NEW ANALYSIS OF A MODEL OF TIME TO BUILD*

Alistair Milne[†] and A Elizabeth Whalley
University of Surrey

February 1996,
revised May 1996

Abstract

We solve a model of time to build, in closed form, for the special case where the only option is commencing investment. The ratio of the optimal to the NPV investment threshold is as in the standard analysis of irreversible investment. We then report numerical solutions for the general case where there is also an option to suspend investment, investigating variation in the time to build, the uncertainty of payoff and the opportunity cost of foregone cashflows. The two options have opposite effects on the optimal investment decision and NPV calculation is sometimes an appropriate guide to investment.

JEL: C63, D92.
Keywords: Irreversible investment, time to build, numerical solution of partial differential equations.

*Financial support from the Leverhulme Trust is gratefully acknowledged.
[†]Corresponding author: Dr A. Milne, Department of Economics, University of Surrey, Guildford, Surrey, GU2 5XH, U.K.. Tel/Fax: +44 (1483) 259380 / 303775. email: a.milne@surrey.ac.uk

1 Introduction

This paper re-examines a model first investigated by Majd and Pindyck (1987). This model amends the standard analysis of the option value of irreversible investment of McDonald and Siegal (1986) by imposing a constraint on the maximum rate of investment, thus requiring that the project takes time to build. The value of the project on completion is uncertain. Analysis of this model is relevant to a number of investment problems where technical requirements constrain the sequencing of investment expenditures. Examples include the development of mining concessions or large building projects subject to fixed construction schedules.

We view the investment opportunities in this model as consisting of two options, commencing investment and suspending investment once the project is underway. We first derive an analytical solution for the special case where, while the project takes time to build, suspension of investment is not possible.

We next provide a new numerical solution of the general model when both options are available. We do this because the solution provided by Majd and Pindyck (1987) fails to enforce all the appropriate boundary conditions. We discuss this point in detail in the main text, but it is quickly apparent from an inconsistency between our analytical solution and the numerical solution of the general model reported by Majd and Pindyck (1987). Our analytical solution satisfies the same partial differential equation and all the boundary conditions enforced by Majd and Pindyck (1987). However, since this is a different model, the two solutions cannot in fact be the same.

Applying our alternative numerical procedure, we find that there are circumstances (when the time to build is long and the opportunity cost of foregone cashflows is high) in which the effects of these two options are to a substantial degree offsetting. In this circumstance net present value calculations are in fact an adequate guide to the investment decision. This point does not appear in Majd and Pindyck (1987). More typically the option to commence investment dominates and, as in the standard analysis of irreversible investment, investment should be postponed until net present value computations yield a return which covers the value of the option.

2 The model and the need for a corrected solution

2.1 Assumptions

A firm can costlessly suspend and restart investment on a project. There is a maximum feasible rate of investment (k) so that the completion of the project takes a minimum period of $\frac{K}{k}$, where K is the amount of capital which is required to complete the project. We shall refer to K/k as the time to build. Investment is irreversible so the rate of investment, I, satisfies $0 \leq I \leq k$.

The project, when complete ($K = 0$), is worth an uncertain amount V which evolves according to the geometric brownian motion:

$$\mathrm{d}V = \alpha V \mathrm{d}t + \sigma V \mathrm{d}z \tag{1}$$

with $\mathrm{d}z$ the standard Wiener process.

Contingent claims analysis can be used to derive the following partial differential equation in the two states V and K, satisfied when investment is undertaken at the rate I (See Dixit and Pindyck (1994) for a detailed discussion):

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + (r - \delta) V F_V - rF + \max(-I(F_K + 1), 0) \equiv \mathcal{L}_V(F) + \max(-I(F_K + 1), 0) = 0 \tag{2}$$

Here r is the risk free rate of return, and δ is the difference between the market risk-adjusted return on owning the project and the expected rate of growth of V. The cost of investment is normalized to unity. δ, the risk adjusted opportunity cost of foregone cashflows for this project, is assumed positive to rule out a pure holding strategy, where the investment is indefinitely postponed to take advantage of the anticipated growth of V. Hereafter we shall, for brevity, refre to δ as the opportunity cost. $\mathcal{L}_V$ is a linear differential operator introduced to simplify subsequent notation.

Since costs are linear, optimal policy is "bang-bang", investing either at rate k (if $F_K + 1 < 0$), or 0 (if $F_K > -1$). Restricting the upper case F to denote the value function when investment takes place, and using lower case f to represent the value function when there is no investment,

(2) becomes:

$$\mathcal{L}_V(F) = k(F_K + 1) \quad (3)$$

$$\mathcal{L}_V(f) = 0 \quad (4)$$

2.2 The model with no suspension of investment

Thus far this setup is exactly that described by Majd and Pindyck (1987). Our departure from their analysis is to view this investment opportunity as involving two options: that of commencing the start of investment and, once investment has started, that of suspending investment. We begin with the special case in which suspension is not possible and investment once started must be undertaken at the maximum rate until completion. (3) then represents the value of the project once investment has commenced, and satisfies three boundary conditions:

(i) On completion of the project

$$F(V, 0) = V; \quad (5)$$

(ii)F is bounded from below for all $(V \geq 0, K)$ (because the committed cost of the project is finite); and

(iii)

$$\lim_{V \to +\infty} \frac{\partial \ln F}{\partial \ln V} = 1. \quad (6)$$

This applies because a 1 percent increase in V increases the final valuation by 1 percent on all sample paths and, when V is large, the costs of investment can be neglected.

Subject to these boundary conditions this equation has the analytical solution (this can be formally derived using the Laplace transform):

$$F(V, K) = \frac{k}{r}[\exp(-r\frac{K}{k}) - 1] + V\exp(-\delta\frac{K}{k}). \quad (7)$$

This solution has a simple economic interpretation. The first term reflects the present discounted cost of current and future investment from the current time t until completion of the project at

$t + K/k$. The second term represents the present discounted value of the (uncertain) final value of the project.

A net present value rule would suggest commencing investing if $F(K_M, V) \geq 0$, leading to a investment trigger given by

$$V^{NPV} = \frac{k}{r}[\exp(\delta \frac{K_M}{k}) - \exp((\delta - r)\frac{K_M}{k})]. \tag{8}$$

The presence of uncertinty leads to the existence of an option value for the investment opportunity prior to actual investment, f, and optimal policy is then to commence investment when the value is greater than a threshold level *i.e.* $V \geq V^c(K_M)$. [1] Before commencement $V < V^c(K_M)$ and the value of the investment opportunity satisfies (4). On commencement, the value of investment in the project is given by (7).

