Does Precommitment Raise Growth?

Dynamic Aspects of Growth and Fiscal Policy

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Abstract

We develop an endogenous growth model driven by externalities of both private capital and public infrastructure. The government levies distortionary taxation to finance a publicly provided consumption good and public infrastructure. Firms face adjustment costs. We first study the steady state, focusing in detail on the non-Ricardian aspects of the model. We then examine the optimal and time-consistent policies in a linear-quadratic approximation of the model. Although the time consistent equilibrium is also sub-optimal in terms of steady-state welfare, it does yield higher growth, through an accumulation of assets by the state and a cut of government consumption.

1 Introduction

This paper studies a dynamic model of fiscal policy with endogenous growth. The model is dynamic in three ways. First it models both private and public capital as stocks. Second it is non-Ricardian with distortionary taxes, finite lives and population growth. The choice between debt or taxation financing of a given path of government spending matters for GDP growth and we do not impose a balanced budget. These first two features imply that the model has transitory dynamics, a characteristic that is absent from many papers in the fiscal policy literature. Third the model allows for dynamic fiscal policy. In particular we allow governments to choose a different tax rate and spending at any time.

The model incorporates a private sector and a government. The government may spend on a publicly provided consumption good and on a flow that enters into the production function of firms. In the steady state the government maintains this input in production as a constant fraction of GDP. Therefore the productivity of capital is bounded away from zero and perpetual growth is possible. This type of model was pioneered by Barro (1990). He models a single-good world where production is a function of labour in inelastic supply, a capital stock that does not depreciate, and the *flow* of public services. There are three key results to his paper. The first is that the maximization of welfare is equivalent to the maximization of the growth rate. Second, the constant optimal tax rate for spending on investment is equal to the share of public investment in national output. And third, when public consumption (exogenous) is taken into account, it is optimal to levy an additional tax to pay for these services and the optimum investment in infrastructure will be unaffected.

These strong conclusions have invited further examination and qualification. Most of these additions have involved changing some of the mechanics of the Barro model. Futagami, Morita, and Shibata (1993) modify a single aspect of the Barro model by modelling government capital as a stock rather than a flow. This considerably improves the realism of the model at the expense of analytical simplicity. The result that growth is maximized when taxation is equal to public investment carries over from Barro to their model; they abstract from government consumption. However welfare maximization is no longer equal to growth maximization because we have transitory dynamics. The authors show that if the tax rate is constant, the steady state is unique and that there is a unique stable path that converges to the equilibrium. In addition they demonstrate that the optimum tax rate is smaller than the one that maximizes growth, because growth maximization implies that future consumption streams are discounted at a rate 0 vis-à-vis current consumption. The analytical solutions of the maximization problem are not addressed because they are to complicated.

Glomm and Ravikumar (1994) present analytical results in the case where both private and public depreciate fully during the period and preferences are logarithmic. Government consumption is absent, but infrastructure spending may exhibit varying degree of non-rivalry. Each individual firm produces with constant returns to scale capital and labour, but production is premultiplied by a term that depends on government spending divided by a Cobb-Douglas type index of private factor usage. Thus government consumption is a shift parameter in private production but the intensity of private factor usage will limit its impact i.e. public services are subject to a congestion effect. The authors then solve for the dynamic programme of the private sector and then study the optimization problem the government. Unfortunately, the quite elegant formulation of congestion has no impact on the optimal growth rate, which is constant and differs from Barro (1990)'s by being premultiplied with the discount factor. The presence of the discount factor can be explained as follows. Barro restricts his policy to time-invariant taxation. In the initial period, capital is predetermined, but infrastructure is not since it is a flow. It is *current* infrastructure that will affect current production. In that case, government spending is allocated to the sole objective of maximizing both current and future output. Glomm and Ravikumar however assume that current production depends on past investment. Therefore output in the first period is predetermined. Increasing taxes today therefore involves a sacrifice in current consumption and the solution becomes dependent on the discount factor. Maximizing welfare, in that scenario, is not equivalent to maximizing growth.

Lau (1995) extends Glomm and Ravikumar (1994) to include government consumption. Like theirs his model is in a permanent steady-growth state. Assuming that preferences are logarithmic, he can compute the optimal—from households' preferences view—share of government spending on consumption and investment in GDP. It turns out that the government consumption is lower under welfare than under growth maximization, and that government consumption is higher. Therefore, assuming that governments are close to the welfare maximizing policy, an increase in government consumption should decrease growth, but an increase in government investment should increase growth, which is what Barro (1991) found in an empirical study. (See Hsieh and Lai (1994) and Lin (1994) for further references to the empirical literature)

A paper in a similar vein is Lee (1992). His production per capita is a Cobb-Douglas in the private capital stock per head and the aggregate public

capital stock. Both stocks do not depreciate. He simultaneously studies government consumption, government investment and lump-sum transfers to private agents. He manages to solve the optimum policy of the government when it acts as a leader over the private sector. He finds that there are two local optima, one with a slow growth rate, high government consumption and high transfers and distortionary taxes, and the other with low taxes, high government investment and low transfers. The fact that the government should finance positive lump-sum transfers in the first equilibrium using distionary taxation appears odd. His results should be taken with caution. There appear computational mistakes in the displayed equation after #10 and after #13.

The effect of fiscal policy in an endogenous growth model has also been examined by Turnovski and Fisher (1995). The basic production framework is the same as Glomm and Ravikumar's, but no specific functional form is assumed for the production function. An additional level of generality is added by assuming that labour supply is elastic. The government finances consumption and infrastructure expenditure through lump-sum taxation. The authors are interested in the effects of permanent and temporary changes in fiscal policy. First, when there is an increase in government consumption spending, the increase in taxation needed to finance it will reduce private sector income and consumption. The marginal utility of income increases, therefore households will increase their labour supply. The increase in labour supply raises the productivity of capital and results in additional capital accumulation, potentially leading to a *raise* in the growth rate. In addition to the effect of taxation, there is a direct effect—through the representative agent's felicity—of government consumption on the marginal rate of substitution between consumption and leisure. This effect could potentially reverse the adverse effect of taxation on the representative consumer's utility. Increased expenditure on infrastructure will have the same taxation effect since it also needs to be financed by tax. In addition, an increase in public infrastructure results in an increase in income that will tend to counter the taxation effect. The total impact of increasing infrastructure on the private capital stock is therefore ambiguous. The authors then show that when technology is Cobb-Douglas, an increase in government consumption will increase the private capital stock by more than an increase in public infrastructure would. However the welfare effect and growth effects of raising one or the other are ambiguous.

The interaction between public expenditure and labour supply decisions are also taken up by Devereux and Love (1995). They show that government spending can have an impact on growth even in the absence of direct government investment into the capital stock. Their model comprises physical capital and human capital. Labour supply is elastic, and human capital accumulation is not taxed. When an permanent increase in government spending occurs, its effect will depend on how the increase is financed. When the government uses a lump-sum tax, the private sector wealth is reduced. Both leisure and consumption are normal goods; therefore there will be a drop in private consumption and an increase in the labour supply. In equilibrium, the rate of return on accumulating human capital increases, and so does the rate of return on physical capital. Therefore the growth rate will rise. This result, which is very close to Turnovski and Fisher hinges on the lump-sum taxation assumption. When the lump-sum tax is replaced by an income tax, Devereux and Love show that a permanent increase in taxation will reduce growth via a reduction in the private capital stock.

Probably the most comprehensive recent study of fiscal policy with optimising government spending is the "Model 3" of Jones, Manuelli, and Rossi (1993). Contrary to Turnovski and Fisher (1995), they study distortionary taxation. There is also an interesting variation on the stock/flow specification of the productive input, where the investment is homogeneous function of degree one in private and public investment[1]. In addition, there is an important feature that is absent in the previous contributions: the government's budget constraint is relaxed to its intertemporal version, i.e. the government may accumulate debt or assets. The government would like to use lump-sum taxation, and even if there is no lump-sum taxation is available, the government can tax the current capital stock. Since this capital stock is predetermined, taxing it mimics a lump-sum tax. Therefore the optimum solution consists in taxing the existing capital stock heavily in the first periods, until a surplus is built up that allows to finance future commitments without the necessity to levy further distortionary taxes. There are two problems with that solution. The first is that the authors need to impose a restriction on the tax rate to prevent it to hit over 100%. The computed trajectory then depends heavily on the restriction that is adopted, in fact when control is implemented, the tax rate jumps to the bound and remains there for many periods. Therefore the bound drives the solution. The second problem is that that solution is not time-consistent. At any point in the future, as long as there is revenue to raise, there remains the temptation to raise taxes again. The authors acknowledge that "This is clearly a problem with the solutions presented in connection with these models" (p. 511) and that " … a more complete treatment of the problem including these issues

[1]A CES specification is chosen for the simulations. "Model 3" does not have a labour/leisure choice but this aspect is addressed in other models of the paper.

would be of considerable interest" (p. 487). This is precisely what we are addressing in this paper.

The rest of the paper is organised as follows. Section 2 sets out our model which combines a Yaari-Blanchard consumption function with a Tobin's q model of private investment that takes adjustment costs into account. Section 3 examines the steady state and derives some analytical results in a simplified model that are confirmed in a calibrated version of the full model. Section 4 compares the optimal precommitment fiscal policy with time-consistent policy and Section 5 concludes the paper.

2 The Model

Our model is closest to Futagami, Morita, and Shibata (1993). Following Yaari (1965), Blanchard (1985) and Weil (1989), we model consumers as having finite lives. The government levies distortionary taxation or sells bonds to finance spending on consumption and infrastructure. Infrastructure has an external effect on labour productivity. Both private and public sector face adjustment costs. The model is in discrete time. The details of the model are as follows.

2.1 Consumption and Savings

We consider an overlapping generations model stretching from the current date into the indefinite future. At each date, some new consumers are born, who gain utility from consumption until they die. We simplify by assuming that the intertemporal utility functions are additively separable, such that the utility today is the discounted sum of current and future felicity. Let $u(c(\mathfrak{t}))$ be the felicity that the consumer derives from consuming $c(\mathfrak{t})$ at time $\mathfrak{t}$ if she is alive. $c(\mathfrak{t})$ could be a vector of any sort of consumables, but for this exposition we will consider that there is only one consumable. When the consumer is alive, felicity is logarithmic in consumption, when she is dead felicity is zero. Let the discount rate be д. Then an individual consumer born at date $\mathfrak{t}'$ maximizes her expected utility $u_{\mathfrak{t}'}(\mathfrak{t})$, that is:

$$u_{\mathfrak{t}'}(\mathfrak{t}) = \sum_{\mathfrak{t}''=\mathfrak{t}}^{\text{death}} \frac{u\big(c(\mathfrak{t}'')\big)}{(1+\text{д})^{\mathfrak{t}''-\mathfrak{t}}} = \sum_{\mathfrak{t}''=\mathfrak{t}}^{\text{death}} \frac{\ln\big(c(\mathfrak{t}'')\big)}{(1+\text{д})^{\mathfrak{t}''-\mathfrak{t}}}$$

We will consider that death can occur, but we are not certain when. To simplify, we assume that the probability of death is constant and denoted by м. For any consumer alive in period $\mathfrak{t}$, the probability of being in period

$t'' > t'$ will be $(1-\text{м})^{t''-t}$. Therefore the expected intertemporal utility becomes

$$\mathrm{E}\ u_{t'}(t) = \mathrm{E} \sum_{t''=t}^{\infty} \ln(c(t'')) \left(\frac{1-\text{м}}{1+\text{д}}\right)^{t''-t} = \mathrm{E}\ u(t) \tag{2.1}$$

i.e we can abstract from the date of birth in this problem. The reason is the formulation of the probability of death as constant over time. In fact from (2.1) we see that the finitely lived consumer's utility function is isomorphic to that of an infinitely lived consumer; the discount factor ϱ is only a fraction $1-\text{м}$ of the discount factor of an infinitely-lived person i.e.

$$\varrho = \frac{1-\text{м}}{1+\text{д}}$$

Every consumer is endowed with a unit of labour that she supplies inelasticly to the market. Labour of people of different ages is a homogeneous good. This means that we abstract from human capital accumulation. For the labour that she supplies at period t, the consumer receives a post-tax wage $w_\tau(t)$. At any period t, we can define her *human* wealth $h_{t'}(t-1)$ as the present value of the current and all future expected wages, discounted at the post-tax interest rate $r_\tau(t''-1)$.

$$h_{t'}(t-1) = \sum_{t''=t}^{\infty} \frac{(1-\text{м})^{t''-t+1}\, w_\tau(t'')}{1+r_\tau(t''-1)} = h(t-1) \tag{2.2}$$

Again, by virtue of the exponential lifetime assumption, human wealth is the same for all living individuals, irrespective of their age, because they all face the same death rate and because the wage is not dependent on age. That does not mean, however that consumers of all ages will have the same consumption, because recently born consumers have no non-human wealth, which they only start accumulating after birth. Non-human wealth $ш(t)$ takes the form of physical capital $k(t)$ or government bonds $d(t)$, and because of arbitrage between both types of assets, they must earn the same return. At time t the consumer born in $t' < t$ has some non-human wealth $ш_{t'}(t-1)$ at her disposal that she accumulated in period $t-1$. It consists of bonds $d_{t'}(t-1)$ and capital $k_{t'}(t-1)$. When the consumer dies she does not leave an *intentional* bequest, but her non-human wealth at the beginning of the period where death occurs. Since modelling the links of each consumer with her heirs would be cumbersome, the following construction is introduced. There is an insurance company that will take the financial post-tax wealth of each dead consumer. It will then distribute these assets as a premium п

paid on the holdings of assets. We assume that the law of great numbers is holding, therefore the assets per period are $\text{м}\,(1+r_\tau(t-1))\, ш_{t'}(t-1)$ and the liabilities are $(1-\text{м})\,(1+r_\tau(t-1))\,\text{п}\, ш_{t'}(t-1)$. If the insurance company has no operating cost, the premium will satisfy the zero-profit condition: $1+\text{п} = 1/(1-\text{м})$. Hence we have the dynamics of non-human wealth as

$$ш_{t'}(t) = \frac{(1+r_\tau(t-1))\, ш_{t'}(t-1)}{1-\text{м}} + w_\tau(t) - c_{t'}(t) \tag{2.3}$$

If we forward the period-to-period budget constraint in (2.3) and make the conventional transversality assumption that the present value of future wealth will tend to zero, the lifetime budget constraint of a consumer is

$$ш_{t'}(t-1) = \sum_{t''=t}^{\infty} \frac{(1-\text{м})^{t''+1-t}\,[w_\tau(t'') - c_{t'}(t'')]}{1+r_{\tau t}(t''-1)} \tag{2.4}$$

where we have defined the interest rate between period t and t'' as:

$$1 + r_{\tau t}(t'') = \prod_{t'''=t}^{t''} (1 + r_\tau(t'')) \tag{2.5}$$

The consumer's problem is to maximize (2.1) under (2.4). The first order condition is

$$\frac{c_{t'}(t''+1)}{c_{t'}(t'')} = \frac{1+r_\tau(t''-1)}{1+\text{д}} \tag{2.6}$$

When substituting the first-order conditions (2.6) in the lifetime budget constraint (2.4), and making use of (2.2), we obtain consumption as

$$c_{t'}(t) = \left(1 - \frac{1-\text{м}}{1+\text{д}}\right)\left(ш_{t'}(t) + h_{t'}(t)\right)$$

The consumer will, at each date consume a fraction of her end-of-period wealth. This completes the study of the individual consumer.

