + \frac{\mu - r^f}{\gamma\sigma^2}\theta\left(\frac{\partial \Gamma_{0,t}}{\partial \mu} + \frac{\partial \Gamma_{1,t}}{\partial \mu}\right) \tag{95}$$

This derivative is strictly positive if:

$$-\theta\left(\frac{\partial \Gamma_{0,t}}{\partial \mu} + \frac{\partial \Gamma_{1,t}}{\partial \mu}\right) < AD_{r,t}\frac{\gamma\sigma^2}{(\mu - r^f)^2} \tag{96}$$

Similarly, differentiating equation (93) with respect to r^f yields:

$$\frac{\partial AD_{r,t}}{\partial r^f} = -\frac{1}{\gamma\sigma^2}\left(1 + \theta(\Gamma_{0,t} + \Gamma_{1,t})\right) + \frac{\mu - r^f}{\gamma\sigma^2}\theta\left(\frac{\partial \Gamma_{0,t}}{\partial r^f} + \frac{\partial \Gamma_{1,t}}{\partial r^f}\right) \tag{97}$$

This derivative is strictly negative if:

$$\theta\left(\frac{\partial \Gamma_{0,t}}{\partial r^f} + \frac{\partial \Gamma_{1,t}}{\partial r^f}\right) < AD_{r,t}\frac{\gamma\sigma^2}{(\mu - r^f)^2} \tag{98}$$

The right hand side is the same in conditions (96) and (98), and it is strictly positive. Let us start with condition (96). It is trivially satisfied if:

$$\left(\frac{\partial \Gamma_{0,t}}{\partial \mu} + \frac{\partial \Gamma_{1,t}}{\partial \mu}\right) \geq 0 \tag{99}$$

Outside of this case, condition (96) is satisfied if:

$$\theta < AD_{r,t}\frac{\gamma\sigma^2}{(\mu - r^f)^2}\left(-\frac{\partial\Gamma_{0,t}}{\partial\mu} - \frac{\partial\Gamma_{1,t}}{\partial\mu}\right)^{-1} \tag{100}$$

Now consider condition (98). Again, this is trivially satisfied if:

$$\left(\frac{\partial\Gamma_{0,t}}{\partial r^f} + \frac{\partial\Gamma_{1,t}}{\partial r^f}\right) \leq 0 \tag{101}$$

Outside of this case, condition (98) is satisfied if:

$$\theta < AD_{r,t}\frac{\gamma\sigma^2}{(\mu - r^f)^2}\left(\frac{\partial\Gamma_{0,t}}{\partial r^f} + \frac{\partial\Gamma_{1,t}}{\partial r^f}\right)^{-1} \tag{102}$$

A sufficient condition for both (96) and (98) to be satisfied is therefore:

$$\theta < \theta^* = AD_{r,t}\frac{\gamma\sigma^2}{(\mu - r^f)^2}\cdot \min\left(\left(-\frac{\partial\Gamma_{0,t}}{\partial\mu} - \frac{\partial\Gamma_{1,t}}{\partial\mu}\right)^{-1}, \left(\frac{\partial\Gamma_{0,t}}{\partial r^f} + \frac{\partial\Gamma_{1,t}}{\partial r^f}\right)^{-1}\right) \tag{103}$$

B Consumption and Saving with Ambiguity Aversion

In the benchmark without ambiguity, Proposition 3 implies that young and middle-aged agents consume less, and save more, when the probability of surviving to old age increases:

$$\frac{dc^*_{j,t}}{d\phi_1} < 0 \tag{104}$$

With ambiguity, we start with equation (28) for $j = \{0, 1\}$, and substitute out for $c^*_{j,t}$ using Proposition 3:

$$c_{0,t} = \frac{\tilde{b}^2}{\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2} w_{0,t} + \theta w_{0,t}^{2-\gamma}\Omega_{c,0} \tag{105}$$

$$c_{1,t+1} = \frac{\tilde{b}}{\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2} R_0^p w_{0,t} + \theta w_{1,t+1}^{2-\gamma}\Omega_{c,1} \tag{106}$$

Differentiating with respect to ϕ_1, we obtain:

$$\frac{dc_{0,t}}{d\phi_1} = -\frac{\tilde{b}^2\phi_1^{\frac{1-\gamma}{\gamma}}}{\gamma(\phi_1^{\frac{1}{\gamma}}+\tilde{b}+\tilde{b}^2)}w_{0,t} + \theta w_{0,t}^{2-\gamma}\frac{d\Omega_{c,0}}{d\phi_1} \tag{107}$$

$$\frac{dc_{1,t+1}}{d\phi_1} = -\frac{\tilde{b}\phi_1^{\frac{1-\gamma}{\gamma}}}{\gamma(\phi_1^{\frac{1}{\gamma}}+\tilde{b}+\tilde{b}^2)}R_0^p w_{0,t} + \theta\left((2-\gamma)w_{1,t+1}^{1-\gamma}\Omega_{c,1}\frac{dw_{1,t+1}}{d\phi_1} + w_{1,t+1}^{2-\gamma}\frac{d\Omega_{c,1}}{d\phi_1}\right) \tag{108}$$

These derivatives are negative whenever:

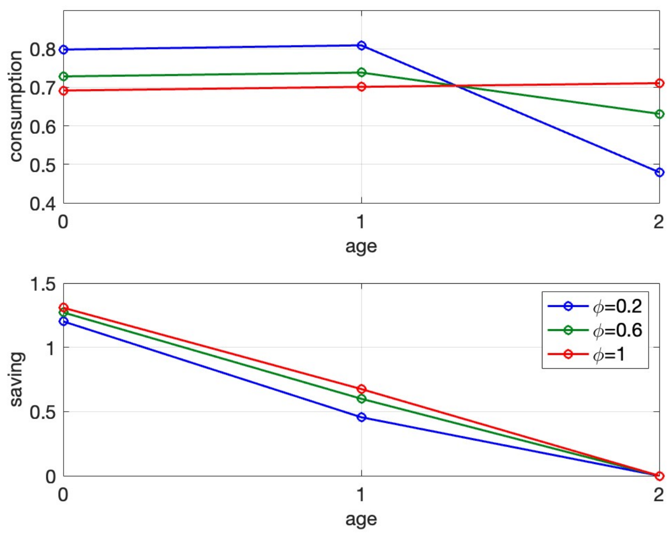
$$\theta\frac{d\Omega_{c,0}}{d\phi_1} < \frac{\tilde{b}^2\phi_1^{\frac{1-\gamma}{\gamma}}}{\gamma(\phi_1^{\frac{1}{\gamma}}+\tilde{b}+\tilde{b}^2)}w_{0,t}^{\gamma-1} \tag{109}$$

$$\theta\left((2-\gamma)\Omega_{c,1}\frac{dw_{1,t+1}}{d\phi_1} + w_{1,t+1}\frac{d\Omega_{c,1}}{d\phi_1}\right) < \frac{\tilde{b}\phi_1^{\frac{1-\gamma}{\gamma}}}{\gamma(\phi_1^{\frac{1}{\gamma}}+\tilde{b}+\tilde{b}^2)}R_0^p w_{0,t}w_{1,t+1}^{\gamma-1} \tag{110}$$

In both of these inequalities, the right hand side is strictly positive. Since $\Omega_{c,j}$ is independent of θ, there is therefore a θ^+ such that $\theta < \theta^+$ is sufficient to ensure both inequalities hold, and consumption of young and middle-aged agents falls as ϕ_1 rises.

Figure 11 shows a numerical example of this. It plots the consumption and saving paths for the ambiguity-averse agents with $J = 2$ studied in Section 3.2, with the same parameters as those in Figures 2 and 3.

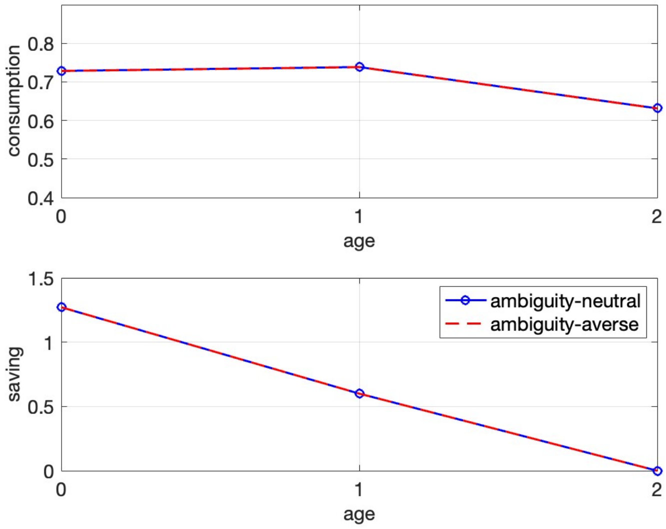
Figure 11: Consumption and saving paths with ambiguity.



Note: Plots constructed using $J = 2$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\theta = 0.045$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 \in (0, 1]$, $w_{0,t} = 2$, and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks.

These paths are very similar to those without ambiguity aversion in Figure 1. Figure 12 plots the paths of consumption for $\phi_1 = 0.6$, with and without ambiguity.

Figure 12: Consumption and saving paths with and without ambiguity, for $\phi_1 = 0.6$.



Note: Plots constructed using $J = 2$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\phi_1 = 0.6$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 \in (0, 1]$, $w_{0,t} = 2$, and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks.

C Section 4 Robustness Tests

Table 4 repeats the analysis of Table 2, this time replacing the basic (unconditional) equity share measure with the transformed measure $\tilde{\alpha}^*_{it}$, defined as:

$$\tilde{\alpha}^*_{it} = \frac{\tilde{\alpha}_{it} \cdot \text{financial wealth}}{\text{total net wealth exc. retirement accounts}} \tag{111}$$

Since the original survey measure $\tilde{\alpha}_{it}$ gives the share of (non-retirement fund) financial wealth invested in stocks, the numerator of the definition of $\tilde{\alpha}^*_{it}$ is the total amount invested in stocks. Dividing that by net wealth gives a measure of the share of overall assets invested in stocks. We exclude retirement accounts (IRAs and 401ks) from this calculation as these may be partly invested in stocks, but we are unable to observe the proportions. The quantitative and qualitative results are as in Table 2.

Tables 5 and 6 then repeat the same analysis using the subjective probability of surviving to age 65 and 75 respectively. The results are qualitatively similar to Table 2, but less precisely estimated because of the smaller variation in these subjective survival probabilities.

Table 4: Regressions on share of total assets invested in equity.

	(1) Equity share	(2) Equity share	(3) Equity share
Age	-0.0922	-0.1168	-0.1164
	(0.0815)	(0.0862)	(0.0919)
Subjective survival prob.	-0.1159*	-0.1221**	-0.1018
	(0.0597)	(0.0612)	(0.0639)
Age × Subjective survival prob.	0.0030**	0.0032**	0.0030*
	(0.0015)	(0.0016)	(0.0016)
Net wealth ($1000s)		0.0002	0.0000
		(0.0001)	(0.0001)
Income ($1000s)		0.0068	0.0064
		(0.0056)	(0.0056)
Period FE	Yes	Yes	Yes
Demographic controls	No	No	Yes
Observations	3440	3422	3414
R^2	0.0021	0.0084	0.0285

Note: Robust standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. *For other regression details see the note to Table 2.*

Table 7 repeats columns 2 and 3 of Table 2 with income and net wealth included in logs, rather than in levels. This entails removing the minority of observations with zero or negative values for these variables. The results are qualitatively and quantitatively robust to this change.