(4) has a general solution

$$f(V) = AV^{\beta_1} + BV^{\beta_2},$$

where β_1, β_2 are the positive and negative roots respectively of the fundamental quadratic $\frac{1}{2}\sigma^2\beta^2 + (r - \delta - \frac{1}{2}\sigma^2)\beta - r = 0$. Application of the boundary condition $f(0) = 0$ (if the project value ever falls to zero, it remains zero from (1), with zero associated option value) gives $B = 0$.

Two further boundary conditions are needed to determine V^c and $f(V^c)$. These are a value matching condition

$$f(V^c) = F(V^c) = \frac{k}{r}[\exp(-r\frac{K_M}{k}) - 1] + V^c \exp(-\delta \frac{K_M}{k}).$$

(this would apply for any commencement threshold of the form $V(K_M)$) and the optimality condition $F_V(V^c) = f_V(V^c)$ (this is the first order condition for optimal commencment of investment). Substituting for f then shows that the optimal threshold $V^c(K_M)$ for commencing investment F satisfies:

$$\beta_1 F(V^c, K_M) = V^c F_V(V^c, K_M) \tag{9}$$

[1] We denote the *total* investment required for the project by K_M. Note that the threshold here is only valid for $K = K_M$, *i.e.* before commencement of the project.

where β_1 is the positive root of the fundamental quadratic. Substitution of (9) into (7) then yields:

$$V^c = \frac{\beta_1}{\beta_1 - 1}\frac{k}{r}[\exp(\delta\frac{K_M}{k}) - \exp((\delta - r)\frac{K_M}{k})] \tag{10}$$

Comparison of (10) and (8) shows that allowing for the option to delay the commencement of the investment project leads to an increase in the threshold for investment from V^{NPV} to $V^c = \frac{\beta_1}{\beta_1 - 1}V^{NPV}$, the same proportionate increase in the threshold for investment that occurs in McDonald and Siegal (1986), where there is no constraint on the maximum rate of investment. Since all expenditures are committed once the project is underway, the analysis is essentially no different from the standard analysis of irreversible investment.

Note that we can rewrite (10) as

$$\frac{V^c}{K_M} = \frac{\beta_1}{\beta_1 - 1}\frac{V^{NPV}}{K_M} = \frac{\beta_1}{\beta_1 - 1}\frac{1}{r}\left(\frac{K_M}{k}\right)^{-1}[\exp(\delta\frac{K_M}{k}) - \exp((\delta - r)\frac{K_M}{k})] \tag{11}$$

showing the optimal investment threshold, and indeed the net present value investment trigger, expressed as a proportion of the total investment K_M, is a function only of the time to build, K_M/k. We shall therefore consider this ratio rather than the threshold itself when we report our results in Section 4.

2.3 The general model

Now consider the general model in which there is an option to suspend investment temporarily as well as the option to delay commencing investment. In this case the value function satisfies (2), reproduced below as (12), for all $K > 0$.

$$\mathcal{L}_V(F) \equiv \frac{1}{2}\sigma^2 V^2 F_{VV} + (r - \delta)VF_V - rF = \max(I(F_K + 1), 0). \tag{12}$$

Since there are no other costs of starting or stopping investment, optimal policy is to invest when $V \geq V^*(K)$ and to suspend investment when $V < V^*(K)$ for some threshold $V^*(K)$ *i.e.*

$$\begin{aligned} \mathcal{L}_V(F) &= k(F_K + 1) & V \geq V^*(K) \\ \mathcal{L}_V(f) &= 0 & V \leq V^*(K) \end{aligned}$$

$V^*(K)$, which is a function of the remaining time to build K, now represents the smallest value of the project for which further investment is worth undertaking. For $V \geq V^*(K)$ the value of the project includes the option value of future suspensions because, unlike the previous case, reduction of V below the threshold V^* leads to suspension of investment and (4) then applies in place of (3). Similarly for $V \leq V^*(K)$ the option value f satisfies (4), but an increase in V to above V^* leads to recommencement of the investment in the project (at the maximum rate k). It is therefore necessary to solve jointly for F, f and V^*. (In the model with no suspension of investment we first solved for F and then jointly for f and V^*.)

The final condition on completion of the project ($F(V,0) = V$) and the boundary conditions for f at 0 ($f(0) = 0$) and F in the limit as V tends to infinity ($\lim_{V\to\infty}(F(V,K)) = V\exp(-\delta K/k) + k(\exp(-rK/k) - 1)/r$ are the same as in the case where suspension is not possible. However the lower limit on V for which (3) holds is now V^*, and so the boundary condition that F be bounded for all $V > 0$, which applies when there is no option to suspend, must be replaced by an additional boundary condition which applies on V^*.

It can be seen by comparing (12), (3) and (4) that this threshold, V^*, is characterised by

$$\begin{aligned} F_K(V) + 1 &\leq 0 \qquad V \geq V^*(K) \\ f_K(V) + 1 &\geq 0 \qquad V \leq V^*(K) \end{aligned}$$

with equality on the boundary. This additional condition,

$$F_K(V^*) + 1 = 0 \tag{13}$$

is a first order condition for optimality which ensures that V^* is the optimal threshold for suspending investment as well as the optimal threshold for commencing investment. This has the economic interpretation that we suspend investment if the marginal value of investment falls to less than its marginal cost (= 1).

The remaining boundary conditions on the optimal investment threshold V^* are those which apply on V^c in the case where there is no option to suspend (*i.e.* continuity of the value function and its first derivative), so (9) now applies on V^* for all K:

$$\beta_1 F(V^*, K) = V^* F_V(V^*, K) \tag{14}$$

(Note that (13) implies that $\mathcal{L}_V(F) = \frac{1}{2}\sigma^2 V^2 F_{VV} + (r-\delta)VF_V - rF = 0$ on V^*, and, comparing this with (4) and noting that we already require $F(V^*) = f(V^*)$ and $F_V(V^*) = f_V(V^*)$ from (14), we see that (13) is equivalent to requiring $F_{VV}(V^*) = f_{VV}(V^*)$. We shall use this to simplify the boundary conditions we impose in our numerical solution.).