All consumers of the same age are identical, but consumers of different ages have different levels of non-human wealth. We therefore need to consider the age structure of the population when aggregating. If $L_{t'}(t)$ the number of people that are born in t' and still alive at t, this is

$$L_{t'}(t) = (1-\text{м})^{t-t'}\, L_{t'}(t')$$

The birth rate б is defined by

$$L_{\mathfrak{t}}(\mathfrak{t}) = \text{б}\,(1-\text{м})\,L(\mathfrak{t}-1)$$

where $L(\mathfrak{t})$ stands for the aggregate labour supply at time $\mathfrak{t}$. Note г is the rate of growth of the *aggregate* population. This means that

$$L(\mathfrak{t}) = (1+\text{г})\,L(\mathfrak{t}-1) = \ldots = (1+\text{г})^{\mathfrak{t}}\,L(0) \tag{2.7}$$

Aggregation is performed for all age groups

$$C(\mathfrak{t}) = \sum_{\mathfrak{t}'=-\infty}^{\mathfrak{t}} L_{\mathfrak{t}'}(\mathfrak{t})\,c_{\mathfrak{t}'}(\mathfrak{t}) \tag{2.8}$$

which, using (2.1) aggregates to

$$C(\mathfrak{t}) = \frac{1-\text{м}}{1+\text{д}}\,(\text{Щ}(\mathfrak{t}) + H(\mathfrak{t})) \tag{2.9}$$

where $H(\mathfrak{t})$ is aggregate human wealth defined as

$$H(\mathfrak{t}) = \sum_{\mathfrak{t}'=-\infty}^{\mathfrak{t}} L_{\mathfrak{t}'}(\mathfrak{t})\,h_{\mathfrak{t}'}(\mathfrak{t}) \tag{2.10}$$

and evolves according to

$$\left(1 - \text{б}\,\frac{1-\text{м}}{1+\text{г}}\right) H(\mathfrak{t}) = (1+r_\tau(\mathfrak{t}-1))\,H(\mathfrak{t}-1) + W_\tau(\mathfrak{t}) \tag{2.11}$$

Aggregate non-human wealth is defined as

$$\text{Щ}(\mathfrak{t}) = \sum_{\mathfrak{t}'=-\infty}^{\mathfrak{t}} L_{\mathfrak{t}'}(\mathfrak{t})\,\text{щ}_{\mathfrak{t}'}(\mathfrak{t}) \tag{2.12}$$

At birth, $\text{щ}_{\mathfrak{t}'}(\mathfrak{t}'-1) = 0$, and non-human wealth accumulates out of each period's savings. Aggregate non-human wealth can be shown to evolve as

$$\text{Щ}(\mathfrak{t}) = (1+r_\tau(\mathfrak{t}-1))\,\text{Щ}(\mathfrak{t}-1) + W_\tau(\mathfrak{t}) - C(\mathfrak{t})$$

We can eliminate human wealth out of (2.9) by taking its first difference and substituting from (2.11). This leads us to the Yaari-Blanchard function

$$\left(\frac{1+\text{д}}{1-\text{м}} - \frac{\text{м}+\text{г}}{1+\text{г}}\right) C(\mathfrak{t}) = [1 + r(\mathfrak{t}-1)\,(1-\tau(\mathfrak{t}))]\,C(\mathfrak{t}-1) - \frac{(\text{м}+\text{д})\,(\text{м}+\text{г})}{(1-\text{м})\,(1+\text{г})}\,[D(\mathfrak{t}) + K(\mathfrak{t})] \tag{2.13}$$

This function represents the consumption/savings choice of consumers. It can be thought of as a intertemporal aggregate demand function in which consumption depends on the expected wealth at the end if the period. Forwarding this relationship in the future, we can show that current consumption depends on the sequence of all current and future interest rates and tax rates.

2.2 Production and Investment

While consumers make consumption and savings decisions, firms make production and investment decisions. We assume that there is a large number of identical firms in the economy. In every date $\mathfrak{t}$, the problem of each firm is to choose investments $\big(i(\mathfrak{t}), i(\mathfrak{t}+1), \ldots\big)$ and employment $\big(l(\mathfrak{t}), l(\mathfrak{t}+1), \ldots\big)$ to maximize its value, which is equal to the discounted sum of future profits

$$\sum_{\mathfrak{t}'=\mathfrak{t}}^{\infty} \frac{\pi(\mathfrak{t}')}{1 + r_{\mathfrak{t}-1}(\mathfrak{t}'-1)}$$

where $\pi(\mathfrak{t}')$ are the profits of the firm in time $\mathfrak{t}'$, and $r_{\mathfrak{t}}(\mathfrak{t}')$ is the rate of interest between the period $\mathfrak{t}$ and $\mathfrak{t}'$.

The firm does not pay taxes, which are paid by its owners on the income they receive. Depreciation is tax deductible, which implies a subsidy paid to the holders of physical capital. With this setup, taxes are immaterial to the firm's problem. Upon investment, it pays adjustment costs. We assume that these costs are a convex function $a(\cdot)$ of investment to the capital stock. Therefore profits are:

$$\pi(\mathfrak{t}') = q(\mathfrak{t}') - w(\mathfrak{t}')\,\epsilon(\mathfrak{t}')\,l(\mathfrak{t}') - i(\mathfrak{t}')\left[1 + a\left(\frac{i(\mathfrak{t}')}{k(\mathfrak{t}'-1)}\right)\right]$$

The value of the firm is maximized under the investment constraint that:

$$k(\mathfrak{t}') - (1-\delta)\,k(\mathfrak{t}'-1) \leq i(\mathfrak{t}') \qquad \forall\,\mathfrak{t}' \geq \mathfrak{t}$$

Associate a series of present-value multipliers $\lambda(\mathfrak{t}')$ with this constraint in time $\mathfrak{t}'$ and define Tobin's $\mathsf{q}(\mathfrak{t}')$ as

$$\mathsf{q}(\mathfrak{t}') = \lambda(\mathfrak{t}')\,(1 + r_{\mathfrak{t}-1}(\mathfrak{t}'))$$

We can then write the Lagrangian

$$\mathcal{L} = \sum_{\mathfrak{t}'=\mathfrak{t}}^{\infty} \frac{\pi(\mathfrak{t}') - \mathsf{q}(\mathfrak{t}')\,(k(\mathfrak{t}') - (1-\delta)\,k(\mathfrak{t}'-1) - i(\mathfrak{t}'))}{1 + r_{\mathfrak{t}-1}(\mathfrak{t}')}$$

All firms are equal therefore the behaviour in the aggregate is identical to the individual behaviour. Using a Cobb-Douglas production function where α is the share of capital, we can write the first-order conditions for all firms as:

$$\begin{aligned} \mathsf{q}(\mathsf{t}) &= 1 + a\left(\frac{I(\mathsf{t})}{K(\mathsf{t}-1)}\right) + \frac{I(\mathsf{t})}{K(\mathsf{t}-1)}\, a'\left(\frac{I(\mathsf{t})}{K(\mathsf{t}-1)}\right) \\ 0 &= \frac{\alpha\, Q(\mathsf{t}+1)}{K(\mathsf{t})} + \frac{I(\mathsf{t}+1)^2}{K(\mathsf{t})^2}\, a'\left(\frac{I(\mathsf{t}+1)}{K(\mathsf{t})}\right) \\ &\quad + (1-\delta)\,\mathsf{q}(\mathsf{t}+1) - (1+r(\mathsf{t}))\,\mathsf{q}(\mathsf{t}) \\ I(\mathsf{t}) &= K(\mathsf{t}) - (1-\delta)\,K(\mathsf{t}-1) \end{aligned} \tag{2.14}$$

When investment is subject to adjustment cost, firms can no longer simply equate marginal product of capital to the interest rate in each period. The problem of firms becomes dynamic. Investment will depend on the sequence of interest rates from t to the indefinite future. If a government wishes to stimulate investment it must make sure that interest rates are kept low at all periods. In equilibrium on the savings market, the government can increase investment by lowering taxation, which reduces the tax distortion between the capital cost of firms and the return available to consumers.

2.3 Government Intervention

The government purchases the amount $G(\mathsf{t})$ of the unique commodity from the private sector. These purchases are split into two parts: first $G^{\mathrm{c}}(\mathsf{t})$ is a publicly provided good that enters directly into the utility of the representative household. We think in this case of public services that have no input into productive services, such as expenditure on defence or cultural and recreational activities. A second part of expenditure $G^{\mathrm{i}}(\mathsf{t})$ is used to augment the productive capacities of the labour force. We think of this expenditure as contributing to a stock $K^{\mathrm{g}}(\mathsf{t})$ that represents the contribution of present and past investment in activities like health and education. This stock evolves according to

$$K^{\mathrm{g}}(\mathsf{t}) = (1-\delta)\,K^{\mathrm{g}}(\mathsf{t}-1) + G^{\mathrm{i}}(\mathsf{t})$$

where δ is the depreciation rate of the public capital stock. We assume that the public capital stock depreciates at the same rate as the private capital stock. The public capital (infrastructure) is also subject to the same adjustment costs as the private capital stock.

The government can finance its expenditure either through taxation $T(\mathsf{t})$ or through issuing single-period real bonds, the stock of which is denoted by $D(\mathsf{t})$. The government's budget identity is

$$D(\mathsf{t}) = (1 + r(\mathsf{t}-1))\,D(\mathsf{t}-1) + G(\mathsf{t}) - T(\mathsf{t})$$

Government spending is the sum of consumption and investment spending, augmented by the adjustment cost that the government pays when it invests

$$G = G^{\mathrm{c}}(\mathsf{t}) + G^{\mathrm{i}}(\mathsf{t})\left[1 + a\left(\frac{G^{\mathrm{i}}(\mathsf{t})}{K^{\mathrm{g}}(\mathsf{t}-1)}\right)\right]$$

Taxation $T(\mathsf{t})$ is modelled as a uniform tax rate applied to the all income. Total income to the population is equal to production minus the depreciation of the capital stock. Therefore

$$T(\mathsf{t}) = \tau(\mathsf{t})\ [Q(\mathsf{t}) - \delta\,K(\mathsf{t}-1)]$$

2.4 Endogenous Growth

We adopt the "learning-by-doing" approach to endogenous growth pioneered by Arrow (1962) and Romer (1986), to allow the productivity of each worker to depend not only on the capital she is using but also on the average capital available to the other firms and the infrastructure put in place by the government. We think of the government capital stock as infrastructure (like roads ports), and any other stock that does is not privately invested but contributes to country's productivity; for example schools, the legal framework, etc. These public goods are rival, because they are subject to congestion. Therefore the *per capita* provision of these goods—rather then their aggregate supply—impact on the productivity of each individual worker. In line with this argument we model the efficiency of the labour force as

$$\epsilon(\mathsf{t}-1) = \bar{\epsilon}^{1/(1-\alpha)}\,\frac{K^{\mathrm{g}}(\mathsf{t}-1)^{\gamma_1}\,K(\mathsf{t}-1)^{1-\gamma_1}}{L(\mathsf{t}-1)}$$

where γ_1 captures the contribution of the public capital stock to the overall measure of capital externality. The aggregate production function is

$$\begin{aligned} Q(\mathsf{t}) &= K(\mathsf{t}-1)\,[\epsilon(\mathsf{t}-1)\,L(\mathsf{t}-1)]^{1-\alpha} \\ &= K(\mathsf{t}-1)^{\alpha}\,\bar{\epsilon}\,K^{\mathrm{g}}(\mathsf{t}-1)^{\gamma_1\,(1-\alpha)}\,K(\mathsf{t}-1)^{(1-\gamma_1)\,(1-\alpha)} \\ &= \bar{\epsilon}\,K^{\mathrm{g}}(\mathsf{t}-1)^{1-\gamma_2}\,K(\mathsf{t}-1)^{\gamma_2} \end{aligned}$$

where we define $\gamma_2 = \alpha + (1-\alpha)(1-\gamma_1)$ as the contribution of private capital to the aggregate production. Now define the growth rate as

$$n(\mathfrak{t}) = \frac{Q(\mathfrak{t}) - Q(\mathfrak{t}-1)}{Q(\mathfrak{t}-1)}$$

This is a model in which GDP is allowed to grow unbounded thus precluding an examination of a steady-state in levels. We will therefore examine all variables in per-GDP terms and examine a balanced-growth steady state where all variables as ratios of GDP do not change. To rewrite the model in per-GDP terms, we define all lower case variable as ratios of GDP[2]. Table 2.1 summarizes the model in the per GDP formulation.

3 Steady-state Analysis

The model of Table 2.4.1, will be used for the numerical simulations in Section 4. In this section we first perform two steps backwards towards simpler models. Within a first simplified model we introduce the analytical conceptual framework. We then further simplify the model to clarify the relationship with other simple analytical growth models. Finally we return to the full model and investigate the comparative statics of its balanced-growth steady state.

3.1 A Simplified Model

Most of the analytical complications of the model come from the existence of adjustment cost of investments. We will ignore these during this subsection. We further simplify the model by ignoring second order terms in г, м, r and linearizing fractions in growth. We also neglect the difference between $n\,(1+d)/\tau$ and $\delta\,k$, which roughly speaking means that when an increase of growth occurs, the incidence on the debt burden is approximately cancelled out against the increase in tax deduction to capital holders[3]. We can then write the following simplified version of the model in a steady state where

[2] *not* per-capita, as we did in sections 2.1 and 2.2

[3] In the calibration of Appendix B, the difference is about .015.

$$\begin{aligned}
0 &= \left(\frac{\text{м}+\text{г}}{1+\text{г}} - \frac{1+\text{д}}{1-\text{м}}\right) c(\mathfrak{t}) + \frac{1+r(\mathfrak{t}-1)\,(1-\tau(\mathfrak{t}))}{1+n(\mathfrak{t})}\, c(\mathfrak{t}-1) \\
&\quad - \frac{(\text{м}+\text{д})\,(\text{м}+\text{г})}{(1-\text{м})(1+\text{г})}\, щ(\mathfrak{t}) && (2.15) \\
n(\mathfrak{t}+1) &= \bar{\epsilon}\, k^{\mathrm{g}}(\mathfrak{t})^{1-\gamma_2}\, k(\mathfrak{t})^{\gamma_2} - 1 && (2.16) \\
k(\mathfrak{t}) &= \frac{1-\delta}{1+n(\mathfrak{t})}\, k(\mathfrak{t}-1) + i(\mathfrak{t}) && (2.17) \\
k^{\mathrm{g}}(\mathfrak{t}) &= \frac{1-\delta}{1+n(\mathfrak{t})}\, k^{\mathrm{g}}(\mathfrak{t}-1) + g^{\mathrm{i}}(\mathfrak{t}) && (2.18) \\
d(\mathfrak{t}) &= \frac{1+r(\mathfrak{t}-1)}{1+n(\mathfrak{t})}\, d(\mathfrak{t}-1) + g(\mathfrak{t}) - t(\mathfrak{t}) && (2.19) \\
g(\mathfrak{t}) &= g^{\mathrm{c}}(\mathfrak{t}) + g^{\mathrm{i}}(\mathfrak{t}) \left[1 + a\left(\frac{g^{\mathrm{i}}(\mathfrak{t})\,(1+n(\mathfrak{t}))}{k^{\mathrm{g}}(\mathfrak{t}-1)}\right)\right] && (2.20) \\
t(\mathfrak{t}) &= \tau(\mathfrak{t}) \left[1 - \frac{\delta\, k(\mathfrak{t}-1)}{1+n(\mathfrak{t})}\right] && (2.21) \\
\mathsf{q}(\mathfrak{t}) &= 1 + a\left(\frac{i(\mathfrak{t})\,(1+n(\mathfrak{t}))}{k(\mathfrak{t}-1)}\right) \\
&\quad + \frac{(1+n(\mathfrak{t}))\, i(\mathfrak{t})}{k(\mathfrak{t}-1)}\, a'\left(\frac{i(\mathfrak{t})\,(1+n(\mathfrak{t}))}{k(\mathfrak{t}-1)}\right) && (2.22) \\
0 &= \frac{\alpha\,(1+n(\mathfrak{t}+1))}{k(\mathfrak{t})} \\
&\quad + \frac{(1+n(\mathfrak{t}+1))^2\, i(\mathfrak{t}+1)^2}{k(\mathfrak{t})^2}\, a'\left(\frac{(1+n(\mathfrak{t}+1))\, i(\mathfrak{t}+1)}{k(\mathfrak{t})}\right) \\
&\quad + (1-\delta)\,\mathsf{q}(\mathfrak{t}+1) - (1+r(\mathfrak{t}))\,\mathsf{q}(\mathfrak{t}) && (2.23) \\
щ(\mathfrak{t}) &= d(\mathfrak{t}) + k(\mathfrak{t}) \\
1 &= c(\mathfrak{t}) + i(\mathfrak{t}) \left[1 + a\left(\frac{i(\mathfrak{t})\,(1+n(\mathfrak{t}))}{k(\mathfrak{t}-1)}\right)\right] + g(\mathfrak{t}) && (2.24)
\end{aligned}$$