Finally, Table 8 repeats Table 2, but with the estimation conducted after winsorizing the subjective survival probability measure. Specifically, to ensure extreme survival probabilities are not driving our results, we drop all households with a subjective survival probability $\tilde{\phi}_{it} \leq 5\%$ (5% of the sample) and with $\tilde{\phi}_{it} \geq 95\%$ (7% of the sample). If anything, our results are stronger after removing these extreme values.

Table 5: Regressions on share of financial assets invested in equity, using subjective survival probability to age 65.

	(1) Equity share	(2) Equity share	(3) Equity share
Age	-0.1015	-0.1723	-0.1489
	(0.2126)	(0.2078)	(0.2046)
Subjective survival prob.	-0.0675	-0.0858	-0.1369
	(0.1104)	(0.1093)	(0.1053)
Age × Subjective survival prob.	0.0026	0.0031	0.0031
	(0.0025)	(0.0024)	(0.0024)
Net wealth ($1000s)		0.0007*	0.0004
		(0.0004)	(0.0003)
Income ($1000s)		0.0024	0.0018
		(0.0017)	(0.0013)
Period FE	Yes	Yes	Yes
Demographic controls	No	No	Yes
Observations	3493	3473	3465
R^2	0.0049	0.0110	0.0733

Note: Robust standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. *For other regression details see the note to Table 2.*

Table 6: Regressions on share of financial assets invested in equity, using subjective survival probability to age 75.

	(1) Equity share	(2) Equity share	(3) Equity share
Age	-0.0043	-0.0409	-0.0286
	(0.1406)	(0.1396)	(0.1349)
Subjective survival prob.	-0.0394	-0.0420	-0.0762
	(0.0847)	(0.0848)	(0.0796)
Age × Subjective survival prob.	0.0019	0.0019	0.0020
	(0.0019)	(0.0019)	(0.0018)
Net wealth ($1000s)		0.0007*	0.0004
		(0.0004)	(0.0003)
Income ($1000s)		0.0023	0.0018
		(0.0017)	(0.0013)
Period FE	Yes	Yes	Yes
Demographic controls	No	No	Yes
Observations	3487	3467	3459
R^2	0.0053	0.0111	0.0729

Note: Robust standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. *For other regression details see the note to Table 2.*

Table 7: Regressions on share of financial assets invested in equity, with income and net wealth in logs.

	(1) Equity share	(2) Equity share
Age	-0.2982***	-0.2082*
	(0.1122)	(0.1148)
Subjective survival prob.	-0.1942**	-0.1849**
	(0.0850)	(0.0843)
Age × Subjective survival prob.	0.0044**	0.0042**
	(0.0018)	(0.0018)
Log net wealth	3.6744***	3.2447***
	(0.4317)	(0.4789)
Log income	1.8458***	1.1855**
	(0.5838)	(0.5902)
Period FE	Yes	Yes
Demographic controls	No	Yes
Observations	2930	2924
R^2	0.0627	0.1001

Note: Robust standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. *For other regression details see the note to Table 2.*

Table 8: Regressions on share of financial assets invested in equity, after winsorizing survival probability.

	(1) Equity share	(2) Equity share	(3) Equity share
Age	-0.1956	-0.2008	-0.1870
	(0.1243)	(0.1235)	(0.1249)
Subjective survival prob.	-0.2490***	-0.2294**	-0.2288**
	(0.0965)	(0.0964)	(0.0957)
Age × Subjective survival prob.	0.0060***	0.0055**	0.0054**
	(0.0022)	(0.0022)	(0.0021)
Net wealth ($1000s)		0.0007*	0.0004
		(0.0004)	(0.0003)
Income ($1000s)		0.0064*	0.0044*
		(0.0034)	(0.0025)
Period FE	Yes	Yes	Yes
Demographic controls	No	No	Yes
Observations	3052	3034	3028
R^2	0.0064	0.0154	0.0791

Note: Robust standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. For other regression details see the note to Table 2.

D Quantitative Model Details

D.1 Model Description

Demographics, Assets, Preferences: Demographics are as in Section 2.1. We also maintain the assumption that there are two assets available for agents to invest in, one safe and one risky. The returns on these assets are as in Section 2.1.

An agent of age j chooses their consumption and portfolio allocation to maximize expected discounted lifetime utility:[32]

$$V_j^\theta(w_{j,t}, P_{j,t}) = \max_{c_{j,t},\tilde{\alpha}_{j,t}} \min_{\nu_{j,t}} \left\{ \frac{c_{j,t}^{1-\gamma}}{1-\gamma} - \beta \left(E_{j,t} \left[-\phi_j \left(\frac{1}{2\theta_{j,t}} \nu_{j,t}^2 + V_{j+1}^\theta(w_{j+1,t+1}, P_{j+1,t+1}) \right) \right]^{1-\chi} \right)^{\frac{1}{1-\chi}} \right\} \quad (112)$$

This is as in equation (21), except that there is an extra state variable (permanent income $P_{j,t}$, defined below) and agents have Epstein-Zin preferences whenever the value of χ differs from 0. This formulation of Epstein-Zin preferences is as in Rudebusch and Swanson (2012), and is equivalent to the form in e.g. Gomes and Michaelides (2005). Here the EIS is equal to γ^{-1}, and the coefficient of relative risk aversion is $1-(1-\chi)(1-\gamma)$. We also allow the ambiguity aversion parameter $\theta_{j,t}$ to vary over time in a specific way to permit a common normalization used in the model solution, described below.