A simple argument establishes that $V^* < V^c$. Write F as the sum of a complementary function F^C (representing the option value of suspending investment) and the particular solution given in (7):

$$F(V,K) = \frac{k}{r}[\exp(-r\frac{K}{k}) - 1] + V\exp(-\delta\frac{K}{k}) + F^C \tag{15}$$

F^C is always positive and declines with V ($F^C > 0, F^C_V < 0$ the second inequality arising because an increase in V delays suspension of investment on all sample paths). Substitution of (15) into (14) then yields:

$$\frac{V^*}{K}\left(1 - \frac{\exp(-\delta\frac{K}{k})}{\beta_1 - 1}F^C_V|_{V^*}\right) + \frac{\beta_1}{\beta_1 - 1}\exp(\delta\frac{K}{k})\frac{F^C}{K} = \frac{V^c}{K} \tag{16}$$

Then since the term multiplying V^*/K is greater than 1 and the second term (involving F^C/K) is positive, we see that V^* is less than the threshold V^c which applies when there is no option to suspend. (10) is an upper bound on the investment threshold in the general case.

As an illustration of these boundaries consider Figure 1 [2] which shows the investment threshold suggested by a net present value computation V^{NPV}/K, the optimal threshold V^c/K when there is no option to suspend, and the optimal threshold V^*/K when both options are available (the numerical computation of V^* is described in the next section). The parameter values are $\delta = 0.05$, $r = 0.02$ and $\sigma = 0.4$ (think of the K/k axis as the number of years required to complete the project (from 0.5 to 12 years)). Note that as K/k increases, V^c/V^* gets larger while V^*/V^{NPV} gets smaller.

[2] Numerical precision puts a lower limit on the value of K/k for which V^*/K can be calculated. The graph contains only numerical computations of V^*/K.

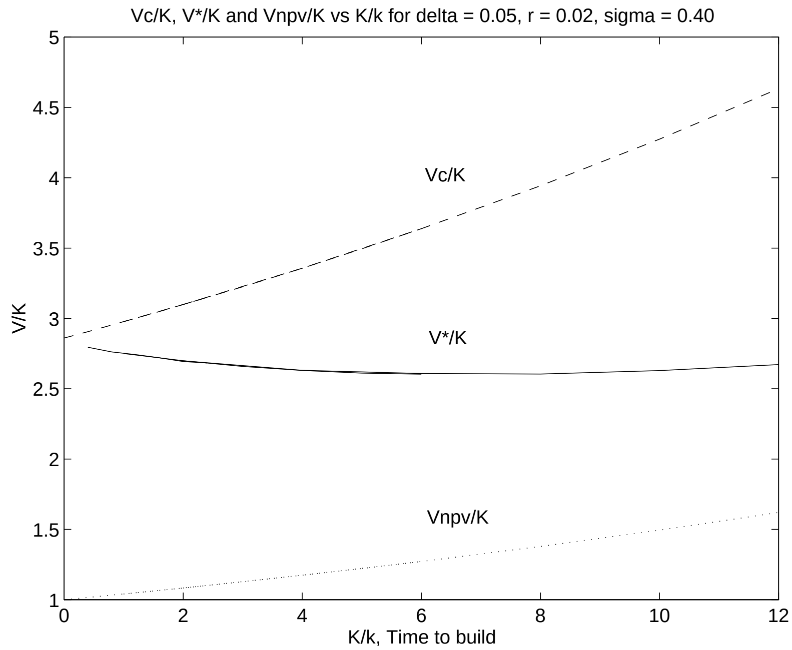

Figure 1: V^*/K, V^c/K and V^{NPV}/K as a function of K/k for $\delta = 0.05$, $r = 0.02$ and $\sigma = 0.4$

2.4 The need to provide a corrected solution

$F(K)$ and V^* must be computed numerically. We provide a corrected procedure for doing this because that applied by Majd and Pindyck (1987) fails to enforce all the appropriate boundary conditions. This is apparent from the following considerations:

- A total of six boundary conditions are needed to solve for f, F and V^*. Majd and Pindyck only supply five.

- In the special case in which suspension of investment is not possible, F satisfies all five boundary conditions stated by Majd and Pindyck (1987) together with a sixth condition that F be bounded from below. However their procedure is supposed to be solving a different model in which suspension is possible and hence in which the value function is *not* given by (7). Thus their statement must omit an alternative sixth condition, which ensures that their numerical results are not the same as (7).

- The boundary condition missing from Majd and Pindyck (1987) is our equation (13). This is *not* implied by the five other boundary conditions. As a counterexample consider the analytical solution of the special case (7), (10). Differentiation shows that this solution satisfies the five other boundary conditions stated by Majd and Pindyck (1987) but not this sixth one. Nor is (13) satisfied by their solution (this is apparent from making approximate calculations using their table 1).

Closer examination of their results also reveals other problems arising from their incorrect numerical procedure:

- They report a value function in their table 1 which, for all values of V above the investment threshold, is slightly smaller than that implied by our equation (7) for the project value where suspension is *not* possible. The introduction of the additional option to suspend investment must however increase, not reduce, the value function. So if (7) is correct their computations are incorrect.

- They report values for V^* in table 1 which to numerical approximation lie exactly on our V^c, and some values in table 2 (for example all $\sigma = 0.4$ values) which exceed our V^c. We have shown however that $V^* < V^c$ (and we will show using our alternative numerical computations that V^* can be considerably less than V^c).

- Their proposed procedure is one in which the value function is computed independently of the boundary (as they describe their procedure it is possible to compute the value function over the entire state space and then return to locate V^* using condition (14)). The value function thus computed cannot correctly incorporate the option value of future suspension.

Why is thescheme applied by Majd and Pindyck (1987), which is a standard solution procedure for the valuation of an American put option, not appropriate for this model of time to build? With the American put there is only one option, exercising for a *known* payoff $f = E - V$ (the difference between the strike price E and the current value of the underlying asset). In that case there are two rather than three boundary conditions which must be satisfied on V^* corresponding to the two unknowns, the value of the option and the position of the boundary itself.

As we point out, in contrast to the American put, the opportunity of time to build involves exercise not of one but two options. The option to suspend investment results in the additional condition (13). An extra condition is necessary because f contains an unknown function of K. The scheme applied in Majd and Pindyck (1987) enforces continuity of the value function and its first derivative (or equivalently (14)) but fails to enforce (13). Comparison of our results with those of Majd and Pindyck (1987) shows that their scheme in fact converges, at least approximately, to V^C. In effect they failed to incorporate the option value of suspension.