Table 2.1: A summary of the model

all aggregates grow at a constant rate:

$$(\text{м} + \text{д})\,(\text{м} + \text{г})\,(d + k) = [r\,(1 - \tau) + \text{г} - \text{д} - n]\,c \tag{3.1}$$

$$(r - n)\,d = \tau - g \tag{3.2}$$

$$k = \frac{\alpha\,(1 + n)}{r + \delta} \tag{3.3}$$

$$1 = c + \frac{\delta\,\alpha}{r + \delta} + g \tag{3.4}$$

$$-\ln \bar{\epsilon} = (1 - \gamma_2)\,\ln\!\left(\frac{g^{\mathrm{i}}}{n + \delta}\right) + \gamma_2\,\ln\!\left(\frac{\alpha}{r + \delta}\right) \tag{3.5}$$

Then substituting for consumption from (3.4), the capital stock from (3.3) in (3.1) and assuming that government spending adjusts to satisfy (3.2) we obtain a relationship between the interest and growth rate. We call this relationship the Yaari-Blanchard (YB) curve.

$$0 = \text{YB} = \text{YB}(r, n, d, \tau)$$

$$0 = 1 - \frac{\alpha\,\delta}{\delta + r} - \tau + (r - n)\,d - \frac{(\text{м} + \text{д})\,(\text{м} + \text{г})\left(d + \frac{\alpha\,(1+n)}{r+\delta}\right)}{r\,(1 - \tau) + \text{г} - \text{д} - n} \tag{3.6}$$

It depends on the fundamental parameters of the model such as the preference rate, the share of capital etc., but also on the variables that indicate fiscal policy. Using the government budget constraint the analysis of the YB curve can be conducted in terms of any two of the fiscal variables τ, d and g. Note that

$$\frac{\partial \text{YB}}{\partial n} < 0 \quad \text{and} \quad \frac{\partial \text{YB}}{\partial r} > 0$$

which implies a rising YB curve in the (n, r) space[4] given d and τ. When the growth rate increases, the income profile of agents becomes steeper. Agents wish to smooth consumption. Therefore they will save less at any rate of interest. Since consumption in various periods are gross substitutes the interest rate rises.

When debt increases we have two effects. Since government spending adjusts, with an increase in debt a decrease in public spending occurs that leaves more room for private expenditure, thereby giving a lower interest

[4]Since

$$\frac{\mathrm{d}r}{\mathrm{d}n} = -\frac{\partial Y/\partial r}{\partial Y/\partial n}$$

along the YB curve.

rate. On the other hand increased debt will lead to a higher supply of bonds, therefore a decrease in their price, thus an interest rate increase. When comparing the two effects for their respective strength—using the calibration in Appendix B—we find that the latter effect is much stronger then the former (.16 > .025); we therefore sign $\partial\text{YB}/\partial d > 0$. This means that the YB curve will shift upwards in the (n, r) space when an increase in debt occurs. When taxation increases to expand government spending at unchanged debt, we find that $\partial\text{YB}/\partial\tau < 0$. An increase in taxes will reduce the return available to bondholders and therefore requires a higher pre-tax interest rate for savings market equilibrium.

An analogous curve to (3.6) can be derived for the case where taxation adjusts to the increase in debt. In that case the curve is

$$0 = \text{YB} = \text{YB}(r, n, g, d)$$

$$0 = \left[1 - \frac{\alpha\,\delta\,(1 + n)}{\delta + r} - g\right] - \frac{(\text{м} + \text{д})\,(\text{м} + \text{г})\left(d + \frac{\alpha\,(1+n)}{r+\delta}\right)}{[r\,(1 - g - ((r - n)\,d) + \text{г} - \text{д} - n]} \tag{3.7}$$

Here the increase in debt will leave the goods market equilibrium unchanged, but the ambiguity of its effect on growth is still with us. On the one hand the increase in debt has the familiar supply-side effect; on the other hand the increase in taxation will raise the pre-tax interest rate. Similarly, the effect will be that the interest rate increases. This version of the YB curve can be used to investigate a tax-financed increase in government spending on the savings market. In this case, $\partial\text{YB}/\partial g > 0$. An increase in government spending reduces consumption on the one hand and cuts the return from savings through increased taxation; therefore an outward shift of the YB curve will occur.

Just as we think of the YB curve as the (intertemporal) demand curve of our model, we can conceptualize the production function and factor supply equations as the supply curve of the economy. If we express government spending on infrastructure is a part σ of total government spending, (i.e. $\sigma = g^{\mathrm{i}}/g$) we can substitute from (3.2) into the production function (3.5). Using (3.3) we obtain a relationship between growth rate and interest rate that we call the CD curve since it is based on the Cobb-Douglas technology

$$0 = \text{CD} = \text{CD}(r, n, \tau, d, \sigma)$$

$$0 = \ln \bar{\epsilon} + \gamma_2\,\ln\!\left(\frac{\alpha}{r + \delta}\right) + (1 - \gamma_2)\,\ln\!\left(\frac{\sigma\,[\tau - (r - n)\,d]}{n + \delta}\right) \tag{3.8}$$

Here we have

$$\frac{\partial \text{CD}}{\partial r} < 0 \qquad \text{and} \qquad \frac{\partial \text{CD}}{\partial n} < 0$$

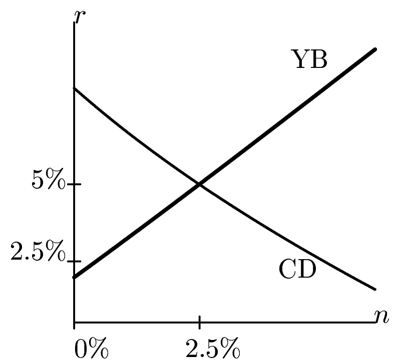


Figure 3.1: YB and CD curves

which makes for a falling CD curve in the (n, r) space, given τ, d and σ. When the interest rate increases, the increased cost of capital decreases the private capital stock. At unchanged government infrastructure expenditure, growth will be lower. As in the case of the YB curve, we can examine the influence of government policy on the position of the curve. When we express the CD curve as (3.8), we can evaluate the impact of debt as $\partial \text{CD}/\partial d < 0$. An increase in debt, compensated by a decrease in government spending will tend to lower public investment. To keep the same level of growth, the interest rate must fall to crowd in private investment. Therefore the CD shifts downwards in the (n, r) space. When taxes increase, this will stimulate government spending on investment, and growth will be higher for every interest rate, therefore we have an outward shift in the CD curve, i.e. $\partial \text{CD}/\partial \tau > 0$. The effect of an increase in σ will be similar and have the same sign.

The equilibrium of the model will be found at the intersection of the YB and CD curves (see Figure 3.1). Since one curve is monotonicly increasing whilst the other is strictly decreasing we are assured of the existence of a unique equilibrium. We can now evaluate the effect of fiscal policy on both growth and interest rate. Differentiating the YB and CD curves, keeping either g and σ or τ and σ fixed, we obtain

$$\left[\frac{\partial n}{\partial d}\right]_{\tau,\sigma,\text{ or } g,\sigma} = \frac{\frac{\partial \text{YB}}{\partial d}\frac{\partial \text{CD}}{\partial r} - \frac{\partial \text{CD}}{\partial d}\frac{\partial \text{YB}}{\partial r}}{\frac{\partial \text{YB}}{\partial r}\frac{\partial \text{CD}}{\partial n} - \frac{\partial \text{CD}}{\partial r}\frac{\partial \text{YB}}{\partial n}} < 0$$

i.e. an increase in government debt leads to a reduction in the growth rate. This result is independent of the choice of adjustment instrument. Keeping all other fiscal variables fixed, we find a positive impact of the composition

of government infrastructure spending on growth

$$\left[\frac{\partial n}{\partial \sigma}\right]_{\tau,g,d} = \frac{-\frac{\partial \text{CD}}{\partial \sigma}\frac{\partial \text{CD}}{\partial r}}{\frac{\partial \text{YB}}{\partial r}\frac{\partial \text{CD}}{\partial n} - \frac{\partial \text{CD}}{\partial r}\frac{\partial \text{YB}}{\partial n}} > 0$$

An increase in the proportion of government spending on infrastructure leads to an outward shift in the CD curve whilst the YB curve is unchanged; therefore we find an increase in the growth rate. However we cannot sign either $\mathrm{d}n/\mathrm{d}\tau$ or $\mathrm{d}n/\mathrm{d}g$ because the signs are ambiguous. By contrast government spending and taxation have unambiguous effects on the interest rate, and we obtain

$$\left[\frac{\partial r}{\partial \tau}\right]_{d,\sigma} = \frac{\frac{\partial \text{YB}}{\partial \tau}\frac{\partial \text{CD}}{\partial n} - \frac{\partial \text{CD}}{\partial \tau}\frac{\partial \text{YB}}{\partial n}}{\frac{\partial \text{YB}}{\partial n}\frac{\partial \text{CD}}{\partial r} - \frac{\partial \text{CD}}{\partial n}\frac{\partial \text{YB}}{\partial r}} > 0$$

$$\left[\frac{\partial r}{\partial g}\right]_{d,\sigma} = \frac{\frac{\partial \text{YB}}{\partial g}\frac{\partial \text{CD}}{\partial n} - \frac{\partial \text{CD}}{\partial g}\frac{\partial \text{YB}}{\partial n}}{\frac{\partial \text{YB}}{\partial n}\frac{\partial \text{CD}}{\partial r} - \frac{\partial \text{CD}}{\partial n}\frac{\partial \text{YB}}{\partial r}} > 0$$

$$\left[\frac{\partial n}{\partial \sigma}\right]_{\tau,g,d} = \frac{-\frac{\partial \text{CD}}{\partial \sigma}\frac{\partial \text{CD}}{\partial n}}{\frac{\partial \text{YB}}{\partial n}\frac{\partial \text{CD}}{\partial r} - \frac{\partial \text{CD}}{\partial n}\frac{\partial \text{YB}}{\partial r}} > 0$$

It is interesting to note that the effect of an increase in debt on the interest rate has an ambiguous effect. When debt increases, there will be an outward shift of the YB curve, because the interest rate required on the savings market will be higher. However, there is also a downwards shift in the CD curve because of a decline in government spending. Unless the increase in debt is entirely compensated by higher taxation—in which case the CD curve does not shift—we cannot be sure that the effect is an increase of the interest rate.

Let us summarize the results for a moment. Growth will depend positively on the fraction spend on investment and negatively on the debt. The interest rate will increase when there is an increase in the size of the public sector in the economy; we are therefore faced with a public/private sector tradeoff since the private capital stock will decline when the interest rate increases.

3.2 Relationship with Simple Models

The idea that there is a trade-off between public and private sector goes right back to Barro (1990), who shows that, in a much simplified version of our model, the growth rate would be maximized at $\tau = 1 - \gamma_2$. Below that the public sector is to small, beyond that tax rate the public sector is to large. We can investigate the optimal size of the public sector analytically if we make further restrictive assumptions. Assume that as in Subsection 3.1, there are no adjustment costs; in addition assume that there is no depreciation of capital and no debt. Further assume an infinitely lived household and no population growth м = г = 0. Let τ^c be a tax to pay for government consumption, and τ^i the part of taxation is spent on investment, with of course $\tau = \tau^i + \tau^c$. We can then write the steady state of the model in *per capital*[5] terms as

$$n\, k^g = \tau^i y \tag{3.9}$$

$$(1+n)\,(1+д) = 1 + r\,(1-\tau^i-\tau^c) \tag{3.10}$$

$$y = k^{g\,(1-\gamma_2)} \tag{3.11}$$

$$r = \alpha\, y \tag{3.12}$$

where n is the steady-state growth rate of capital. Such a steady-state growth rate will exist if

$$,\,(k^g) = 1 + \alpha\, k^{g\,1-\gamma_2}\,(1-\tau^i-\tau^c) - (1+д)\,(1+\tau^i k^{g\,-\gamma_2}) = 0$$

The function $,\,(\cdot)$ has the properties that

$$\lim_{k^g\to 0}\,,\,(k^g) = -\infty, \quad \lim_{k^g\to\infty}\,,\,(k^g) = +\infty, \quad ,\,'(k^g) > 0$$

which ensures the existence of a unique steady-state growth rate. We are now ready to examine

PROPOSITION 1: For any given consumption tax rate τ^c, the steady-state growth rate is maximized when the share of public investment in GDP is equal to the share of infrastructure in production multiplied by the share of GDP not allocated to public consumption

$$\tau^i = (1-\gamma_2)\,(1-\tau^c) \tag{3.13}$$

PROOF: If we differentiate equations (3.9)–(3.12) totally and impose the first order condition for growth maximization $\mathrm{d}n/\mathrm{d}\tau^i = 0$, we get

$$n\,\mathrm{d}k = y + \tau^i \mathrm{d}y \tag{3.14}$$

$$0 = (1-\tau^i-\tau^c)\,\mathrm{d}r - r \tag{3.15}$$

$$\frac{\mathrm{d}y}{y} = (1-\gamma_2)\,\frac{\mathrm{d}k}{k} \tag{3.16}$$

$$\mathrm{d}r = \alpha\,\mathrm{d}y \tag{3.17}$$

[5]This may be somewhat unconventional, but it considerably simplifies the calculations.

(putting $\mathrm{d}\tau^i = 1$). Combining (3.17), (3.15) and (3.12) we obtain

$$\frac{\mathrm{d}y}{y} = \frac{1}{1-\tau^i-\tau^c} \tag{3.18}$$

On the other hand we can use (3.14), (3.16) and (3.9) to get

$$\frac{\mathrm{d}y}{y} = (1-\gamma_2)\left(\frac{\mathrm{d}y}{y} + \frac{1}{\tau^i}\right) \tag{3.19}$$

Substituting out $\mathrm{d}y/y$ between (3.18) and (3.19) yields (3.13). Q.E.D.
We can the obtain Barro (1990) as a corollary:

COROLLARY: When there is no government consumption spending, the growth maximizing investment tax is $\tau^i = 1-\gamma_2$.

The crucial aspect in the model that drives this result is that the elasticity of national product with respect to public infrastructure is constant. Modelling infrastructure as a stock or a flow does not matter, neither it is important how the remainder of output is distributed between various factors.