For robustness exercises below, we also consider a version of this objective with a bequest motive, where the form of the bequest motive follows Gomes and Michaelides (2005):[33]

$$V_j^\theta(w_{j,t}, P_{j,t}) = \max_{c_{j,t},\tilde{\alpha}_{j,t}} \min_{\nu_{j,t}} \left\{ \frac{c_{j,t}^{1-\gamma}}{1-\gamma} - \beta \left(E_{j,t} \left[-\left(\frac{1}{2\theta_{j,t}} \nu_{j,t}^2 + \phi_j V_{j+1}^\theta(w_{j+1,t+1}, P_{j+1,t+1}) + (1-\phi_j) \frac{b^\gamma}{1-\gamma} w_{j+1,t+1} \right) \right]^{1-\chi} \right)^{\frac{1}{1-\chi}} \right\} \quad (113)$$

[32] As in Section 4, $\tilde{\alpha}_{j,t}$ refers to the fraction of an agent's financial assets invested in the risky asset.

[33] Note that there is a small change in the specification of ambiguity aversion in this case, as the penalty to belief distortions is no longer multiplied by ϕ_j (i.e. the penalty is still paid if the agent dies). If we did not make this change, then distortions in the terminal period would be cost-free, but there would still be a benefit to distorting beliefs through the impact on the expected utility from bequests. The optimal belief distortion would therefore be infinite. To avoid this pathological solution, we therefore change the ambiguity specification accordingly for this robustness check.

Income and Wealth: Agents are born at age $j = 20$ with no assets. Until the age of 66, they are employed, earning risky labor income $y_{j,t}$ with the law of motion:

$$y_{j,t} = \underbrace{\exp[f_j + q_{j,t}]}_{\equiv P_{j,t}} \cdot \exp[u_{j,t}] \tag{114}$$

$$q_{j,t} = q_{j-1,t-1} + z_{j,t} \tag{115}$$

where f_j is a deterministic function of age, which we set to a cubic with parameters as in Cocco et al. (2005); $q_{j,t}$ is a persistent shock, $u_{j,t}$ and $z_{j,t}$ are transitory shocks with i.i.d. mean-zero Gaussian distributions, and $P_{j,t}$ is permanent income. A more detailed description of the income process is found in Gomes (2020) - equation 114 here reproduces his equation 18.

At age 67, agents retire. From then until their death they earn a risk-free retirement income, given by:

$$y_{j,t} = \exp[\lambda(f_K + q_{K,t_r})] \tag{116}$$

where retirement occured in period t_r at age K, and λ is a constant determining the generosity of retirement income. Log retirement income is therefore a constant fraction of the log of the persistent components of income in the final period of the agent's working life.

Wealth then accumulates according to:

$$w_{j+1,t+1} = \begin{cases} (R^f + (R_{t+1} - R^f + \sigma_1 \nu_{j,t})\tilde{\alpha}_{j,t}) \cdot (w_{j,t} - c_{j,t}) + y_{j+1,t+1} \cdot (1 - F) & \text{if } \tilde{\alpha}_{j,t} > 0 \\ R^f \cdot (w_{j,t} - c_{j,t}) + y_{j+1,t+1} & \text{if } \tilde{\alpha}_{j,t} = 0 \end{cases} \tag{117}$$

where F is a participation cost, set proportional to income to reflect that much of the cost is in terms of the agent's time. These processes for income and wealth follow the standard form in the portfolio choice literature (see e.g. the discussion in Gomes, 2020). Indeed, the only differences to Gomes and Michaelides (2005) in these processes are that there is no housing expenditure, and that equity market participation costs are paid per-period (as in e.g. Catherine, 2022) rather than once in an agent's lifetime. The only difference to Cocco et al. (2005) is that they do not include a participation cost.

Normalization: The model is solved with standard numerical techniques. As in Cocco et al. (2005), we take advantage of the fact that the utility function and constraints can be scaled by permanent income $P_{j,t}$. This removes one state variable from the model and so makes the solution substantially faster.

Specifically, define consumption, income, and wealth relative to permanent income as $\hat{c}_{j,t} = \frac{c_{j,t}}{P_{j,t}}$, $\hat{y}_{j,t} = \frac{y_{j,t}}{P_{j,t}}$, and $\hat{w}_{j,t} = \frac{w_{j,t}}{P_{j,t}}$. To enable this normalization, we also assume that $\theta_{j,t} = \frac{\hat{\theta}}{P_{j,t}^{1-\gamma}}$ for some constant $\hat{\theta}$.[34] With these definitions, the objective function and constraints can be written as:

$$\hat{V}_j^{\theta}(\hat{w}_{j,t}) = \max_{\hat{c}_{j,t},\tilde{\alpha}_{j,t}} \min_{\nu_{j,t}} \left\{ \frac{\hat{c}_{j,t}^{1-\gamma}}{1-\gamma} - \beta \left(E_{j,t}\left[-\phi_j \left(\frac{1}{2\hat{\theta}}\nu_{j,t}^2 + g_{j+1,t+1}^{1-\gamma}\hat{V}_{j+1}^{\theta}(\hat{w}_{j+1,t+1}) \right) \right]^{1-\chi} \right)^{\frac{1}{1-\chi}} \right\} \tag{118}$$

$$\hat{w}_{j+1,t+1} = \begin{cases} (R^f + (R_{t+1} - R^f + \sigma_1 \nu_{j,t})\tilde{\alpha}_{j,t}) \cdot \frac{(\hat{w}_{j,t} - \hat{c}_{j,t})}{g_{j+1,t+1}} + \hat{y}_{j+1,t+1} \cdot (1-F) & \text{if } \tilde{\alpha}_{j,t} > 0 \\ R^f \cdot \frac{(\hat{w}_{j,t} - \hat{c}_{j,t})}{g_{j+1,t+1}} + \hat{y}_{j+1,t+1} & \text{if } \tilde{\alpha}_{j,t} = 0 \end{cases} \tag{119}$$

where $\hat{V}_j^{\theta}(\hat{w}_{j,t}) \equiv \frac{V_j^{\theta}(w_{j,t},P_{j,t})^{\frac{1}{1-\gamma}}}{P_{j,t}}$, and $g_{j+1,t+1} \equiv \frac{P_{j+1,t+1}}{P_{j,t}}$.