We could have enforced (13) using a finite difference scheme similar to that of Majd and Pindyck (1987) but using an iterative search to find the value of $V^*(K)$ which satisfied all three boundary conditions. As we show in the next section these conditions can be imposed more conveniently using an alternative numerical procedure, the analytic method of lines. This procedure involves a joint computation of the value function and the investment threshold and yields a non- negative option value for suspension and a threshold V^c which is strictly less than V^* for positive K. We are satisfied that this procedure is reliable within reasonable margins of numerical accuracy.

3 Numerical solution

3.1 General method

We solved (3) with associated boundary conditions using the analytic method of lines, which has been shown to give efficient and accurate solutions for the related parabolic moving boundary problem of American option valuation (see Carr and Faguet (1994)).

This method involves discretising the K variable so that the partial differential equation (3) is replaced by a series of second order ordinary differential equations on the region $V_n^* \leq V \leq \infty$. Each ordinary differential equation is valid on a line K = constant, and the lines are indexed by the remaining investment. The solution is obtained by stepping successively backwards from the known final solution with no remaining investment (5). At each stage we need to derive the ordinary differential equation, which depends on the solution to the previous stage, find its general closed-form solution, and then apply the boundary conditions at either end ((6) at infinity and (13) and (14) at the lower threshold) to obtain the coefficients of each term in the solution. The solutions become increasingly more complicated as the number of steps increases; however we use Richardson extrapolation to improve the speed of convergence of the method. With extrapolation, we obtain accurate results with only five iterations and find that the error correction in performing the last two iterations is negligible for the range of K we investigate.

Specifically, we let the change in investment between successive approximations be ΔK, and replace the K derivative in (3), F_K, by the backwards finite difference approximation $(F(K) - F(K - \Delta K))/(\Delta K)$. For ease of solution we also transform to log variables *i.e.* $X = \ln(V)$ (or $V = e^X$). This transforms the resulting ordinary differential equation to one with constant coefficients. After these transformations (3) becomes

$$\frac{\sigma^2}{2} F_{XX}^{(n)} + \left(r - \delta - \frac{\sigma^2}{2}\right) F_X^{(n)} - rF^{(n)} = k\left(1 + \frac{F^{(n)} - F^{(n-1)}}{\Delta K}\right), \tag{17}$$

or rearranging,

$$\frac{\sigma^2 \Delta K}{2} F_{XX}^{(n)} + \left(r - \delta - \frac{\sigma^2}{2}\right) \Delta K F_X^{(n)} - (r\Delta K + k)F^{(n)} = k\left(\Delta K - F^{(n-1)}\right), \tag{18}$$

where $F^{(i)}$ is the ith approximation to F and $-_X$ and $-_{XX}$ denote the first and second derivatives with respect to X. The initial and boundary conditions (5), (6), (13) and (14) become

$$F^{(0)}(X) = e^X \tag{19}$$

$$F^{(n)}(X) \rightarrow e^{x - \delta \frac{K}{k}} + \frac{k}{r}(e^{-r\frac{K}{k}} - 1) \quad \text{as} \quad X \rightarrow \infty \tag{20}$$

$$F^{(n)}(X^*) = A^{(n)} e^{\beta_1 X} \tag{21}$$

$$F_X^{(n)}(X^*) = \beta_1 A^{(n)} e^{\beta_1 X} \tag{22}$$

$$F_{XX}^{(n)}(X^*) = \beta_1^2 A^{(n)} e^{\beta_1 X} \tag{23}$$

where

$$\beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

is the positive root of the fundamental equation $0.5\sigma^2\beta^2 + (r - \delta - 0.5\sigma^2)\beta - r = 0$, and $A^{(n)}$ is as yet unknown. (21) and (22), which enforce continuity in the value function and its first derivative with respect to V (or equivalently X) across the optimal investment threshold, are equivalent to (14). As mentioned in the previous section, the third boundary condition (13) implies $\mathcal{L}_V(F) = 0$ on V^*. Noting that $\mathcal{L}_V(f) = 0$ for $V \leq V^*$ and using (21) and (22), we see that (13) is equivalent to enforcing continuity of the second derivative with respect to X at V^*,*i.e.* (23), which is significantly easier to apply.

The general solution for (18) is composed of the general solution to the related homogenous equation plus the particular solution due to the inhomogeneous term (the right-hand side).

The homogeneous equation related to (18) is

$$\frac{\sigma^2 \Delta K}{2k} F_{XX}^{(n)} + \left(r - \delta - \frac{\sigma^2}{2}\right) \frac{\Delta K}{k} F_X^{(n)} - \left(r\frac{\Delta K}{k} + 1\right) F^{(n)} = 0. \tag{24}$$

This has a general solution of the form

$$F^{(n)}(X) = a^{(n)} e^{\mu_1 X} + b^{(n)} e^{\mu_2 X}$$

where

$$\mu_{1,2} = \frac{1}{2} - \frac{r-\delta}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\left(1 + \frac{k}{r\Delta K}\right)}$$

are the positive and negative roots respectively of the fundamental equation $0.5\sigma^2\Delta K\mu^2 + (r - \delta - 0.5\sigma^2)\Delta K\mu - (r\Delta K + k) = 0$, and $a^{(n)}, b^{(n)}$ are to be determined.

3.2 Solution for first iteration

The particular solution for the inhomogeneous equation depends on the inhomogeneous term $k(\Delta K - F^{(n-1)})$ and hence on the previous iteration's solution. For the first iteration, $F^{(1)}$, the inhomogeneous term in the equation is $k(\Delta K - e^X)$ (since $F^{(0)}(X) = e^X$ by (19)). We therefore try a solution of the form $c^{(1)}e^X + d^{(1)}$ and find that the particular solution is

$$\frac{1}{1+D}e^X - \frac{\Delta K}{1+R},$$

where $D = \delta\Delta K/k$ and $R = r\Delta K/k$.