As noted by Lau (1995), growth maximization is not equivalent to welfare maximization. In his model, welfare maximization occurs when[6]

$$\tau^i = \frac{1-\gamma_2}{1+\theta} \qquad \text{and} \qquad \tau^c = \frac{д}{1+д}\,\eta$$

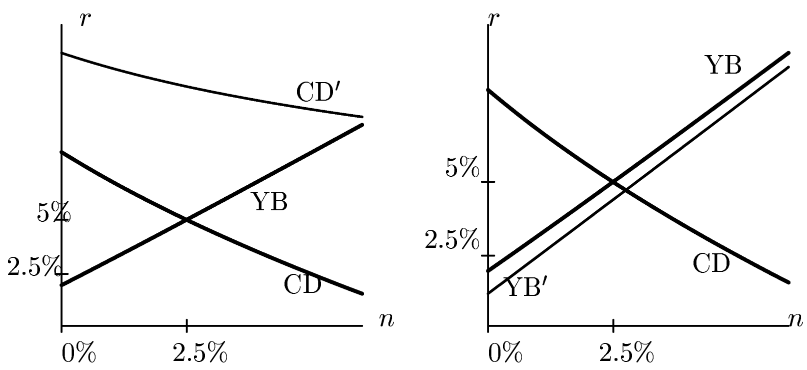
In our model, the presence of stock variables implies that there are transitional dynamics. Therefore it is not possible to compute analytical solutions for the optimal policy, unless one would make the extreme assumption that the government could command the steady state of the economy and simply pick any steady state regardless of the transitional cost. This computation can be done, but leads to complicated expressions that give no further insight into the problem. However it is quite straightforward to show

PROPOSITION 2: The growth rate for an optimal fiscal policy will be higher the lower is τ^c.

PROOF Using (3.13) in the model (3.9)–(3.12), we find a relationship between government consumption and growth as

$$n = (1-\gamma_2)\,(1-\tau^c)\left(\frac{д + n\,(1+д)}{\alpha\,(1-(1-\gamma_2)\,(1-\tau^c)-\tau^c)}\right)^{-\gamma_2/(1-\gamma_2)} \tag{3.20}$$

[6]In our calibration, this gives $g^c = \tau^c = .3\%$, which is tiny when compared to our calibrated Figure $g^c = 10\%$.



(a) without q   (b) infinite lives

Figure 3.2: Effects of simplification

Differentiating (3.20) totally, we obtain

$$\frac{\mathrm{d}n}{\mathrm{d}\tau^{\mathrm{c}}} < 0$$

Q.E.D.

This proposition is crucial to understanding the nature of the optimization problem. There is a trade-off between government consumption and government investment. When a government invests it will increase production possibilities in the future and when it consumes it increases current utility. The problem is intrinsicly dynamic, and therefore prone to time-inconsistency problems. As we shall see, the optimal and time-consistent policy differ substantially; but before looking at this issue, let us dwell on the steady state to get a feel for the effects of steady-state policy changes.

3.3 Numerical Results

In this section, we investigate the steady state of the "full" model to check if the earlier results on the simple model still hold and to derive more results that depend on the values of parameters. We are also interested in obtaining a feel for how large the impact of fiscal policy will be on the growth rate and interest rate.

First recall that we conceptualize the equilibrium of our model as the intersection between YB and CD curve. For the full model, we compute the YB and CD curves and draw them on Figure 3.1. The basic results from

the simplified model are carried over to the full model, i.e. the YB curve is rising and the CD curve is falling. This ensures that a unique intersection of the two curves will exist. Following the calibration, detailed in Appendix B, the equilibrium is at $n = 2.5\%$ and $r = 5\%$.

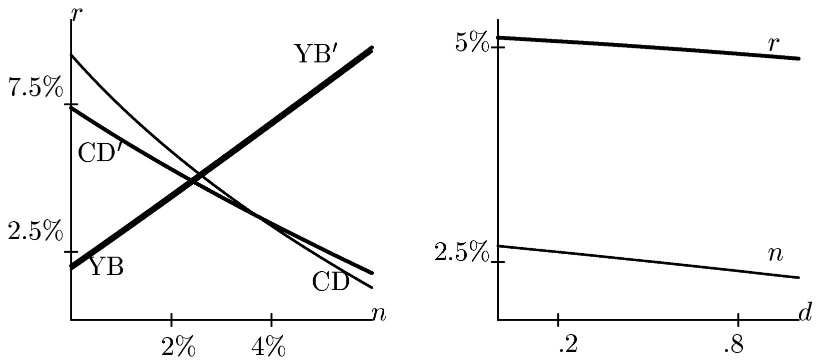
The simplified steady state (3.1) is clearly based on rather stringent assumptions. To evaluate the importance of Tobin's q we compare the full model with a calibration where Tobin's $\mathrm{q} = 0$ while keeping all other parameters at the values of Appendix B. The YB curve will not be affected, because it does not depend on adjustment costs. The CD curve without the adjustment costs is the thin downward sloping curve in Figure 3.2.a. Without the adjustment cost, there would be a substantial gain in growth, in fact the growth rate would be at 6%. We also note that the CD curve would be much flatter without adjustment costs. Roughly speaking, to achieve an increase in the growth rate, we need a smaller decrease in the interest rate, because firms do not have the additional spending on adjustment costs when investing.

Figure 3.2.b shows the impact of finite lives in the model. The figure shows two YB curves, one fat for the case where lives are calibrated as in Appendix B, and a thin one for infinite lives. The finite live aspect of the model reduces the supply of savings since at each period there is a risk that the proceeds of savings alloted to the insurance company. Therefore the YB curve with infinite lives is lower than the finite-live YB curve.

We now turn to the comparative statics of the model, using two diagrams. The one on the left shows the YB curve (rising) and CD curves (falling) for two separate values of government policies. The second value is the one for which we use the thick line, and the curves receive a ′ label. This diagram illustrates the comparative statics of steady-state policies. The growth and interest rate can then be read from the intersection. On the right-hand diagram, we show the evolution of the interest rate (thick line) and the growth rate (thin line), as the policy variable (on the first axis) changes. Note that the four policy variables g, τ, d are linked by the steady-state budget constraint.

$\Delta\tau = 0$		$\mathrm{q} = 1$	$\mathrm{M} = \mathrm{R} = 0$	$\mathrm{q} - 1 = \mathrm{M} = \mathrm{R} = 0$
$\mathrm{d}n/\mathrm{d}d$	$-.0037\%$	$-.0057\%$	$-.0028\%$	$-.0042\%$
$\mathrm{d}r/\mathrm{d}d$	$-.0025\%$	$-.0047\%$	$-.0034\%$	$-.0052\%$

Table 3.1: An increase in debt that reduces spending, (τ, σ) fixed



(a) d =20% vis $d = 80\%$ (b) $n(d)$ and $r(d)$

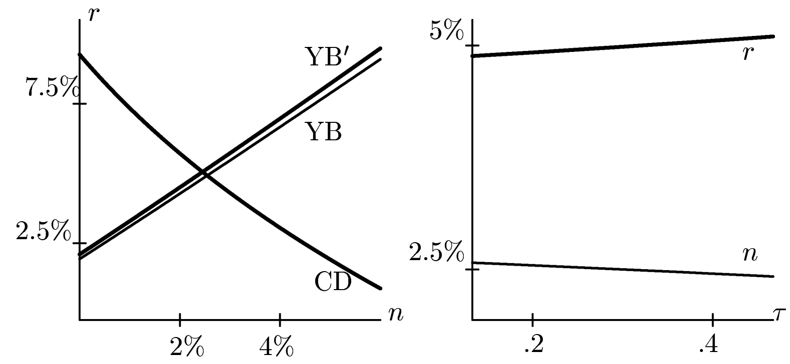
Figure 3.3: An increase in debt that reduces spending

In Figure 3.3 we consider an increase in debt, compensated by a decrease in government spending at unchanged taxes, keeping σ fixed. There is a small—hardly visible—upward movement[7] of the YB curve. The CD curve rotates anti-clockwise around the point where $r = n$. In the region where the economy is dynamically efficient $r > n$, the increase in debt will be compensated by a decline in spending and therefore a reduction in infrastructure. In the dynamically inefficient region increasing debt will allow an expansion of government spending. As long as this case is ruled out, we will have an unambiguous decline of the growth rate. The interest rate falls as well, but not by as much as the growth rate because of the upward pressure from the non-Ricardian effect.

In Table 3.1 we present numbers for the multiplier of debt. For the full model, we predict a fall of growth by .0037% in the growth rate when debt increases by one percent of GDP. If Tobin's q is ignored, the fall in the growth rate would be larger, because the CD curve is more elastic in that case. If we withdraw the finite life aspect from the model the impact of reduced government spending would not be affected by much—the reduction of growth is about one quarter smaller than in a model with finite lives.

The picture is different when there is an increase in debt and taxation adjusts. At first sight, Figure 3.4.b seems to be contradicting the Yaari-Blanchard relationship between interest rates and growth rates, because growth falls, and the interest rate increases. But Figure 3.4.a explains the

[7] With an infinitely lived agent without population growth, there would be none.



(a) d =20% vis $d = 80\%$ (b) $n(d)$ and $r(d)$

Figure 3.4: An increase in debt sustained by higher taxation, (g, σ) fixed

$\Delta g = 0$		q = 1	M = R = 0	q − 1 = M = R = 0
dn/dd	−.0016%	−.0025%	−.0007%	−.0011%
dr/dd	.0021%	.0013%	.0010%	.0005%

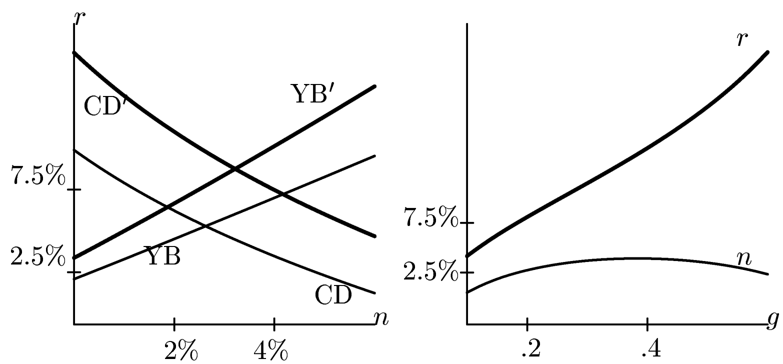
Table 3.2: An increase in debt sustained by higher taxation, (g, σ) fixed

apparent paradox, and illustrates the usefulness of the YB-CD framework. Because there is no change in government spending, the CD curve does not shift. The only shift is the outward shift of the YB curve. Therefore at the intersection, there will be an increase in the interest rate and a decline in the growth rate. The example shows that there is a substantial difference in the way an increase in debt acts on growth, depending on the way the additional debt is financed.

This can further be illustrated by the multiplier of debt on interest and growth rate. Our calculations from Table 3.2 suggest that the impact in the growth rate is twice as large when spending is reduced rather than taxation increased. However in the latter case the private capital stock will fall under the impact of the increase in the interest rate. If adjustment costs are absent then the impact of debt through a reduction on growth will be higher. The CD curve is flatter in that case, thus the leftward shift in the YB curve will have more impact on growth as compared to the interest rate. Our figures for the full calibration suggest a strong increase in the interest rate. Under infinite lives, the impact is only half of what it would be under finite lives.

Assuming away adjustment costs would also contribute to the interest rate impact being underestimated.

In Figure 3.5 we consider an increase of taxes to increase government spending at unchanged debt. The upward shift in the YB curve is due to the distortionary effect in taxes. The outward shift in CD is due to the effect of increased government spending on infrastructure. At the intersection, there is a substantial crowding out effect, which makes for an ambiguous effect of the spending programme on the growth rate. If the YB curve shifts out by more than the CD curve, the result of would be a decline in the growth rate. As the right-hand diagram suggests, this would be the case for a strong increase in taxation, that brings taxation to over 50%.



(a) g =20% vis $g = 40\%$ (b) $n(g)$ and $r(g)$

Figure 3.5: An increase of spending financed by taxation, (τ, σ) fixed

$\Delta d = 0$		q = 1	M = r = 0	q − 1 = M = r = 0
dn/dg	7.4%	10.2%	07.7%	10.2%
dr/dg	16.0%	19.0%	15.6%	19.0%

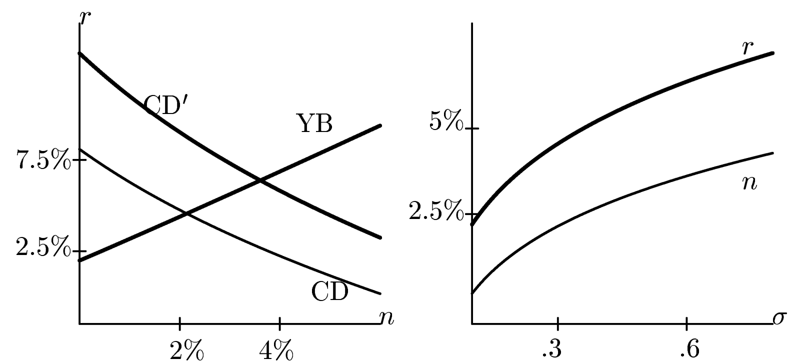
Table 3.3: An increase of spending financed by taxation, (τ, σ) fixed

The computations in Table 3.3 suggest that for the calibration of Appendix B, an increase of government spending by one percent of GDP will increase the rate of growth by .07%. Unfortunately the interest rate increases by more than that, which depresses the supply of private capital. For higher

tax rates, the growth rate will eventually fall. The finite-life aspect of the model hardly matters but Tobin's q exerts a substantial dampening effect on both growth and interest rate movements.

Up until now we have kept the part of investment expenditure in total government expenditure, σ, fixed. We now study a shift in the fraction σ. In Figure 3.6 the YB curve does not shift since it does not depend on σ, because it is only affected by the total size of government expenditure, but not by its decomposition. For each level of investment, private capital will be more productive when infrastructure has been increased, therefore at any given interest rate the growth will be higher and there is an outward shift in the CD curve. The growth rate and interest rate increase.

In Table 3.4 we present the multipliers at the calibrated steady state. Again the inclusion of Tobin's q has a moderating impact on the changes in growth and interest rate.



(a) σ =30% vis $\sigma = 60\%$ (b) $n(\sigma)$ and $r(\sigma)$

Figure 3.6: Increase in the fraction spent on infrastructure, (τ, g, d) fixed

$\Delta g = 0$		q = 1	M = r = 0	q − 1 = M = r = 0
dn/dσ	5.7%	7.7%	5.6%	7.3%
dr/dσ	6.7%	9.4%	7.2%	9.6%

Table 3.4: Increase in the fraction spent on infrastructure, (τ, g, d) fixed

4 Intertemporal Aspects and the Time-Inconsistency Problem

Until now the paper has focused on the steady state of an economy in which consumers are intertemporal optimizers. Fiscal policy has been introduced in an ad hoc fashion ignoring the consequences of treating the government too as an intertemporal optimizer. Although the government may not necessarily be able to stick to pre-announced plans, we assume that it is perfectly benevolent and chooses a utility function which reflects that of a "representative consumer". Note that we assume that the government uses the same discount rate as the individual household; however there is no representative consumer in our overlapping generations model, but rather a spectrum of young and old consumers and those yet to be born. We get round this by using aggregate consumption to represent households of different generations.

$$U(t) = \sum_{t'=0}^{\infty} \left(\frac{1-м}{1+д}\right)^{t'} [\ln(C(t+t')) + \eta \ln(G^c(t+t'))] \tag{4.1}$$

This approach has been suggested by Calvo and Obstfeld (1988). They showed that a general optimization problem that takes account of generational diversity could be broken down into a problem of maximizing a function of aggregate consumption and a second problem of distributing aggregate consumption between generations. By using a social welfare function which aggregates consumption across all households of different ages we can formulate the optimization problem in a state-space linear-quadratic form and so utilize the Markov-perfect concept of a time-consistent equilibrium set out in Appendix C.