In the model with bequests, the constraint remains as above, but the objective function becomes:

$$\hat{V}_j^{\theta}(\hat{w}_{j,t}) = \max_{\hat{c}_{j,t},\tilde{\alpha}_{j,t}} \min_{\nu_{j,t}} \left\{ \frac{\hat{c}_{j,t}^{1-\gamma}}{1-\gamma} - \beta \left(E_{j,t}\left[\frac{1}{2\hat{\theta}}\nu_{j,t}^2 - \phi_j g_{j+1,t+1}^{1-\gamma}\hat{V}_{j+1}^{\theta}(\hat{w}_{j+1,t+1}) - (1-\phi_j)\frac{b^{\gamma}}{1-\gamma} g_{j+1,t+1}^{1-\gamma}\hat{w}_{j+1,t+1}^{1-\gamma} \right]^{1-\chi} \right)^{\frac{1}{1-\chi}} \right\} \tag{120}$$

Inner Minimization: To solve for agent decision rules, it is helpful to derive the first order condition for the optimal belief distortion $\nu_{j,t}$. This also provides useful intuition for the model results. The resulting first order condition is:

$$\nu_{j,t} = -\hat{\theta}\sigma_1\tilde{\alpha}_{j,t}(\hat{w}_{j,t} - \hat{c}_{j,t}) \cdot \frac{1}{E_{j,t}Z_{j,t}^{-\chi}} \cdot E_{j,t}Z_{j,t}^{-\chi} g_{j+1,t+1}^{-\gamma} \frac{\partial \hat{V}_{j+1}^{\theta}(\hat{w}_{j+1,t+1})}{\partial \hat{w}_{j+1,t+1}} \tag{121}$$

[34] This implies that the degree of ambiguity aversion is increasing in permanent income. The main consequences of ambiguity aversion identified in the simple model without this assumption continue to hold here.

where:

$$Z_{j,t} = -\phi_j \left(\frac{\nu_{j,t}^2}{2\hat{\theta}} + g_{j+1,t+1}^{1-\gamma} \hat{V}_{j+1}^{\theta}(\hat{w}_{j+1,t+1}) \right) \tag{122}$$

Notice that setting $\chi = 0$ (CRRA utility) and $g_{j+1,t+1} = 1$ (removing risky income growth) returns us to the form of the inner minimization FOC in the simple model (equation (41)). The same forces driving the reaction of belief distortions to changes in survival probabilities therefore operate in this richer quantitative model as in the simple model of Section 3.

In the case with a bequest motive we have:

$$\nu_{j,t} = -\hat{\theta}\sigma_1 \tilde{\alpha}_{j,t}(\hat{w}_{j,t} - \hat{c}_{j,t}) \cdot \frac{1}{E_{j,t}\tilde{Z}_{j,t}^{-\chi}} \cdot E_{j,t}\tilde{Z}_{j,t}^{-\chi} g_{j+1,t+1}^{-\gamma} \left[\phi_j \frac{\partial \hat{V}_{j+1}^{\theta}(\hat{w}_{j+1,t+1})}{\partial \hat{w}_{j+1,t+1}} + (1-\phi_j) b^{\gamma} \hat{w}_{j+1,t+1}^{-\gamma} \right] \tag{123}$$

where:

$$\tilde{Z}_{j,t} = -\left(\frac{\nu_{j,t}^2}{2\hat{\theta}} + \phi_j g_{j+1,t+1}^{1-\gamma} \hat{V}_{j+1}^{\theta}(\hat{w}_{j+1,t+1}) + (1-\phi_j) \frac{b^{\gamma}}{1-\gamma} g_{j+1,t+1}^{1-\gamma} \hat{w}_{j+1,t+1}^{1-\gamma} \right) \tag{124}$$

D.2 Data Construction and Calibration

Standard parameters: The majority of model parameters are taken from Gomes (2020), which is itself based on Cocco et al. (2005) and Gomes and Michaelides (2005). As in those papers, the age-dependent component of income is assumed to be given by:

$$f_j = aa + b1 \times j + b2 \times j^2 + b3 \times j^3 \tag{125}$$

These calibrated parameters are listed in Table 9. The only ones different to Gomes (2020) are γ and χ: where Gomes (2020) assumes CRRA utility for his baseline analysis (and so $\chi = 0$), we separate risk aversion and the elasticity of intertemporal substitution. We select γ and χ so that the EIS is 0.35 and the coefficient of relative risk aversion is 5. These are common values in this literature (see e.g. Foltyn, 2020).

Mortality: For the main calibration, we take mortality rates by age group from the Office of the Chief Actuary at the Social Security Administration, and compute ϕ_j as $1 - \text{mortality rate}_j$. For projections to 2100 in Section 5.3, we use projected mortality rates from the same data source. They project mortality rates for males and females separately, at every year of age up to 119. To compute aggregate survival rates from these series, we first take data on sex ratios in the US in 2021 from the UN World Population Prospects 2022. This data is reported at selected ages,[35] so for the ages missing from the sex ratio data we assume the sex ratio is equal to that of the nearest cohort for which there is data. We then generate projected sex ratios at each age for all years up to 2100, by combining the mortality rates from the Office of the Chief Actuary for each sex, age, and year, with the assumption that the sex rate at birth will remain the same as it was in 2021. These projections are important, as in 2019 the female-male ratio rose sharply at older age groups due to women having longer life expectancy in the US. However, this life expectancy gap is projected to narrow in coming decades, which will reduce the female-male ratio in older age groups. Finally, we use the projected sex ratios to combine female and male mortality rates to give an aggregate mortality rate. 1- this mortality rate for each age group then gives projected ϕ_j in 2100.