The general solution for the first iteration is

$$F^{(1)}(X) = a^{(1)}e^{\mu_1 X} + b^{(1)}e^{\mu_2 X} + \frac{1}{1+D}e^X - \frac{\Delta K}{1+R}.$$

If we now apply the boundary condition at infinity, (20), we see that $a^{(1)} = 0$. (It turns out that $a^{(n)} = 0$ for all n for the same reason). The boundary conditions at X^* give us three equations for three unknowns, $A^{(1)}, b^{(1)}$ and X_1^* itself:

$$\begin{aligned}
A^{(1)}e^{\beta_1 X_1^*} &= b^{(1)}e^{\mu_2 X_1^*} + \frac{1}{1+D}e^{X_1^*} - \frac{\Delta K}{1+R}, \\
\beta_1 A^{(1)}e^{\beta_1 X_1^*} &= \mu_2 b^{(1)}e^{\mu_2 X_1^*} + \frac{1}{1+D}e^{X_1^*}, \\
\beta_1^2 A^{(1)}e^{\beta_1 X_1^*} &= \mu_2^2 b^{(1)}e^{\mu_2 X_1^*} + \frac{1}{1+D}e^{X_1^*},
\end{aligned}$$

After some manipulation these can be solved explicitly to give

$$\begin{aligned}
A^{(1)} &= \frac{\mu_2}{(1+R)(\mu_2-\beta_1)(\beta_1-1)}\Delta K e^{-\beta_1 X_1^*}, \\
b^{(1)} &= \frac{\beta_1}{(1+R)(\mu_2-\beta_1)(\mu_2-1)}\Delta K e^{-\mu_2 X_1^*}, \\
e^{X_1^*} &= \frac{\mu_2\beta_1(1+D)}{(1+R)(\mu_2-1)(\beta_1-1)}\Delta K,.
\end{aligned}$$

This means the full solution for the first iteration is

$$\begin{aligned}
f^{(1)}(X) &= \tfrac{\mu_2}{(1+R)(\mu_2-\beta_1)(\beta_1-1)}\Delta K e^{\beta_1(X-X_1^*)}, & X \le X_1^* \\
F^{(1)}(X) &= \tfrac{\beta_1}{(1+R)(\mu_2-\beta_1)(\mu_2-1)}\Delta K e^{\mu_2(X-X_1^*)} + \tfrac{1}{1+D}e^X - \tfrac{\Delta K}{1+R}, & X \ge X_1^*
\end{aligned}$$

with

$$e^{X_1^*} = \frac{\mu_2\beta_1(1+D)\Delta K}{(1+R)(\mu_2-1)(\beta_1-1)}.$$

3.3 Second and subsequent iterations

For the second iteration we must solve

$$\frac{\sigma^2 \Delta K}{2} F^{(2)}_{XX} + \left(r - \delta - \frac{\sigma^2}{2}\right) \Delta K F^{(2)}_X - (r\Delta K + k) F^{(2)} = k\left(\Delta K - F^{(1)}\right),$$

in (X_2^*, ∞), where $F^{(1)}$ is now known. Note that because we are only solving this in (X_2^*, ∞) and $X_2^*(2\Delta K) > X_1^*(\Delta K)$, we need only consider the form of $F^{(1)}$ for $X \geq X_1^*$, *i.e.*

$$F^{(1)}(X) = \frac{\beta_1}{(1+R)(\mu_2 - \beta_1)(\mu_2 - 1)} \Delta K e^{\mu_2 (X - X_1^*)} + \frac{1}{1+D} e^X - \frac{\Delta K}{1+R}.$$

However, when we consider the inhomogeneous term

$$k(\Delta K - F^{(1)}) = -\frac{k\beta_1 \Delta K e^{-\mu_2 X_1^*}}{(1+R)(\mu_2 - \beta_1)(\mu_2 - 1)} e^{\mu_2 X} - \frac{k}{1+D} e^X - \Delta K\left(k + \frac{1}{1+R}\right).$$

we see that since it has a term in $e^{\mu_2 X}$, we must seek a particular solution of the general form $c^{(2)} e^X + d^{(2)} + g^{(2)} X e^{\mu_2 X}$.

The general solution for the homogeneous equation is, as for the first iteration,

$$\begin{aligned} f^{(2)}(X) = &\quad a^{(2)} e^{\mu_1 X} + b^{(2)} e^{\mu_2 X} \qquad X \geq X_2^* \\ F^{(2)}(X) = &\quad A^{(2)} e^{\beta_1 X} \qquad X \leq X_2^* \end{aligned}$$

and $a^{(2)} = 0$ from the boundary condition at infinity (20). The general solution for the second approximation is

$$\begin{aligned} F^{(2)}(X) = &\quad b^{(2)} e^{\mu_2 X} - \tfrac{k\beta_1}{s(1+R)(\mu_2 - \beta_1)(\mu_2 - 1)} X e^{\mu_2 (X - X_1^*)} \\ & \quad - \tfrac{1}{(1+D)^2} e^X - \tfrac{\Delta K (2+R)}{(1+R)^2} \qquad X \geq X_2^* \\ f^{(2)}(X) = &\quad A^{(2)} e^{\beta_1 X} \qquad X \leq X_2^* \end{aligned} \tag{25}$$

As for the first approximation, the boundary conditions at X_2^* give three equations for the three unknowns, $b^{(2)}, A^{(2)}$, and X_2^*:

$$\begin{aligned}
A^{(2)}e^{\beta_1 X} &= b^{(2)}e^{\mu_2 X_2^*} - \frac{k\beta_1}{s(1+R)(\mu_2-\beta_1)(\mu_2-1)}X_2^* e^{\mu_2(X_2^*-X_1^*)} \\
&\quad - \frac{1}{(1+D)^2}e^{X_2^*} - \frac{\Delta K(2+R)}{(1+R)^2} \\
\beta_1 A^{(2)}e^{\beta_1 X} &= \mu_2 b^{(2)}e^{\mu_2 X_2^*} - \frac{k\beta_1}{s(1+R)(\mu_2-\beta_1)(\mu_2-1)}e^{\mu_2(X_2^*-X_1^*)} \\
&\quad - \frac{k\beta_1\mu_2}{s(1+R)(\mu_2-\beta_1)(\mu_2-1)}X_2^* e^{\mu_2(X_2^*-X_1^*)} - \frac{1}{(1+D)^2}e^{X_2^*} \\
\beta_1^2 A^{(2)}e^{\beta_1 X} &= \mu_2^2 b^{(2)}e^{\mu_2 X_2^*} - \frac{2k\beta_1\mu_2}{s(1+R)(\mu_2-\beta_1)(\mu_2-1)}e^{\mu_2(X_2^*-X_1^*)} \\
&\quad - \frac{k\beta_1\mu_2^2}{s(1+R)(\mu_2-\beta_1)(\mu_2-1)}X_2^* e^{\mu_2(X_2^*-X_1^*)} - \frac{1}{(1+D)^2}e^{X_2^*}.
\end{aligned}$$

From these we can find explicit formulae for $b^{(2)}$ and $A^{(2)}$, and the equation satisfied by X_2^*, which must be solved numerically [3]

$$\frac{(\beta_1-1)(\mu_2-1)}{(1+D)^2}e^X - \frac{\Delta K\mu_2\beta_1(2+R)}{(1+R)^2} - \frac{k\beta_1 e^{-\mu_2 X_1^*}}{s(1+R)(\mu_2-1)}e^{\mu_2 X} = 0. \tag{26}$$

Following a similar procedure we can find the third, fourth and fifth approximations, V_3^*, V_4^* and V_5^* for the optimal investment threshold, and $F^{(3)}, F^{(4)}$ and $F^{(5)}$, and $f^{(3)}, f^{(4)}$ and $f^{(5)}$ respectively for the value of the investment project. These are given in Appendix A.