However a consequence of using this welfare criterion is that it embodies the policymaker's desired distribution across present and future generations and is dependent upon the authorities' discount factor. In particular, welfare improvement with respect to our chosen social welfare function is not necessarily Pareto improving with respect to present and future generations; for example, an increase in long-run growth can be at the expense of the current generation (See Saint-Paul (1992)). In our model with private capital externalities and tax distortions, there are potential efficiency gains, but these cannot be disentangled from an increase in social welfare measured by (4.1) which arises from redistribution between generations. Nonetheless, bearing in mind the distinction between welfare improvement using (4.1) as the criterion, and Pareto improvement across all generations, we formulate the governments problem as the maximization of (4.1) with respect to fiscal

instruments, given the model summarised in Table 2.1.

4.1 Solvency Considerations

Let $\rho(t) = (1+r(t))/(1+n(t)) - 1$ be the "growth-adjusted" real interest rate over $[t, t+1]$. Then solving the government budget identity

$$d(t) = (1+r(t-1))\, d(t-1) + g(t) - t(t) \tag{4.2}$$

forward in time we transform the budget identity into a solvency constraint at time t

$$d(t-1) = \frac{t(t) - g(t)}{(1+\rho(t))\,(1+\rho(t+1))\ldots(1+\rho(t+t'))} \tag{4.3}$$

where $t(t) - g(t)$ is the primary deficit at time t, provided that the transversality or "no-Ponzi" condition

$$\lim_{t'\to\infty} \frac{d(t+t')}{(1+\rho(t))\,(1+\rho(t+1))\ldots(1+\rho(t+t'))} = 0 \tag{4.4}$$

holds. In (4.3) and (4.4) we assume that eventually $\rho(t) > 0$. This is a feature of the Yaari-Blanchard consumption/savings model and rules out dynamic inefficiency. According to (4.3) a government in debt with $d(0) > 0$ must, sometime in the future, run primary surpluses to be solvent.

It should be noted that the transversality condition (4.4) does not require a *stable* debt/GDP ratio but merely that, in the long run, it does not increase faster than the growth adjusted real interest rate $\rho(t)$. Stability of is sufficient but not necessary to ensure solvency. However in a world with even very small departures from perfectly functioning capital markets, the notion of unbounded government debt/GDP ratios does not appeal. A stronger concept of solvency is that debt/GDP ratios do stabilize. We shall refer to the transversality condition (4.4) and the latter stability condition as *weak* and *strong* solvency conditions respectively.[8] In this paper we adopt the strong condition and enforce it through a small penalty attached to debt in the government's loss function which reflects the costs of issuing debt (or acquiring assets if d is negative) and of collecting taxes we modify the social welfare function (4.1) as follows. The single-period welfare function becomes

$$u(t) = \ln C(t) + \eta \ln G^c(t) - \eta_d\,(d(t))^2 - \eta_\tau\,(\tau(t))^2 - \eta_{\Delta\tau}\,(\Delta\tau(t))^2 \tag{4.5}$$

[8] Buiter and Patel (1990) provide an interesting discussion of this distinction.

The third term in (4.5) with a small value for η_d is sufficient to ensure a stable debt/GDP ratio, i.e. strong solvency. The final two terms penalize both large changes and large levels in the tax rate. We think of the inclusion of these extra terms as imposing a constraint on the liabilities or assets the government can acquire and on the extent of taxation it can impose in any one period. All these terms cover features not directly included in the model.

4.2 Expectations and Time Inconsistency

The credibility of policies and the associated problem of time inconsistency is potentially a major issue in any policy debate. In the model of this paper time inconsistency originates from two basic sources. First, when taxes are distortionary the time inconsistency of optimal tax-smoothing over time are well-known.[9] Second, we have both a private and public capital externality to address, which requires some combination of public investment and inducements to increase private investment and private savings. One such inducement is the reduction of taxation so it is immediately apparent there is a policy trade-off between a public sector investment programme which implies higher, taxation and a strategy aimed at increasing private sector investment.

To explore why these considerations lead to a time-inconsistency problem let us consider the dynamic behaviour of consumption and of Tobin's q. In linearized form[10] these can be written as

$$c_t = \alpha_1 c^e_{t+1} + \alpha_2 n_{t+1} + \alpha_3 \tau^e_{t+1} + \alpha_4 w^e_{t+1} - \alpha_5 r_t \quad (4.6)$$
$$= \alpha_1 c^e_{t+1} + \Theta_t \quad (4.7)$$

say, where $\Theta(\cdot)$ is an increasing function of next period's growth rate and expectations of next period's tax rate and non-human capital, and a decreasing function of the real interest rate. Similarly for Tobin's q we have

$$\mathsf{q}_t = \beta_1 \mathsf{q}^e_{t+1} + \beta_2 n_t - \beta_3 k_t - \beta_4 r_t \quad (4.8)$$
$$= \beta_1 \mathsf{q}^e_{t+1} + \Phi_t \quad (4.9)$$

say, where $\Phi(\cdot)$ is an increasing function of the current growth rate and a decreasing function of the real interest rate and the private capital-labour ratio. Solving forward in time we then have

$$c_t = \sum_{t'=0}^{\infty} \Theta^e_{t+1} \quad \text{and} \quad \mathsf{q}_t = \sum_{t'=0}^{\infty} \Phi^e_{t+1} \quad (4.10)$$

[9]See Faig (1995) for a recent discussion of these issues.

[10]We use time as a subscript for differences with the steady state around which we linearize, e.g. $c_t = c(t) - c$.

giving consumption and Tobin's q as a function of expected future endogenous variables which themselves depend on instrument settings.

Given these features of the model and rational expectations we can distinguish between the cases when an authority has or does not have a reputation for precommitment. A fiscal or monetary authority which enjoys reputation in this sense can exercise the greatest leverage over the private sector in that an announced path of instrument settings would be credible and would effect private sector behaviour immediately in the desired way. For instance the announcement of low taxes in the distant future will immediately raise savings, lower the real interest rate and increase private investment.

When a government cannot precommit itself to a future policy, it must act each period to maximize its welfare function, given that a similar optimization problem will be carried out in the next period. Formally, the policymaker maximizes at time t a welfare function $U(t)$ such that

$$U(t) = u(t) + \varrho U(t+1) \quad (4.11)$$

where u_t is the single-period welfare given above and U_t is evaluated on the assumption that an identical optimization exercise is carried out from time $t+1$ onwards. The solution to this problem is found by dynamic programming and, unlike the precommitment policy leads to a time consistent trajectory or rule for instruments. The simulations reported below use a linearized form of the model and a Taylor series quadratic approximation to the social welfare function valid in the vicinity of the original steady-state corresponding to the calibration in Appendix B. This approximation reduces the optimization problems under both precommitment and discretion to a manageable linear-quadratic form details of which are to be found in Appendix sec:linear.

4.3 Simulation Results

We first report results for the central values of the calibrated model and then we consider some interesting variations. First, however, we need to choose the parameters η_τ, $\eta_{\Delta\tau}$, and η_d in (4.5). If we put $\eta_\tau = \eta_{\Delta\tau} = 0$ which implies no constraint on the size of the tax rate in any one period, but enforce strong solvency by setting η_d equal to a small value (in fact we find that 0.1 is sufficient for this purpose) we obtained optimal trajectories under precommitment for which the tax rate in the first period is over 100%, though the tax rate falls sharply thereafter. This oddity reflects a number of deficiencies in our model including the absence of other tax distortions, the absence of explicit modelling of collection costs, political constraints on high tax rates etc, as well as the shortcomings of a linear-quadratic approximation. Fortunately quite small values of η_τ and $\eta_{\Delta\tau}$ easily[2] remedy this feature of the simulation.

Columns (2) and (3) of Table 4.1 report the steady-state values of key variables for the precommitment (P) and time consistent (TC) regimes and Figures 4.1 and 4.2 show the trajectories. All variables are reported in deviation form about their baseline values. The welfare losses are in percent growth equivalents, again relative to the baseline i.e. a welfare gain of 1% is equivalent to a permanent increase in growth of 1%.

What then do our results tell us about the benefits of precommitment when fiscal policy affects long-run growth? For our central parameter values this is summarized in terms of the transitional and steady-state values of the welfare loss U_0 and U_∞ respectively. For the former regime TC is inferior by

[2] We choose $\eta_\tau = \eta_{\Delta\tau} = 1$. In fact these are small values because in our quadratic approximation the marginal rate of substitution between the consumption/GDP ratio c and τ along the modified utility curve is $-\eta_\tau \tau^* c^2/c^* = .12\eta_\tau$ for our calibration.

	central calibration		same calibration except $\gamma_2 = .33$		
	P Regime	TC Regime	P Regime	TC Regime $\varrho = \frac{(1-M)}{(1+Д)}$	TC Regime $\varrho = \frac{(1-M)}{2(1+Д)}$
n_∞	0.58	2.12	0.9	2.4	0.18
r_∞	0.21	0.39	1.0	1.6	0.53
c_∞	0.34	13	−0.23	12	5.54
d_∞	−95	−180	−94	−188	−40
τ_∞	−4.9	−27	−4.8	−27	−10
g^c_∞	−2.1	−9.9	−2.2	−9.9	−7.0
g^i_∞	0.82	2.2	1.3	2.9	1.5
k_∞	−2.4	−0.72	−14	−18	−5.7
k^g_∞	4.1	6.5	4.8	7.9	8.5
q_∞	4.1	14	7.9	19	6.6
U_0	0.74	0.17	1.2	0.5	0.19
U_∞	0.50	0.17	0.75	0.32	0.31

Table 4.1: Precommitment (P) and Time-Consistent (TC) Policies. Variables are in % and measured as deviations about the original steady-state. For example, $n_t = n(t) - n$ where $n(t)$ is actual and n is steady-state growth. All are (new) steady-state values except for U_0 which is the transitional welfare loss from the baseline values to the new steady state. U_0 and U_∞ are expressed as growth rate equivalents relative to the baseline.

a 0.57% growth equivalent, and for the latter 0.33%. This is not insignificant, but not spectacular either. The manner in which this welfare difference comes about is of interest. The long-run growth rate under regime TC is actually *greater* than under P, i.e. precommitment does not raise growth. This underlines the fact that welfare maximization in models with transitional dynamics is by no means equivalent to growth maximization. The main reason for this outcome is that under TC government consumption is lowered to a level close to zero. The negative welfare implications of the latter mean that despite the higher growth TC is inferior to P in a welfare sense.

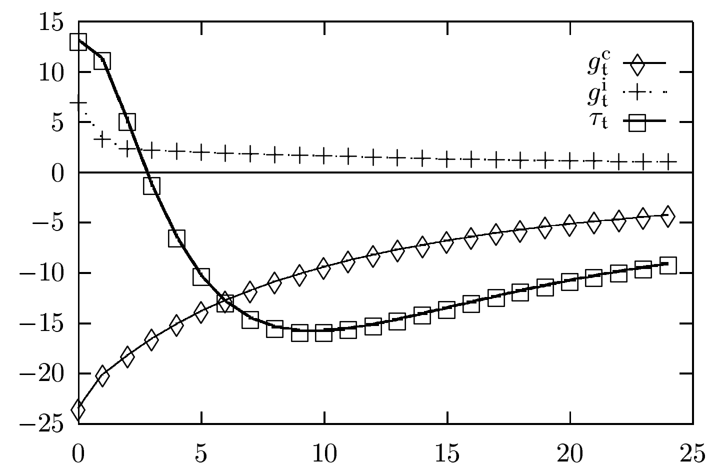


Figure 4.1: Fiscal Policy Under Precommitment

From Figure 4.1 and 4.2 and the steady-state values we can see that under both regimes there is an initial burst of high taxation which gives way eventually to the tax rate falling below the baseline. Government consumption also initially falls under both regimes and government investment rises. A combination of a tax increases and a reduction in spending reduces government debt and eventually when d drops below −53.5%, the baseline debt/GDP ratio, the government begins to acquire assets. The main differences between P and TC is that first, under the latter, government consumption gradually rises to an eventual steady-state close to its baseline whereas under TC the fall is permanent. Second, the TC regimes lowers the tax rate by far more and indeed when τ drops below the baseline tax rate of 22% it becomes a

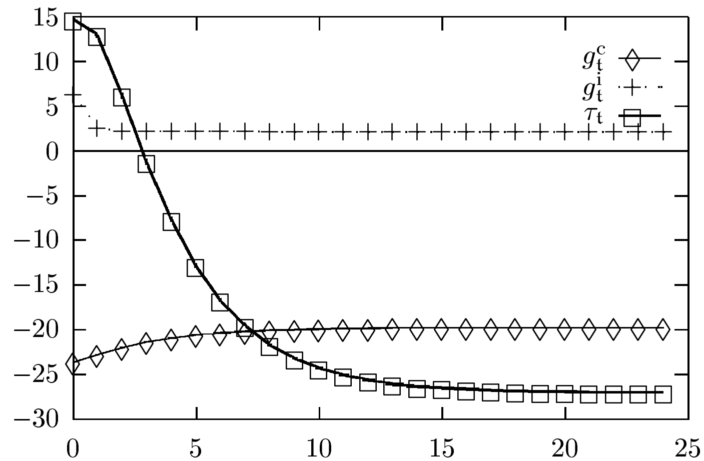


Figure 4.2: Fiscal Policy in the Time Consistent Regime

subsidy to the private sector financed by returns from publicly-owned assets. Both regimes see the government acquiring assets, but these are much higher under TC. As a result under that regime both private and public investment is higher and this explains the higher growth rate.

We know from the comparative statics analysis of Section 3 why the two sets of fiscal policies have the effects we see. It is less straightforward to explain why the differences occur. The nature of the time-inconsistency of the initially optimal policy, regime P, is dynamic and does not arise from permanent benefits from policy surprises as in the Lucas surprise-inflation type of model. The only surprise occurs at the beginning of the regimes; we assume that the baseline policy is a rational expectations equilibrium and the regimes P and TC are not anticipated. But thereafter there is no scope for any more surprises. Broadly speaking, time-consistent policies have flatter profiles for trajectories and a fairly stationary welfare loss over time. Time-inconsistent policies promise long-run policies which are quite different from those in the short and medium term. It is this feature which renders the fiscal policy trajectories under P time inconsistent i.e. if re-optimization were to occur at any time, the new policies would not imply a continuation of the existing trajectories. This pattern is observed in Figure 4.1 and Figure 4.2. Both regimes require an initial increase in the tax rate in order to finance government investment. This is followed by a reduction in the tax rate that facilitates increased private sector savings and investment. Under regime P

the fiscal instruments arrive at the new steady-state relatively early.

Our second policy exercise considers a variation in the crucial parameter γ_2 in the Cobb-Douglas production function. The central choice assumes that observed levels of public sector capital stock corresponding to our baseline are such that the marginal product of public and private capital are the same. Then we have $\gamma_2 = .7$, and the contribution of the public capital stock to the overall capital externality is given by $\gamma_1 \approx 50\%$. Now suppose that the public sector provides the *only* externality i.e. $\gamma_1 = 1$, and $\gamma_2 = \alpha = 33\%$ for our calibration. Columns (4) and (5) of Table 4.1 presents results for this case in the steady-state of the two regimes. Both regimes now involve a much greater increase in public investment and a greater increase in the growth rate relative to the baseline. The gains from optimization are correspondingly greater. The broad qualitative features we observed before remain, and the benefits from precommitment are of the same order of magnitude.