Targets: We choose the risky asset participation cost to match the average participation rate in the 2019 wave of the Survey of Consumer Finances, which was 0.53 (Bhutta et al., 2020). We choose θ to match β_2 and β_3 in column 3 of Table 3, as described in Section 5.1. This step involves choosing a value ζ that regulates the heterogeneity in subjective life expectancy in the simulated data used for the regressions. This is chosen to match the standard deviation of subjective probabilities of surviving to age 85 among respondents in the SCE who answered the question on their equity share (28%). These targets imply parameter values of $F = 0.036$, $\zeta = 0.88$, and $\theta = 0.19$.[36]

Table 9: Calibrated parameters.

Parameter	Value	Parameter	Value
aa	0.530339	γ	$\frac{20}{7}$
b1	0.16818	λ	0.68212
b2	-0.00323371	μ	1.055
b3	0.000019704	σ_1	0.2
R^f	1.015	σ_u^2	0.1
χ	$-\frac{15}{13}$	σ_z^2	0.1
δ	0.97		

Note: Calibrated parameters used in the baseline model, taken from Gomes (2020), with the exception of γ and χ, which are taken to generate an EIS of 0.35 and a coefficient of relative risk aversion of 5, as in Foltyn (2020).

[35]These ages are 15, 20, 30, 40, 50, 60, 70, 80, 90, 100.

[36]When evaluating the value of ζ, recall that annual mortality rates are typically small numbers. The mortality rate of 60-year-olds in 2019 was 0.90%. Our ζ calibration implies that the perceived mortality rate at 60 varies between 0.11% and 1.71%. This is why a large ζ value is required to generate the substantial heterogeneity in survival probabilities reported in the SCE data.

D.3 Detection Error Probabilities

As is common in the literature on ambiguity aversion, we use model detection error probabilities (DEP) to infer whether agents are hedging against the models that are empirically plausible to generate the data we observe. Intuitively, we treat agents as statisticians using likelihood ratio test to discriminate among models. DEP measures how far the alternative models can deviate from the approximating one without being discarded. Low values of DEP means that agents are unwilling to discard very different alternative models, which could be easily discriminated given observed data.

DEP assigns equal initial priors to the approximating and distorted models, hence it is the average of the probabilities of Type I and Type II errors. A Type I error occurs when the likelihood ratio test chooses the distorting model when the approximating model is the true data generating process. A Type II error is the reverse. Formally, DEP is defined as:

$$DEP = \frac{1}{2}Prob(\ln(\frac{L_A}{L_B}) < 0|A) + \frac{1}{2}Prob(\ln(\frac{L_B}{L_A}) > 0|B) \tag{126}$$

where A denotes the approximating model and B is the distorting model.

As discussed by Anderson et al. (2003), the following bound on the average error in using a likelihood ratio test to discriminate between the approximating and distorted models is useful when the data is of a continuous record with length T. The DEP bound in this discrete-time model can be approximated in the following way:

$$avg\ DEP \leq \frac{1}{2}E\ exp\{-\frac{1}{8}\int_{t=0}^{T}\nu^2(w_t)\,dt\} \tag{127}$$

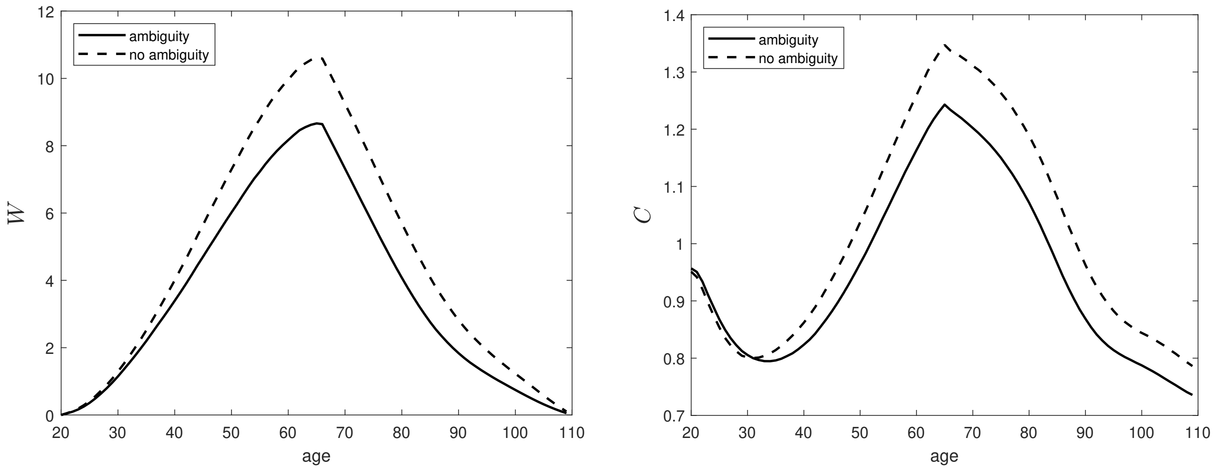
$$\approx \frac{1}{2}E\ exp\{-\frac{1}{8}\sum_{j=20}^{J}\nu^2(w_j)\} \tag{128}$$

In the calibration, we take $J = 109$, which is consistent with the maximum lifespan in our calibration, and set the initial age to $j = 20$. With the calibrated value of $\theta = 0.19$, we obtain a large average DEP value of 0.487. Besides, if instead we use the distortion for the poorest individual in each age group, we obtain a DEP value of 0.462, which acts as a lower bound for all agents. This implies that based on the observed data, it is not easy to distinguish the distorted model from the approximating model.