We can, however, considerably improve the accuracy of our results with a relatively small number of iterations by using the technique of Richardson extrapolation. This uses a combination of approximations chosen to eliminate the errors between the successive approximations and the true result. The Richardson extrapolation with two components is

$$\begin{aligned}
V_{(12)}^* &= 2V_1^* - V_2^*, \\
F^{(12)}(X) &= 2F^{(1)} - F^{(2)};
\end{aligned}$$

the Richardson extrapolations with three, four and five components are given in Appendix A. These were found to be give results which were extremely close together up to large values of K ($K = 6$).

[3] For our numerical simulations we used a simple bisection method, in spite of slightly slower speed, because it eliminated the problems associated with the value of this function for low values of X

In the results presented in the next section, we have used the Richardson extrapolation on five approximations, $F^{(12345)}$ and X^*_{12345}.

4 Findings

As mentioned in Section 2, all findings we report are in terms of the proportional investment thresholds with respect to the investment K, *i.e.* we report the investment threshold divided by K. This facilitates the analysis because this proportionate investment thresholds $(V^c/K, V^{NPV}/K, V^*/K)$ are functions of K, the remaining investment, and k, the maximum investment rate, only through their ratio, K/k, which represents the (minimum) time to complete the project.

4.1 Dependence on K/k

We first consider how the proportional optimal investment threshold, V^*/K, varies with the time to build K/k. As shown in Section 2, the ratio of the optimal investment threshold with no suspension of investment, (V^c/K from (10)), to the investment threshold based on a Net Present Value rule, (V^{NPV}/K from (8)), is

$$\frac{V^c/K}{V^{NPV}/K} = \frac{\beta_1}{\beta_1 - 1} \geq 1. \tag{27}$$

This ratio is independent of K/k.

Figure 1 shows how V^*/K, V^c/K and V^{NPV}/K change with the time to build K/k. Whilst the investment thresholds V^c/K and V^{NPV}/K always rise with increasing time to build, the investment threshold *with* the option to suspend, V^*/K, falls when the time to build is small before reaching a minimum and starting to rise as a function of the time to build.

An expansion of V^*/K for small times to build $K/k \ll 1$ shows that to leading order

$$\frac{V^*}{K} = \frac{\beta_1}{\beta_1 - 1}\left(1 - \frac{\sigma}{\sqrt{2}}\left(\frac{K}{k}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\delta - r + \frac{\sigma^2}{2}\right)\left(\frac{K}{k}\right) + O\left(\left(\frac{K}{k}\right)^{\frac{3}{2}}\right)\right). \tag{28}$$

This indicates that V^*/K always falls for for times to build very close to 0 (where the first term $-\sigma/2^{1/2}(K/k)^{1/2} < 0$ dominates). This initial decline in V^*/K for small K/k is independent of the other parameter values. The value of K/k for which the minimum of V^*/K occurs is sensitive to δ, σ and r. Our computations of V^*/K (as in Figure 1) confirm that, as suggested by (28), the position of this minimum increases when δ and σ increase and when r decreases.

The ratio V^*/V^{NPV} always falls as time to build increases, reflecting the greater significance of the option to suspend as the time over which it may be exercised increases. As anticipated, we find $V^{NPV}/K \leq V^*/K \leq V^c/K$ for all δ, σ, k computed, and that $V^*/K \to V^c/K$ as the time to build decreases ($K/k \to 0$). Additionally we find that as time to build increases ($K/k \to \infty$), $V^*/K \to V^{NPV}/K$.

4.2 The ratio of optimal investment thresholds

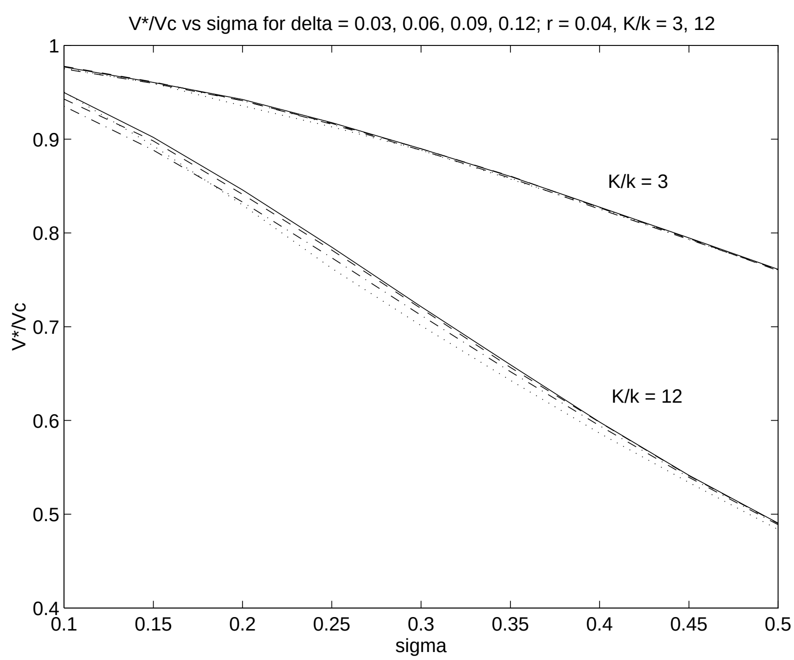

Figure 2: V^*/V^c as a function of σ for $\delta = 0.03, 0.06, 0.09, 0.12$ and $K/k = 12$ and 3

A measure of the relative effects of the two options, commencing and suspending the investment, is the ratio of the optimal investment thresholds, V^*/V^c. Figure 2 plots this ratio against changes in the uncertainty, measured by σ, for a range of values of δ, and for two times to build K/k.