The last column of the table shows results for our last experiment. Given that we do not seem to ever observe governments accumulating assets rather than debt it seems worthwhile to generate this outcome in our model. We concentrate on the TC regime and consider the outcome if the policymaker discounts the future very heavily. This could capture a "political equilibrium" in which the government in a two-party democracy faces the prospect of losing office with a probability of ϕ per period and accordingly discounts at a rate $1 - \phi$ times that of the social planner. The last column shows this case with $\phi = 1/2$. The welfare loss reported is that of the social planner with $\phi = 0$ as before. Now we observe $d > -54\%$ so in the new equilibrium the government still has positive debt (in fact at 13% of GDP). The growth rate is now considerably less than that in the P regime and the transitional welfare benefit of precommitment at the social rate of discount is now over 1%. In fact in the steady state the transitional welfare outcome of the TC regime is worse than that of the baseline "do nothing different" situation!

5 Conclusion

We believe this paper to be the first to study time-inconsistency under a model of endogenous growth. Our main result is that precommitment can actually lead to *lower* long-run growth and the time-consistent solution is associated with an overaccumulation of assets by the government.

The overall profile of taxation and expenditure under the optimal (time-inconsistent) policy is reminiscent of the one discussed by Chamley (1986). A large burst of taxation in the first periods is followed by a decline in the tax rate. However we also see a later increase in taxation, such that the limiting tax rate is still positive due to discounting. The explanation for this profile is quite familiar. The installed capital stock is predetermined at the beginning of the control period, i.e. the start of control is not expected by the private sector. Therefore a tax on that stock mimics a lump-sum tax. Using a heavy tax in the beginning therefore minimises the overall cost of tax collection.[11] When we take account of the time-consistency constraint, the incentive to raise taxes persists through all periods until no more taxes are needed to finance expenditure. Obstfeld (1991) is an early contribution that established this result. Our study shows that the essence of Obstfeld's results carries over to a much more developed model incorporating endogenous growth. If we believe that the accumulation of debt is an important feature of observed economic policy, considering time-consistent policies does not bring the predictions of the model closer to the empirical facts; in fact it drives them away since asset accumulation of the government is larger.

The new element that we add to the picture is the decision between government consumption and investment expenditure. A naïve view would be to blame time-consistency for insufficient investment. Our numerical experiments suggest that this is not correct and in fact the time-consistent policy overaccumulates public capital. Loosely speaking we are adding another layer of overinvestment into the dynamic behaviour. For any given path of government expenditure, the time-consistent policy overaccumulates financial assets (with respect to the optimal policy). When we free government spending we also have overinvestment in physical assets. If we believe that "out there in the real world" governments in fact underinvest, we can not take comfort from the time-consistency approach when searching for a theoretical underpinning for this view, unless we allow the government to discount *much* more heavily than the private sector.

These results must be qualified when we take into account of the most important limiting feature of the model, which is that the economy is closed. When the economy is open there are externalities from one country's fiscal deficit on the others. In particular debt becomes more attractive since an increase in one country's debt will only raise the common interest rate. There is also the question of the externality of one country's government expenditure on the growth rate on the other. These issues are left for further research.

[11]Note that this result is not dependent on the finite-life aspects of the model. In fact the impact of the finite life aspect is rather small. All that matters is that taxation is distortionary.

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A The linear-quadratic framework

In this appendix we present the linear-quadratic form of our model. It is written in differences from the steady-growth state. For any variable x, we introduce the notation

$$x_{\mathfrak{t}} = x(\mathfrak{t}) - x$$

where $x(\mathfrak{t})$ is the actual value, and x the steady-growth value. For the target variables (see below) we need to reintroduce levels; we do that with a special variable $1_{\mathfrak{t}} = 1, \forall \mathfrak{t}$.

When writing down the model, it is important to grasp the sequence of events though time. The capital and infrastructure stocks at the beginning of the period $\mathfrak{t}$ are predetermined as $k_{\mathfrak{t}-1}$ and $k^{\mathrm{g}}_{\mathfrak{t}-1}$. Therefore output in period $\mathfrak{t}$ and the growth rate in period $\mathfrak{t}$ are equally predetermined. However the split of output between various usages is not predetermined. From the Yaari-Blanchard consumption function, we can see that actual consumption depends on the expected value of wealth and the taxation in the next period. Its linearized version (A.5) can be forwarded in time to demonstrate that present consumption depends on the path of interest rates—therefore on government spending—and tax rates from $\mathfrak{t}$ to the indefinite future. To model this dependency we define a forward looking variable τ^{f} such that $\tau^{\mathrm{f}}_{\mathfrak{t}} = -\mu\,\tau^{\mathrm{f}}_{\mathfrak{t}+1} + g_{\mathfrak{t}}$. We can define the rational expectations of government spending as

$$\tau^{\mathrm{e}}_{\mathfrak{t}+1} = \lim_{\mu \to 0} \tau^{\mathrm{f}}_{\mathfrak{t}+1}$$

The same procedure is followed for the tax rate, and μ is given a very small value. To linearize the adjustment costs we assume that $a(\cdot)$ is linear and define $\psi = a(x)/x\,\forall\, x$ The equation where adjustment costs occur are written as

$$\begin{aligned}
g(\mathfrak{t}) &= g^{\mathrm{c}}(\mathfrak{t}) + g^{\mathrm{i}}(\mathfrak{t})\left[1 + \psi\,\frac{g^{\mathrm{i}}(\mathfrak{t})\,(1+n(\mathfrak{t}))}{k^{\mathrm{g}}(\mathfrak{t}-1)}\right] \\
\mathfrak{q}(\mathfrak{t}) &= 1 + 2\,\psi\left(\frac{i(\mathfrak{t})\,(1+n(\mathfrak{t}))}{k(\mathfrak{t}-1)}\right) \\
0 &= \frac{\alpha\,(1+n(\mathfrak{t}+1))}{k(\mathfrak{t})} + \psi\,\frac{(1+n(\mathfrak{t}+1))^2\,i(\mathfrak{t}+1)^2}{k(\mathfrak{t})^2} \\
&\quad + (1-\delta)\,\mathfrak{q}(\mathfrak{t}+1) - (1+r(\mathfrak{t}))\,\mathfrak{q}(\mathfrak{t}) \\
1 &= c(\mathfrak{t}) + i(\mathfrak{t})\left[1 + \psi\,\left(\frac{i(\mathfrak{t})\,(1+n(\mathfrak{t}))}{k(\mathfrak{t}-1)}\right)\right] + g(\mathfrak{t})
\end{aligned} \tag{A.1}$$

The rest of the linearization is straightforward. The state equations are

$$k^{g}_{t} = \frac{1-\delta}{1+n}\,k^{g}_{t-1} - \frac{(1-\delta)\,k^{g}}{(1+n)^2)}\,n_t + g^{i}_{t} \tag{A.2}$$

$$d_t = \frac{1+r}{1+n}\,d_{t-1} - \frac{(1+r)\,d}{(1+n)^2}\,n_t + \frac{d}{1+n}\,r_{t-1} + g_t - t_t \tag{A.3}$$

$$k_t = \frac{1-\delta}{1+n}\,k_{t-1} - \frac{(1-\delta)\,k}{(1+n)^2}\,n_t + i_t \tag{A.4}$$

$$\begin{aligned} 0 = {} & \left(\frac{1+д}{1-м} - \frac{м+г}{1+г}\right) c_{t+1} - \frac{1+r\,(1-\tau)}{1+n}\,c_t - \frac{c\,(1+r\,(1-\tau))}{(1+n)^2}\,n_{t+1} \\ & + \frac{c\,(1-\tau)}{1+n}\,r_t - \frac{c\,r}{1+n}\,\tau_{t+1} - \frac{(м+г)\,(м+д)}{(1-м)\,(1+г)}\,ш_{t+1} \end{aligned} \tag{A.5}$$

$$\mu\, g^{f}_{t+1} = g^{f}_{t} - g_t \tag{A.6}$$

$$\mu\, \tau^{f}_{t+1} = \tau^{f}_{t} - \tau_t \tag{A.7}$$

The measurement equations are:

$$-t_t = -\left(1 - \frac{\delta\,k}{1+n}\right)\tau_t - \frac{\delta\,k\,\tau}{(1+n)^2}\,n_t + \frac{\tau\,\delta}{1+n}\,k_{t-1} \tag{A.8}$$

$$\begin{aligned} -g_t = {} & -\left(1 + \frac{2\,\psi\,g^{i}(1+n)}{k^{g}}\right) g^{i}_{t} - g^{c}_{t} - \frac{\psi\,g^{i^2}}{k^{g}}\,n_t \\ & + \frac{g^{i^2}\,(1+n)}{{k^{g}}^2}\,k^{g}_{t-1} \end{aligned} \tag{A.9}$$

$$-t_{t+1} = -\left(1 - \frac{\delta\,k}{1+n}\right)\tau_{t+1} - \frac{\delta\,k\,\tau}{(1+n)^2}\,n_{t+1} + \frac{\tau\,\delta}{1+n}\,k_t \tag{A.10}$$

$$-\frac{n_{t+1}}{1+n} = -\frac{\gamma_2}{k}\,k_t - \frac{1-\gamma_2}{k^{g}}\,k^{g}_{t} \tag{A.11}$$

$$\begin{aligned} -ш_{t+1} = {} & -\frac{1+r}{1+n}\,d_t + \left[\frac{(1+r)\,d}{(1+n)^2} + \frac{(1-\delta)\,k}{(1+n)^2}\right] n_{t+1} \\ & - \frac{1-\delta}{1+n}\,k_t - \frac{d}{1+n}\,r_t - g_{t+1} + t_{t+1} - i_{t+1} \end{aligned} \tag{A.12}$$

$$\mathrm{q}_t = \frac{2\,\psi\,(1+n)}{k}\,i_t - \frac{2\,\psi\,i\,(1+n)}{k^2}\,k_{t-1} + \frac{2\,\psi\,i}{k}\,n_t \tag{A.13}$$

$$\tag{A.14}$$

$$\mathrm{q}_{t+1} = \frac{2\,\psi\,(1+n)}{k}\,i_{t+1} - \frac{2\,\psi\,i\,(1+n)}{k^2}\,k_t + \frac{2\,\psi\,i}{k}\,n_{t+1} \tag{A.15}$$

$$\begin{aligned} 0 = {} & -\frac{\alpha}{k}\,n_{t+1} + \frac{\alpha\,(1+n)}{k^2}\,k_t - (1+n)\,\mathrm{q}_{t+1} \\ & + (1+r)\,\mathrm{q}_t + \mathrm{q}\,r_t \end{aligned} \tag{A.16}$$

$$-\Delta\tau_t = -\tau_t + \tau_{t-1} \tag{A.17}$$

$$-g_{t+1} = -\frac{g^{f}_{t}}{\mu} + \frac{g_t}{\mu} \tag{A.18}$$

$$\left(1 + \frac{2\,\psi\,i\,(1+n)}{k}\right) i_{t+1} = -\frac{\psi\,i^2}{k}\,n_{t+1} + \frac{\psi\,i^2\,(1+n)}{k^2}\,k_t - c_{t+1} - g_{t+1} \tag{A.19}$$

$$\left(1 + \frac{2\,\psi\,i\,(1+n)}{k}\right) i_t = -\frac{\psi\,i^2}{k}\,n_t + \frac{\psi\,i^2\,(1+n)}{k^2}\,k_{t-1} - c_t - g_t \tag{A.20}$$

$$-d^{*}_{t} = d\,1_t + d_t \tag{A.21}$$

$$-t^{*}_{t} = t\,1_t + t_t \tag{A.22}$$

$$-c^{*}_{t} = -c_t + c\,1_t \tag{A.23}$$

$$-g^{c*}_{t} = -g^{c}_{t} + g^{c}\,1_t \tag{A.24}$$

$$-n^{*}_{t} = -n_t + (1+n)\,1_t \tag{A.25}$$

$$-\kappa_t = -\frac{k_t}{k} + \frac{k^{g}_{t}}{k^{g}} \tag{A.26}$$

$$-\tau_{t+1} = -\frac{\tau^{f}_{t}}{\mu} + \frac{\tau_t}{\mu} \tag{A.27}$$

Here the starred variables denote target values, and the Δ stands for changes in time. This type of variables are needed to formalize the extensions to government's objective. This is a quadratic approximation of the welfare criterion (4.1). The term in private consumption may be written as

$$\sum_{t'=0}^{\infty} \varrho^{t'} \left[\ln\!\left(\frac{C(t+t')}{Y(t+t')}\right) + \ln(Y(t+t'))\right] \tag{A.28}$$

There first term of the sum (A.28) is:

$$\sum_{t'=0}^{\infty} \varrho^{t'} \ln\!\left(\frac{C(t+t')}{Y(t)}\right) \approx -\frac{1}{2\,c^2} \sum_{t'=0}^{\infty} \varrho^{t'}\,(c - c_{t+t'})^2 + \text{constant} \tag{A.29}$$

The second term of (A.28) can be expanded as:

$$\begin{aligned}\sum_{t'=0}^{\infty} \varrho^{t'} \ln(Q(t+t')) &= \ln(Q(t)) \\ &\quad + \varrho \ln(Q(t)) + \varrho\,(1+n(t)) \\ &\quad + \varrho^2 \ln(Q(t)) + \varrho^2 \ln(1+n(t)) + \varrho^2 \ln(1+n(t+1)) \\ &\quad \vdots \\ &= \frac{\ln(Q(t))}{1-\varrho} + \frac{\varrho}{1-\varrho} \sum_{t'=0}^{\infty} \varrho^{t'} \ln(1+n(t+t')) \\ &= \frac{\varrho}{1-\varrho} \sum_{t'=0}^{\infty} \varrho^{t'} \ln(1+n(t+t')) + \text{constant} \\ &\approx \frac{\varrho}{1-\varrho} \sum_{t'=0}^{\infty} \varrho^{t'} \left[\frac{n_{t+t'}}{1+n} - \frac{n_{t+t'}^2}{2\,(1+n)^2}\right] + \text{constant}\end{aligned}$$

Since the growth rate has a high coefficient attached to it, we include squares in the deviation of the growth rate. The equation that defines the growth rate can be approximated as

$$\frac{n_t}{1+n} = \frac{\gamma_2}{k}\, k_{t-1} + \frac{1-\gamma_2}{k^g}\, k^g_{t-1} - \frac{(1-\gamma_2)\,\gamma_2}{2} \left[\frac{k_{t-1}}{k} - \frac{k^g_{t-1}}{k^g}\right]^2$$

The last term can be incorporated in our approximation of the growth effect.

$$\begin{aligned}\sum_{t'=0}^{\infty} \varrho^{t'} \ln(Y_{t+t'}) &= \sum_{t'=0}^{\infty} \varrho^{t'} \Bigg(\frac{n_{t+t'}}{1+n} - \frac{n_{t+t'}^2}{2\,(1+n)^2} \\ &\quad - \frac{(1-\gamma_2)\,\gamma_2}{2} \left[\frac{k_{t+t'-1}}{k} - \frac{k^g_{t+t'-1}}{k^g}\right]^2\Bigg) + \text{constant} \\ &= -\frac{1}{2\,(1+n)^2}\,\frac{\varrho}{1-\varrho} \sum_{t'=0}^{\infty} \varrho^{t'}\,(n-n_{t+t'})^2 \\ &\quad - \frac{\varrho}{1-\varrho}\,\frac{(1-\gamma_2)\,\gamma_2}{2} \sum_{t'=0}^{\infty} \varrho^{t'}\,\kappa_t^2 + \text{constant}\end{aligned} \tag{A.30}$$

where we define

$$\kappa_t \equiv \frac{k_{t-1}}{k} - \frac{k^g_{t-1}}{k^g}$$

The same split as in (A.28) for private consumption can then be made for government consumption. We also incorporate penalties on taxation and changes to taxation, (to reflect tax collection costs), debt (to reflect the cost of administering the debt) and on changes in Tobin's q. To sum up all the components, the instantaneous loss of the government is

$$\begin{aligned}u_t = \frac{{c^*_t}^2}{c^2} + \frac{\eta\, {g^{c*}_t}^2}{g^{c2}} + \frac{\varrho\,(1+\eta)}{(1-\varrho)\,(1+n)^2}\, {n^*_t}^2 + \frac{\gamma_2\,(1-\gamma_2)\,\varrho\,(1+\eta)}{(1-\varrho)}\,\kappa_t^2 \\ + \eta_{\Delta\tau}\,\Delta\tau_t^2 + \eta_d\, {d^*_t}^2 + \eta_\tau\, {\tau^*_t}^2\end{aligned} \tag{A.31}$$

and the intertemporal loss is

$$U_t = \sum_{t'=t}^{\infty} \varrho^{t'-t}\,\frac{u_t}{2}$$

B Calibration

The model is calibrated around a steady-growth state fitted to the economy of the United States in 1990. There are two possible approaches on how to calibrate a model like ours. The first consists in collecting data about observable variables like debt, growth, consumption etc, and deduce variables that are not observed from the steady state of the model. A second approach would do the opposite, i.e. use different scenarios of the unobserved variables to see whether in the steady state these will give values for the observed variable that conform to observation. This method has the advantage to allow for "what if" simulations to study the effect of changes in the unobserved exogenous parameters.