D.4 Other life cycle profiles

Figure 13 shows the average wealth and consumption over agent life-cycles in the model, with and without ambiguity. Both variables are expressed relative to permanent income, as in Gomes (2020) Figure 3a.[37] As in the simple model, ambiguity does not affect the qualitative shape of these profiles. The main effect of ambiguity is to reduce risky asset shares, particularly for younger agents (Figure 8), reducing average portfolio returns, and thus limiting wealth accumulation somewhat. To compensate for this lower wealth accumulation, agents consume less than in the no-ambiguity case.

Figure 13: Further life-cycle profiles I.



(a) Wealth **(b)** Consumption

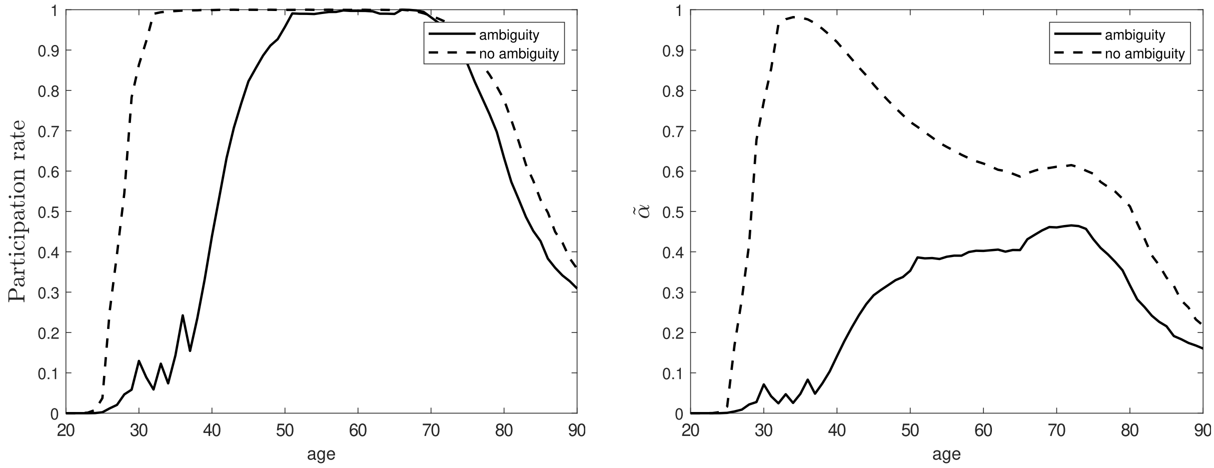
Note: Plots constructed using the calibration and data described in Section 5.1 and Appendix D.2. All variables are expressed after normalizing by permanent income $P_{j,t}$.

Figure 14 shows the risky asset participation rate, and the average unconditional risky asset share, over the life-cycle. The participation rate inherits the hump-shaped profile from the no-ambiguity model. Ambiguity does however substantially delay the rise in participation at young ages. Empirically, participation rates in the 2019 SCF display a similar hump-shaped profile, with a peak at age 45-54. Studying the SCF from 1998-2007, Chang et al. (2018) find a hump-shaped profile with a peak around age 60. While we therefore match the approximate location of peak participation (unlike the model without ambiguity), the hump-shape in our model is too pronounced relative to the data, where the maximum participation rate is typically around 60%.

As with the conditional risky asset share in Figure 8, ambiguity lowers the unconditional risky asset share, especially for young agents. It also makes the profile upward-sloping for most of the life-cycle. This upward slope is consistent with the SCE data (see Figure 6), and the overall profile is qualitatively consistent with estimates from the SCF – which similarly finds a gradual upward slope, with a decline in later years (see Chang et al., 2018, Figure 1B).

[37] The initial decline in consumption therefore occurs because permanent income grows faster than consumption at early ages, not because the level of consumption declines. The same path is present in Gomes (2020) Figure 3a.

Figure 14: Further life-cycle profiles II.



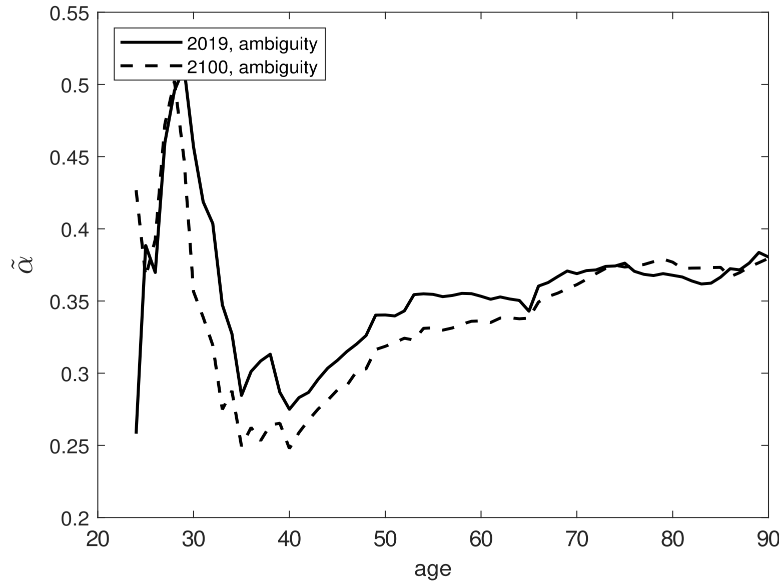
(a) Participation rate **(b)** Unconditional risky asset share

Note: Plots constructed using the calibration and data described in Section 5.1 and Appendix D.2.