There are two new points demonstrated by this graph. First, changing δ from low to high values leaves the ratio V^*/V^c unchanged (within the limits of numerical accuracy) *i.e.* a change in the opportunity cost has proportionately the same effect on the two investment thresholds. Second, V^*/K decreases as a proportion of V^c/K as the volatility increases, demonstrating the hedging nature of the option to suspend and its effect in limiting the increase in initial investment threshold. Finally, as also appears in Figure 1, V^*/V^c decreases as the time to build increases.

4.3 Effect of opportunity cost δ on $V^*/K, V^c/K$

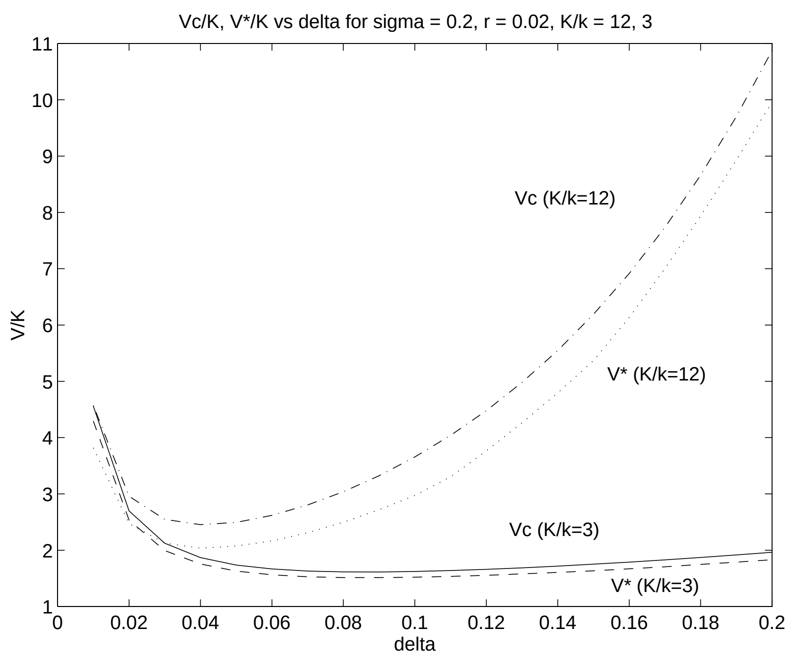

Figure 3: V^*/K and V^c/K as functions of δ for $\sigma = 0.2$ and $K/k = 12$ and 3

Figure 3 plots the optimal investment thresholds, V^*/K and V^c/K, against the rate of opportunity cost, δ, for various values of K/k. This confirms one of the main results of Majd and Pindyck

(1987) and their intuition that for investment both with and without the suspension option there are two competing effects of a change in δ. The first is that increasing the opportunity cost (of foregone cash-flows) lowers the values of the options to commence and suspend investment, and hence lowers the investment threshold. This dominates for small values of δ. The second is that, because of the time taken to complete the investment project, increasing the opportunity cost decreases the value of the cashflows arising from the project (since they are discounted at a higher rate). This makes the project less attractive and so raises the investment threshold. As expected, this second effect is more pronounced for larger values of δ and when time to build is long. Since the ratio V^*/V^c is independent of δ, the minimum of this graph is the same as reported by Majd and Pindyck (1987).

4.4 Implications for investment criteria

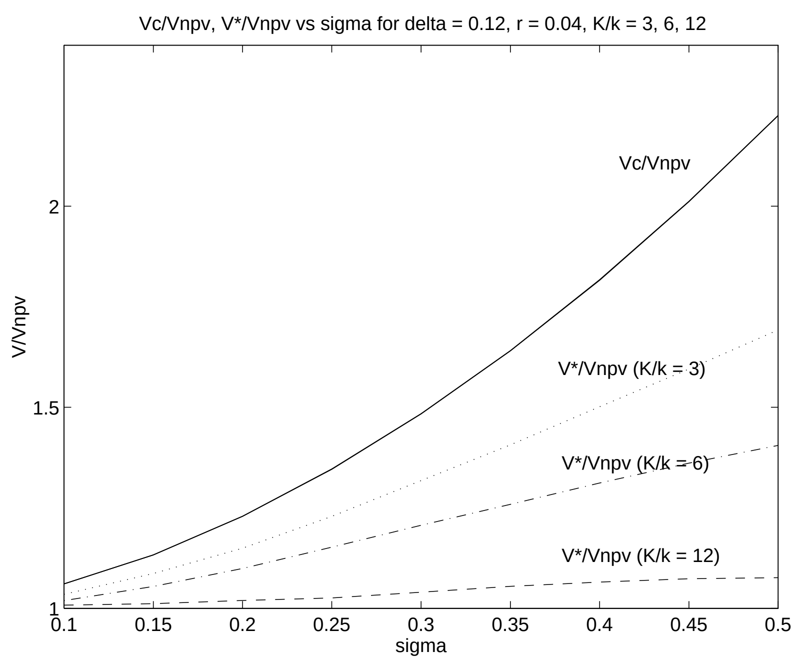

Figure 4: V^c/V^{NPV} and V^*/V^{NPV} as a function of σ for $\delta = 0.12$, $r = 0.04$ and $K/k = 3, 6$ and 12

(27) shows that when suspension of the investment is not possible, the introduction of uncertainty fundamentally alters the investment criteria from a naive NPV rule. This has greatest effect (and hence the largest potential errors in using a NPV framework for decision making) for large values of σ or small values of δ. This is illustrated by the top line in Figure 4 which shows the ratio V^c/V^{NPV} (which is independent of the time to build K/k) as a function of the uncertainty σ. Note this is for a relatively large value of δ.

With large δ and long time to build, the introduction of the option to suspend means that the NPV criteria is once again an appropriate guide to decision making. This is illustrated by the bottom line of Figure 4 which shows the ratio V^*/V^{NPV} for a time to build $K/k = 12$ and $\delta = 0.12$. In this case the optimal threshold taking into account the uncertainty, V^*/K, is less than 10% greater than the NPV threshold, V^{NPV}/K, even for 50% volatility. The remaining dotted lines show the ratio V^*/V^{NPV} for shorter times to build. As the time to build decreases or the uncertainty increases, this ratio (and hence the errors incurred by use of the NPV rule) increases and, especially for large values of σ, becomes large (it produces an optimal threshold 60% greater than the NPV threshold when the time to build is 3 years and volatility is 50%).