We have taken a hybrid pragmatic approach. From the model it is apparent that the growth rate is the central variable of the model whose deduction as an endogenous variable would be subject to multiple numerical solutions. Therefore we first choose $n = 2.5\%$. We also fix $r = 5\%$. For the population we chose $\text{M} = 2\%$ and overall population growth $\text{r} = 1\%$. This is to take account of immigration.

For the capital stock, we have data available from OECD (1994) about the net capital stock $K = 9650.3$ billion \$US. in 1990. This is the figure we choose for the private capital stock[12] i.e. $k \approx 1.8$. We also collect the following figures from the US department of commerce mirror at `gopher://una.lib.umich.e du:70/11/ebb`, all in millions of dollars.

[12] This figure excludes public infrastructure but data on infrastructure is not available.

Taking i to stand for fixed investment and observed figure for the capital stock k we calibrate the depreciation rate as

$$\delta = \frac{(1+n)\,i}{k} - n \approx 6\%$$

To estimate the stock of infrastructure we use our assumption that the rate of depreciation of private and public capital are equal to deduct the public infrastructure stock from the government expenditure on infrastructure. To estimate that expenditure, we collect data for various categories of expenditure numbered 1–14 in Table B.2. We assume that the categories 4, 5 and 12 are the expenditure contributing to the capital stock of the government. We can then compute σ, the proportion of investment expenditure, as $\sigma \approx 36\%$. This is used to find the stock of infrastructure

$$k^{g} = \frac{\sigma\, g\,(1+n)}{n+\delta} \approx 79\%$$

The constant $\bar{\epsilon}$ is obtained from the identity between production and national income

$$\bar{\epsilon} = (1+n)\, k^{g-\gamma_2}\, k^{\gamma_2 - 1} \approx 73\%$$

Let $a(x) = \psi\, x$, for the adjustment parameter we choose $\psi = 3$. Tobin's q is then derived from

$$\mathsf{q} = 1 + 2\,\psi\,(n+\delta) \approx 1.5$$

which appears to be on the high side, i.e. slightly above the range suggested in the recent empirical study by Blanchard, Rhee, and Summers (1993), but when we use the equation for the desired capital stock we find

$$\alpha = \psi\,\frac{(i+n)^2}{1+n} + \mathsf{q}\,\frac{(r+\delta)\,k}{1+n} \approx 33\%$$

Gross domestic Product	5250.8
Personal Consumption Expenditures	3523.1
Gross Private Domestic Investment	832.3
Fixed Investment	798.9
Government Purchases	975.2
Gross Domestic Purchases	5330.5
Final Sales to Domestic Purchasers	5297.2

Table B.1: US national accounts in 1989 (source: US Bureau of Commerce)

which is in line with received wisdom that suggests that capital's share is about one third of output.

To calibrate the relative efficiency of the private vis-à-vis the public sector, we use the relative share of the private capital stock

$$\gamma_2 = \frac{k}{k^{g} + k} \approx 69\%$$

Private consumption is found as

$$c = 1 - i\,[1 + \psi\,(\delta + n)] - g \approx 63\%$$

This is somewhat lower than the direct national income figures suggest, because consumption is a residual when investment and government spending *including* adjustment costs are subtracted from output. It is a widely held view that national account systems overestimate consumption, therefore this approach to integrate adjustment costs into the national accounting identity seems appropriate.

	Category	A	B	C
0	Total expenditure	1194.60	501.66	438.77
1	General public services	78.71	14.25	23.63
2	Defense	293.54		
3	Public order and safety	10.57	20.01	40.83
4	Education	21.50	169.14	191.57
5	Health	154.19	87.30	36.27
6	Social security & welfare	317.82	82.74	32.39
7	Housing and community amenities	32.62	3.61	13.01
8	Recreation, cultural & religious affairs	3.21	2.67	13.69
9	Fuel & energy	5.41	.26	2.14
10	Agriculture, forestry, fishing & hunting	21.89	8.99	2.42
11	Mining, manufacturing & construction	.64		
12	Transports & communications	29.89	46.74	27.23
13	Other economic affairs & services	38.29	7.04	2.20
14	Other expenditures	187.17	58.91	53.39

Table B.2: US Government expenditure by Category. Column A is "consolidated central government" expenditure, B is "state region and province government", and C is "local government" (source: IMF Government Finance Statistics Yearbook)

Government consumption is the part that is left from total government spending once government investment has been taking place and adjustment costs have been paid

$$g^c = g\,(1 - \sigma - \sigma\,\psi\,(n + \delta)) \approx 10\%$$

Now we look at the financing of that expenditure. Here we combine expenditure data from Table B.1 and debt data from Table B.3. Adding the debt components and dividing by GNP gives a ratio of 53.5%. The overall tax rate in the steady-state can then be determined from the government's budget constraint

$$\tau = \frac{d\,(r - n) + (1 + n)\,g}{1 + n - \delta\,k} \approx 22\%$$

Using that value in the consumption function, we can deduce the time preference parameter out of the observed data

$$\begin{aligned} д &= (1 + n)\,(d + k(\text{м}\,(1 + \text{г}) + c\big[r\,(1 - \tau)\,(1 + \text{г})\,(1 - \text{м}) + (1 - \text{м}) \\ &\quad (\text{г} - \text{м}) + (\text{г} + \text{м})\,\text{м}\,(1 - n)\big] / \left[c\,(1 + \text{г}) + (\text{м} + \text{г})\,(k + d)\right] / (1 + n) \\ &\approx 1.8\% \end{aligned}$$

Finally we calibrate the government felicity parameter η on the ratio of public versus private consumption

$$\eta = \frac{g^c}{c} \approx 17\%$$

Consolidated Central Government Debt	2207.46
State, Region & Province Government Debt 1989	283.31
Local Government Debt	361.37

Table B.3: US government debt in 1989 (source: IMF Government Finance Statistics Yearbook)

C The Solution Procedures

C.1 Setting Up the Linear Version

The model of Appendix A can be expressed in state-space form as

$$\begin{bmatrix} \mathbf{z}_{t+1} \\ \mathbf{x}^e_{t+1,t} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{z}_t \\ \mathbf{x}_t \end{bmatrix} + \mathbf{B}\,\mathbf{w}_t \tag{C.1}$$

$$\mathbf{s}_t = \mathbf{E}_1 \begin{bmatrix} \mathbf{z}_t \\ \mathbf{x}_t \end{bmatrix} + \mathbf{E}_2\,\mathbf{w}_t \tag{C.2}$$

where $\mathbf{z}_t$ is an $n_p \times 1$ vector of predetermined variables at time t. $\mathbf{x}_t$ is an $n_f \times 1$ vector of free variables, and $\mathbf{x}^e_{t+1,t}$ denotes rational expectations of $\mathbf{x}_{t+1}$. In our model there are only three non-predetermined variables, consumption and the two forward-looking variables for government policy, therefore $n_f = 3$. $\mathbf{s}_t$ is an $n_t \times 1$ vector of target variables, expressed as deviation from a bliss point. In our model the bliss point for consumption and government consumption are 100% of GDP, and the bliss point for growth is 100% as well. Of course these points can not be achieved simultaneously at any date, therefore the welfare loss will be strictly positive. The loss of the government is written as

$$U_t = \frac{1}{2} \sum_{t'=0}^{\infty} \varrho^{t'}\, \mathbf{s}^\top_{t+t'}\, \boldsymbol{\eta}\, \mathbf{s}_{t+t'} \tag{C.3}$$

where $\boldsymbol{\eta}$ is a symmetric and positive definite matrix of weights and $\varrho > 0$ is the discount factor. The policymaker's optimization problem is to minimize U_t subject to the model (C.1) and the initial vector $\mathbf{z}_t$. Substituting (C.2) in (C.3) will give the following form of the welfare loss

$$\begin{aligned} U_t = \frac{1}{2} \sum_{t'=0}^{\infty} \varrho^{t'} \big[&\mathbf{y}^\top_{t+t'}\,\mathbf{Q}\,\mathbf{y}_{t+t'} + 2\,\mathbf{y}^\top_{t+t'}\,\mathbf{U}\,\mathbf{w}_{t+t'} \\ &+ \mathbf{w}^\top_{t+t'}\,\mathbf{R}\,\mathbf{w}_{t+t'}\big] \end{aligned} \tag{C.4}$$

Where we use the definitions $\mathbf{Q} = \mathbf{E}_1^\top\,\boldsymbol{\eta}\,\mathbf{E}_1$, $\mathbf{U} = \mathbf{E}_1^\top\,\boldsymbol{\eta}\,\mathbf{E}_2$, and $\mathbf{R} = \mathbf{E}_2^\top\,\boldsymbol{\eta}\,\mathbf{E}_2$. We also introduce the notation $\mathbf{y}_t^\top = [\mathbf{z}_t^\top, \mathbf{x}_t^\top]$ as the state vector, of dimension $n_s \times 1$, where $n_s = n_p + n_f$. For the vectors that have the dimension $n_s \times 1$, it is convenient to partition the vector into the first n_p elements and the n_f elements that follow. Using this notation, for example

$$\mathbf{y}_t \equiv \begin{bmatrix} \mathbf{y}_{p,t} \\ \mathbf{y}_{f,t} \end{bmatrix} \tag{C.5}$$

where here of course $\mathbf{y}_{p,t} = \mathbf{z}_t$ and $\mathbf{y}_p = \mathbf{z}_t$. It is also inconvenient to introduce a similar notation for matrices. Let $\mathbf{X}$ be any matrix of dimension $n_s \times n_s$, then write

$$\mathbf{X} \equiv \begin{bmatrix} \mathbf{X}_{p,p} & \mathbf{X}_{p,f} \\ \mathbf{X}_{f,p} & \mathbf{X}_{f,f} \end{bmatrix} \tag{C.6}$$

such that $\mathbf{X}_{p,p}$ is of dimension $n_p \times n_p$ $\mathbf{X}_{f,p}$ is of dimension $n_f \times n_p$ $\mathbf{X}_{p,f}$ is of dimension $n_p \times n_f$ and $\mathbf{X}_{f,f}$ is of dimension $n_f \times n_f$. We will make repeated use of this notation in the remainder of the appendix, when we develop the solution procedures for both the precommitment and the time consistent case.

C.2 The Optimal Policy With Precommitment

To find the optimum policy under precommitment, consider the government's ex-ante optimum policy at $t = 0$ under the assumption that precommitment is possible. By standard theory of Lagrangian multipliers, we then minimize the Lagrangian

$$\mathcal{L}_0 = U_0 + \sum_{t=0}^{\infty} \varrho^t \boldsymbol{\lambda}_{t'} \left[\mathbf{A}\,\mathbf{y}_t + \mathbf{B}\,\mathbf{w}_t - \mathbf{y}_{t+1} \right] \tag{C.7}$$

with respect to $\{\mathbf{y}_t\}_{t=0}^{\infty}$, $\{\boldsymbol{\lambda}_t\}_{t=0}^{\infty}$, and $\{\mathbf{w}_t\}_{t=0}^{\infty}$, for a given $\mathbf{z}_0$. This gives the first order conditions that

$$\mathbf{w}_t = -\mathbf{R}^{-1} \left[\varrho\,\mathbf{B}^\top \boldsymbol{\lambda}_{t+1} + \mathbf{U}^\top \mathbf{y}_t \right] \tag{C.8}$$

$$\mathbf{U}\,\mathbf{w}_t = \boldsymbol{\lambda}_t - \varrho\,\mathbf{A}^\top \boldsymbol{\lambda}_{t+1} - \mathbf{Q}\,\mathbf{y}_t \tag{C.9}$$

together with the original constraint

$$\mathbf{y}_{t+1} = \mathbf{A}\,\mathbf{y}_t + \mathbf{B}\,\mathbf{w}_t \tag{C.10}$$

Equations (C.8), (C.9) and (C.10) hold for $t \geq 1$. They can be written in state-space form as

$$\begin{bmatrix} \mathbf{I} & \varrho\,\mathbf{B}\,\mathbf{R}^{-1}\mathbf{B}^\top \\ \mathbf{0} & \varrho\,(\mathbf{A}^\top - \mathbf{U}\,\mathbf{R}^{-1}\mathbf{B}^\top) \end{bmatrix} \begin{bmatrix} \mathbf{y}_{t+1} \\ \boldsymbol{\lambda}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\,\mathbf{R}^{-1}\mathbf{U}^\top & \mathbf{0} \\ -\mathbf{Q} + \mathbf{U}\,\mathbf{R}^{-1}\mathbf{U}^\top & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{y}_t \\ \boldsymbol{\lambda}_t \end{bmatrix} \tag{C.11}$$

The solution to (C.11) requires $2\,n_s$ boundary conditions. The first order condition in $t = 0$, requires that

$$\boldsymbol{\lambda}_0^\top\, d\,\mathbf{y}_0 = 0 \tag{C.12}$$

Within $\mathbf{y}_0$ the first n_p elements are predetermined, therefore $d\mathbf{y}_0^p = 0$, whilst the n_f elements that follow are free and therefore require from (C.12) that

$$\boldsymbol{\lambda}_{f,0} = \mathbf{0} \tag{C.13}$$

This gives n_f boundary conditions to solve (C.11). The initial value $\mathbf{z}_0$ gives n_p more conditions. Finally the transversality condition

$$\lim_{t \to \infty} \varrho^t\, \boldsymbol{\lambda}_t = \mathbf{0} \tag{C.14}$$

provides n_s more conditions, which complete to the required $2\,n_s$ boundary conditions. The solution takes the form

$$\boldsymbol{\lambda}_t = \mathbf{S}\,\mathbf{y}_t \tag{C.15}$$

Substituting into (C.9) we get

$$\begin{aligned} \mathbf{w}_t &= -\left(\mathbf{R} + \mathbf{B}^\top \mathbf{S}\,\mathbf{B}\right)^{-1} \left(\mathbf{B}^\top \mathbf{S}\,\mathbf{A} + \mathbf{U}^\top\right) \mathbf{y}_t \\ &= -\mathbf{F}\,\mathbf{y}_t \end{aligned} \tag{C.16}$$

say, where $\mathbf{S}$ is the solution to the Ricatti matrix equation

$$\begin{aligned} \mathbf{S} = \mathbf{Q} - \mathbf{U}\,\mathbf{F} - \mathbf{F}^\top \mathbf{U}^\top + \mathbf{F}^\top \mathbf{R}\,\mathbf{F} \\ + (\mathbf{A} - \varrho\,\mathbf{B}\,\mathbf{F})^\top \mathbf{S}\,(\mathbf{A} - \mathbf{B}\,\mathbf{F}) \end{aligned} \tag{C.17}$$