D.5 Results with a bequest motive

Figure 15 reproduces Figure 10a with a bequest motive in utility, as in equation (113). The bequest parameter b is set to 2.5, as in Gomes and Michaelides (2005). Although the overall life-cycle profile is less strongly upward-sloping that in Figure 10, especially at older ages, the key qualitative results remain the same. The life-cycle profile still slopes upwards from early middle age onward, and increased longevity makes the gradient steeper.

Figure 15: Model-implied conditional risky asset shares, 2019 and 2100, with a bequest motive.

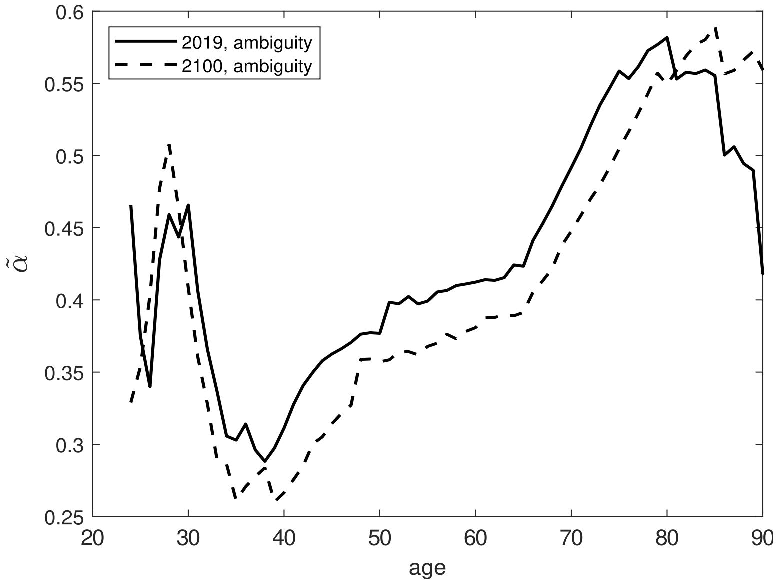


Note: Plots constructed using the calibration and data described in Section 5.1 and Appendix D.2, with a bequest motive in utility as described in Appendix D.1.

D.6 Results with homogeneous mortality rates

Figure 16 reproduces Figure 10a with homogeneous mortality rates. All agents have accurate beliefs about their mortality rates, set to the US data described above. The key qualitative results remain the same. The life-cycle profile is upward sloping from early middle age onward, and increased longevity makes the gradient steeper.

Figure 16: Model-implied conditional risky asset shares, 2019 and 2100, with homogeneous perceived mortality.



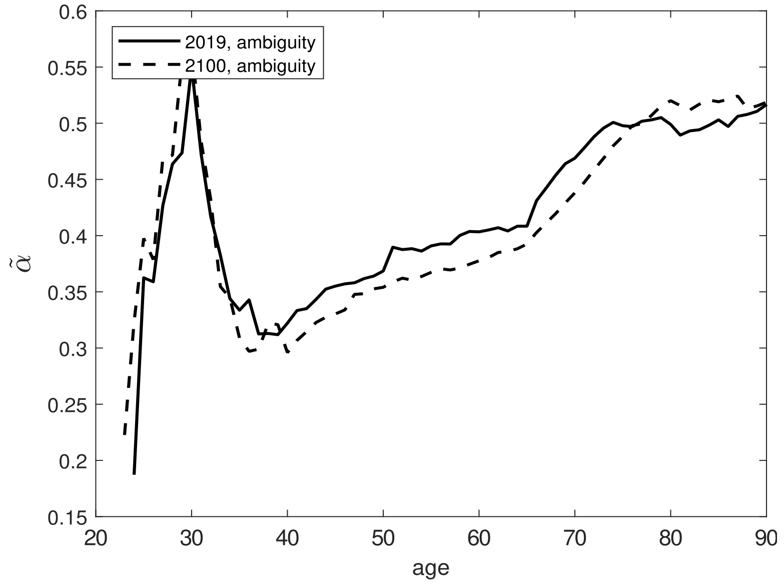
Note: Plots constructed using the calibration and data described in Section 5.1 and Appendix D.2, without any heterogeneity in perceived mortality rates.

D.7 Endogenous equity premium

Figure 17 reproduces Figure 10a, allowing for an endogenous response of the equity premium to the increase in life expectancy. Specifically, we assume that the relative supply of risky assets is fixed as in Section 3.5 (equation (54)). We calibrate the relative supply $\bar{S}$ such that the calibrated equity premium used in Section 5 clears the asset market in 2019. We then hold this fixed, and allow the equity premium to vary to ensure that the asset market continues to clear at this relative supply in 2100. To achieve this, we hold the risk-free rate constant, and let the expected risky asset return adjust.

Following the logic laid out for the simple model in Section 3.5, the increase in life expectancy to 2100 implies a modest increase in the equity premium, by 7 basis points. Since this change is small, Figure 17 is very similar to Figure 10a. In the analysis in Section 5.3 with a fixed equity premium, the gap between the conditional risky asset shares of those aged 80 and 35 rises from 16.6 p.p. in 2019 to 20.9 p.p. in 2100. With this endogenous response of the equity premium response, the 2100 gap becomes 21.1 p.p..

Figure 17: Model-implied conditional risky asset shares, 2019 and 2100, with endogenous equity premium.



Note: Plots constructed using the calibration and data described in Section 5.1 and Appendix D.2. For the '2100' line, μ is varied such that the relative aggregate demand for risky assets remains at its 2019 level. This involves a rise from the original calibration ($\mu = r^f + 0.04$) to $\mu = r^f + 0.04066$.