To complete the solution we express the non-predetermined variables at time t, $\left[\boldsymbol{\lambda}_{p,t}^\top \quad \boldsymbol{\lambda}_{p,t}^\top\right]^\top$ in terms of the predetermined variables $\left[\mathbf{z}_t^\top \quad \boldsymbol{\lambda}_{p,t}^\top\right]^\top$. Rearranging (C.15), we obtain

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\lambda}_{p,t} \\ \mathbf{x}_t \end{bmatrix} &= \begin{bmatrix} \mathbf{S}_{p,p} - \mathbf{S}_{f,f}^{-1}\,\mathbf{S}_{f,p} & \mathbf{S}_{p,f}\,\mathbf{S}_{f,f}^{-1} \\ -\mathbf{S}_{f,f}^{-1}\,\mathbf{S}_{f,p} & \mathbf{S}_{f,f}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t} \end{bmatrix} \\ &= -\mathbf{N} \begin{bmatrix} \mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t} \end{bmatrix} \end{aligned} \tag{C.18}$$

say. Substituting into (C.16) gives

$$\begin{aligned} \mathbf{w}_t &= -\mathbf{F} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{N}_{f,p} & -\mathbf{N}_{f,f} \end{bmatrix} \begin{bmatrix} \mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t} \end{bmatrix} \\ &= \mathbf{G} \begin{bmatrix} \mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t} \end{bmatrix} \end{aligned} \tag{C.19}$$

say, and combining (C.10), (C.16) and (C.18) gives

$$\begin{bmatrix} \mathbf{z}_{t+1} \\ \boldsymbol{\lambda}_{p,t+1} \end{bmatrix} = \mathbf{T}\,(\mathbf{A} - \mathbf{B}\,\mathbf{F})\,\mathbf{T}^{-1} \begin{bmatrix} \mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t} \end{bmatrix} \qquad \text{where} \quad \mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{S}_{f,p} & \mathbf{S}_{f,f} \end{bmatrix}$$
$$= \mathbf{H} \begin{bmatrix} \mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t} \end{bmatrix} \tag{C.20}$$

say. Given the solution $\mathbf{S}$ to the Ricatti equation (C.17), equations (C.18) to (C.20) completely characterize the solution to the optimization problem. The solution can be expressed as a feedback on the history of the state vectors. At $t = 0$, this feedback is simply given by (C.19). To find the feedback for the following periods, use (C.20) to write

$$\boldsymbol{\lambda}_{p,t+1} = \mathbf{H}_{f,p}\,\mathbf{z}_t + \mathbf{H}_{f,f}\,\boldsymbol{\lambda}_{p,t} \tag{C.21}$$

Solving (C.21) and using (C.12), we find

$$\boldsymbol{\lambda}_{p,t+1} = \mathbf{H}_{f,p} \sum_{t'=0}^{t} (\mathbf{H}_{f,f})^{t'}\,\mathbf{z}_{t-t'} \tag{C.22}$$

Hence the feedback form of the rule $\mathbf{w}_t = \mathbf{G}_p\,\mathbf{z}_t + \mathbf{G}_f\,\boldsymbol{\lambda}_{f,t}$ can be expressed solely in terms of the (at time t) predetermined variables $\mathbf{z}_t$.

Finally let us evaluate the welfare loss along the trajectory or "cost-to-go". From the envelope theorem and the first order condition (C.12), we have that

$$\frac{dU}{d\mathbf{y}_0} = \frac{d\mathcal{L}_0}{d\mathbf{y}_0} = \boldsymbol{\lambda}_0^\top \tag{C.23}$$

Hence from (C.15) on integration we have

$$U_0 = \frac{1}{2}\,\mathbf{y}_0^\top\,\mathbf{S}\,\mathbf{y}_0 \tag{C.24}$$

at time $t' = 0$. At time t this becomes

$$U_t = \frac{1}{2}\,\mathbf{y}_t^\top\,\mathbf{S}\,\mathbf{y}_t \tag{C.25}$$

Another way of expressing U_t, which will proof useful, is found by eliminating $\mathbf{x}_t$ in (C.25) using (C.18). We obtain

$$U_t = -\frac{1}{2}\left[\text{trace}(\mathbf{N}_{p,p}\,\mathbf{z}_t\,\mathbf{z}_t^\top) + \text{trace}(\mathbf{N}_{f,f}\,\boldsymbol{\lambda}_{p,t}\,\boldsymbol{\lambda}_{p,t}^\top)\right] \tag{C.26}$$

which at $t = 0$, using (C.13) becomes

$$U_0 = -\frac{1}{2}\left[\text{trace}(\mathbf{N}_{p,p}\,\mathbf{z}_0\,\mathbf{z}_0^\top)\right] \tag{C.27}$$

C.3 The Time Consistent (Markov-Perfect) Solution

The precommitment solutions takes the feedback form of a rule (C.19) which as we have seen from (C.21) is a rule with memory. The time-inconsistency of this solution is best seen by examining the cost-to-go (C.25). Re-optimising at time t and reneging on the commitment given at time 0 involves putting $\boldsymbol{\lambda}_{p,t} = \mathbf{0}$. Thus the gains from reneging are $-\text{trace}(\mathbf{N}_{f,f}\,\boldsymbol{\lambda}_{p,t}\,\boldsymbol{\lambda}_{p,t}^\top)$. Since it can be shown that $\mathbf{N}_{f,f}$ is negative definite (Currie and Levine (1994), chapter 5, page 145 for a formal proof), it follows that everywhere along the trajectory at which $\boldsymbol{\lambda}_{f,t} \neq \mathbf{0}$ there will be gains from reneging and the ex ante optimal policy will be suboptimal ex post.

In order to construct a time-consistent policy we employ dynamic programming and seek a Markov-perfect equilibrium in which instruments are still allowed to depend on the past history, but only through a feedback on the *current* value of the state variables. This precludes feedback as in (C.21) which involves memory. Thus we seek a stationary solution $\mathbf{w}_t = \mathbf{G}\,\mathbf{z}_t$ in which U_t is minimized at the time t subject to the model (C.2) in the knowledge that an identical procedure will be used to determine U_{t+1} at time $t+1$. Other features of the solution are the $\mathbf{x}_{t'} = -\mathbf{N}\,\mathbf{z}_{t'}$, which we know is true of saddle-path stable solutions to rational expectations models under a rule $\mathbf{w}_{t'} = -\mathbf{F}\,\mathbf{z}_{t'}$, and $U_t = \mathbf{z}_t^\top\,\mathbf{S}\,\mathbf{z}_t$. Notice that all three solution features follow from the precommitment solution with $\boldsymbol{\lambda}_{p,t} = \mathbf{0}$ for all t. The solution is completely characterized by the matrices $\mathbf{F}$, $\mathbf{N}$ and $\mathbf{S}$. We now derive an iterative procedure and sequences $\mathbf{F}_t$, $\mathbf{N}_t$ and $\mathbf{S}_t$ which—if convergent—converge to the these stationary values. Suppose that from time $t+1$ onwards,

$$\mathbf{x}_{t+t'} = -\mathbf{N}_{t+1}\,\mathbf{z}_{t+t'} \qquad \forall\, t' \geq 1 \tag{C.28}$$

Then from (C.1)

$$\begin{aligned} \mathbf{x}_{t+1} &= -\mathbf{N}_{t+1}\,(\mathbf{A}_{p,p}\,\mathbf{z}_t + \mathbf{A}_{p,f}\,\mathbf{x}_t + \mathbf{B}_p\,\mathbf{w}_t) \\ &= \mathbf{A}_{f,p}\,\mathbf{z}_t + \mathbf{A}_{f,f}\,\mathbf{x}_t + \mathbf{B}_f\,\mathbf{w}_t \end{aligned} \tag{C.29}$$

Thus

$$\mathbf{x}_t = \mathbf{J}_t\,\mathbf{z}_t + \mathbf{K}_t\,\mathbf{w}_t \tag{C.30}$$

where

$$\mathbf{J}_t = -(\mathbf{A}_{f,f} + \mathbf{N}_{t+1}\,\mathbf{A}_{p,f})^{-1}\,(\mathbf{N}_{t+1}\,\mathbf{A}_{p,p} + \mathbf{A}_{f,p}) \tag{C.31}$$
$$\mathbf{K}_t = -(\mathbf{A}_{f,f} + \mathbf{N}_{t+1}\,\mathbf{A}_{p,f})^{-1}\,(\mathbf{N}_{t+1}\,\mathbf{B}_p + \mathbf{B}_f) \tag{C.32}$$

Rewrite (C.4) as

$$U_t = \frac{1}{2}\left(\mathbf{y}_t^\top \mathbf{Q}\,\mathbf{y}_t + 2\,\mathbf{y}_t^\top \mathbf{U}\,\mathbf{w}_t + \mathbf{w}_t^\top \mathbf{R}\,\mathbf{w}_t\right) + \varrho\,U_{t+1} \tag{C.33}$$

then putting $U_{t+1} = \mathbf{z}_{t+1}^\top \mathbf{S}_{t+1}\,\mathbf{z}_{t+1}/2$, and substituting for $\mathbf{x}_t$ from (C.30), we obtain

$$U_t = \frac{1}{2}\left(\mathbf{z}_t^\top \bar{\mathbf{Q}}_t\,\mathbf{z}_t + 2\,\mathbf{z}_t^\top \bar{\mathbf{U}}_t\,\mathbf{w}_t + \mathbf{w}_t^\top \bar{\mathbf{R}}_t\,\mathbf{w}_t\right) + \frac{\varrho\,\mathbf{z}_{t+1}^\top \mathbf{S}_{t+1}\,\mathbf{z}_{t+1}}{2} \tag{C.34}$$

where

$$\bar{\mathbf{Q}}_t = \mathbf{Q}_{p,p} + \mathbf{J}_t^\top \mathbf{Q}_{f,p} + \mathbf{Q}_{p,f}\,\mathbf{J}_t + \mathbf{J}_t^\top \mathbf{Q}_{f,f}\,\mathbf{J}_t \tag{C.35}$$

$$\bar{\mathbf{U}}_t = \mathbf{U}_p + \mathbf{Q}_{p,f}\,\mathbf{K}_t + \mathbf{J}_t^\top \mathbf{U}_f + \mathbf{J}_t^\top \mathbf{Q}_{f,f}\,\mathbf{J}_t \tag{C.36}$$

$$\bar{\mathbf{R}} = \mathbf{R} + \mathbf{U}_f^\top \mathbf{K}_t + \mathbf{K}_t^\top \mathbf{U}_f + \mathbf{K}_t^\top \mathbf{Q}_{f,f}\,\mathbf{K}_t \tag{C.37}$$

Similarly eliminate $\mathbf{x}_t$ from (C.1) to obtain

$$\mathbf{z}_{t+1} = \bar{\mathbf{A}}_t\,\mathbf{z}_t + \bar{\mathbf{B}}_t\,\mathbf{w}_t \tag{C.38}$$

where

$$\bar{\mathbf{A}}_t = \mathbf{A}_{p,p} + \mathbf{A}_{p,f}\,\mathbf{J}_t \tag{C.39}$$

$$\bar{\mathbf{B}}_t = \mathbf{B}_p + \mathbf{A}_{p,f}\,\mathbf{K}_t \tag{C.40}$$

Hence substituting (C.38) into (C.34) we arrive at

$$\begin{aligned} U_t = \frac{1}{2}\big[&\mathbf{z}_t^\top (\bar{\mathbf{Q}}_t + \varrho\,\bar{\mathbf{A}}_t\,\mathbf{S}_{t+1}\,\bar{\mathbf{A}}_t)\,\mathbf{z}_t \\ &+ 2\,\mathbf{z}_t^\top (\bar{\mathbf{U}}_t + \varrho\,\bar{\mathbf{A}}_t^\top \mathbf{S}_{t+1}\,\bar{\mathbf{B}}_t)\,\mathbf{w}_t \\ &+ \mathbf{w}_t^\top (\bar{\mathbf{R}}_t + \varrho\,\bar{\mathbf{B}}_t^\top \mathbf{S}_{t+1}\,\bar{\mathbf{B}}_t)\,\mathbf{w}_t\big] \end{aligned} \tag{C.41}$$

The control problem is now to minimize U_t with respect to $\mathbf{w}_t$ given the current state $\mathbf{z}_t$. and given $\mathbf{S}_{t+1}$ and $\mathbf{N}_{t+1}$ which are determined by subsequent reoptimizations. The first order condition is then

$$\begin{aligned} \mathbf{w}_t &= (\bar{\mathbf{R}}_t + \varrho\,\bar{\mathbf{B}}_t^\top \mathbf{S}_{t+1}\,\bar{\mathbf{B}}_t)^{-1}(\bar{\mathbf{U}}_t^\top + \varrho\,\bar{\mathbf{A}}_t^\top \mathbf{S}_{t+1}\,\bar{\mathbf{B}}_t)\,\mathbf{z}_t \\ &= \mathbf{G}_t\,\mathbf{z}_t \end{aligned} \tag{C.42}$$

say. Then combining (C.30) and (C.42) we have

$$\mathbf{x}_t = (\mathbf{J}_t - \mathbf{K}_t\,\mathbf{G}_t)\,\mathbf{z}_t \tag{C.43}$$

$$= -\mathbf{N}_t\,\mathbf{z}_t \tag{C.44}$$

say. Substituting (C.42) into (C.41) and equating the quadratic terms in $\mathbf{z}_t$ gives

$$\begin{aligned} \mathbf{S}_t = {}& \bar{\mathbf{Q}}_t + \bar{\mathbf{U}}_t\,\mathbf{G}_t + \mathbf{G}_t^\top \bar{\mathbf{U}}_t^\top + \mathbf{G}_t^\top \bar{\mathbf{R}}_t\,\mathbf{G}_t \\ &+ (\bar{\mathbf{A}}_t + \bar{\mathbf{B}}_t\,\mathbf{G}_t)^\top \mathbf{S}_{t+1}(\varrho\,\bar{\mathbf{A}}_t + \bar{\mathbf{B}}_t\,\mathbf{G}_t) \end{aligned} \tag{C.45}$$

Given $\mathbf{S}_{t+1}$ and $\mathbf{N}_{t+1}$ equations (C.42), (C.43) and (C.45) give $\mathbf{F}_t$, $\mathbf{N}_t$, and $\mathbf{S}_t$ defining our iterative process. If these converge[13] to stationary values $\mathbf{F}$, $\mathbf{N}$ and $\mathbf{S}$, then we have a time-consistent optimal rule $\mathbf{w}_t = \mathbf{G}\,\mathbf{z}_t$, with cost to go

$$U_t = \frac{1}{2}\,\mathbf{z}_t^\top \mathbf{S}\,\mathbf{z}_t = \frac{1}{2}\,\mathrm{trace}(\mathbf{S}\,\mathbf{Z}_t) \tag{C.46}$$

[13]We have not found any problems with convergence for a wide range of models, including that in this paper