5 Conclusions

Our paper has sought to improve on the analysis of time to build offered by Majd and Pindyck (1987), distinguishing the separate options of commencing and suspending investment. We provide a closed form solution for the case where the only option available is that of commencing investment and show that the proportionate increase in the optimal threshold for commencing investment, relative to that suggested by a net present value calculation, is exactly that which applies to an irreversible investment project which can be completed instanteneously (the standard analysis of McDonald and Siegal (1986)). Investment is postponed until the point at which the net present value equals the value of exercising the option to invest.

Turning to the general case where both options are available, we find that the finite difference numerical solution method applied by Majd and Pindyck (1987) does not enforce all the required boundary conditions. We provide an alternative numerical procedure which uses the analytical method of lines to enforce all these conditions.

Comparing our results with those of Majd and Pindyck (1987) yields a new economic insight: the two options to commence and suspend investment alter the optimal investment threshold in different directions and, for some combinations of parameters, are largely offsetting. Specifically, when the opportunity cost of the foregone cashflows from the project is large and the time to build is long, the optimal investment threshold is close to that suggested by a net present value calculation. In this case a net present value criterion, which does not take account of the values of the options to commence and suspend investment, is an appropriate guide to decision making.

For intermediate parameter values the optimal investment threshold lies between that suggested by a net present value computation and that which emerges when the only option is commencing investment. In the limit, as the time to build grows longer or the opportunity cost grows higher, the ratio of the optimal threshold to that suggested by a net present value criteria tends to unity. For most parameter values it is necessary to take account of both options in assessing the investment decision. We also find that when the time to build is increased, but other parameters are held constant, the optimal investment threshold as a proportion of the total investment, V^*/K, initially falls before eventually rising as K/k becomes sufficiently large; the value of K/k which minimizes V^*/K is sensitive to the values of σ, δ and r.

In other respects our analysis confirms that of Majd and Pindyck (1987). Like them we find that an increase in the opportunity cost from a low level reduces the optimal investment threshold because it reduces the value of the option to invest relative to the expected net benefits of investment. Further increases in the opportunity cost eventually increase the investment threshold because of the greater relative effect of the net benefits of the investment. Increasing uncertainty raises the optimal investment threshold, but when both options are taken into consideration does so by less than reported by Majd and Pindyck (1987).

References

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[2] Dixit, Avinash K and Robert S Pindyck (1994) *Investment under Uncertainty*, Princeton University Press, Princeton, NJ.

[3] Majd, Saman and Robert S Pindyck (1987), Time to Build, Option Value and Investment Decisions, *Journal of Financial Economics* 18, 7-27, North-Holland.

[4] McDonald, Robert and Daniel Siegal (1986), The Value of Waiting to Invest, *Quarterly Journal of Economics* 101, 707-727.

A Approximations for X_n^* and $F^{(n)}$

The first approximation for the value function is

$$\begin{aligned} F^{(1)}(X) = &\quad b^{(1)}e^{\mu_2 X} + c^{(1)}e^X + d^{(1)} \qquad && X \geq X_1^* \\ f^{(1)}(X) = &\quad A^{(1)}e^{\beta_1 X} && X \leq X_1^* \end{aligned}$$

where

$$\begin{aligned} A^{(1)} &= \frac{\mu_2}{(1+R)(\mu_2-\beta_1)(\beta_1-1)}\Delta K e^{-\beta_1 X_1^*}, \\ b^{(1)} &= \frac{\beta_1}{(1+R)(\mu_2-\beta_1)(\mu_2-1)}\Delta K e^{-\mu_2 X_1^*}, \end{aligned}$$

$$\begin{aligned} c^{(1)} &= \frac{1}{1+D}, \\ d^{(1)} &= -\frac{\Delta K}{1+R}, \end{aligned}$$

and where we have used the notation $D = \delta\Delta K/k$ and $R = r\Delta K/k$.

The first approximation for the optimal investment threshold is

$$V_1^* = e^{X_1^*} = \frac{\mu_2\beta_1(1+D)\Delta K}{(1+R)(\mu_2-1)(\beta_1-1)}.$$

The second approximations are given by

$$F^{(2)}(X) = (b^{(2)} + g^{(2)}X)e^{\mu_2 X} + c^{(2)}e^X + d^{(2)} \qquad X \geq X_2^*$$

$$f^{(2)}(X) = A^{(2)}e^{\beta_1 X} \qquad X \leq X_2^*$$

where

$$\begin{aligned}
A^{(2)} &= \frac{(\mu_2 - 1)c^{(2)}}{\beta_1(\mu_2 - \beta_1)}e^{X_2^*(1-\beta_1)} + \frac{\mu_2 g^{(2)}}{\beta_1(\mu_2 - \beta_1)}e^{X_2^*(\mu_2-\beta_1)} \\
b^{(2)} &= \frac{(\beta_1 - 1)c^{(2)}}{\mu_2(\mu_2 - \beta_1)}e^{X_2^*(1-\mu_2)} - g^{(2)}\left(X_2^* + \frac{2\mu_2 - \beta_1}{\mu_2(\mu_2 - \beta_1)}\right) \\
c^{(2)} &= \frac{c^{(1)}}{1 + D} \\
d^{(2)} &= -\frac{(\Delta K - d^{(1)})}{1 + R} \\
g^{(2)} &= -\frac{kb^{(1)}}{s\Delta K}
\end{aligned}$$

and where we have used the notation $s = (\mu_2 - 0.5)\sigma^2 + r - \delta$, and $V_2^* = e^{X_2^*}$ is the solution of

$$(\beta_1 - 1)(\mu_2 - 1)c^{(2)}e^X + \mu_2\beta_1 d^{(2)} + (\mu_2 - \beta_1)g^{(2)}e^{\mu_2 X} = 0$$

The third approximation is given by

$$F^{(3)}(X) = (b^{(3)} + g^{(3)}X + h^{(3)}X^2)e^{\mu_2 X} + c^{(3)}e^X + d^{(3)} \qquad X \geq X_3^*$$

$$f^{(3)}(X) = A^{(3)}e^{\beta_1 X} \qquad X \leq X_3^*$$

where

$$\begin{aligned}
A^{(3)} &= \frac{(\mu_2 - 1)c^{(3)}}{\beta_1(\mu_2 - \beta_1)}e^{X_3^*(1-\beta_1)} + \frac{e^{X_2^*(\mu_2-\beta_1)}}{\beta_1(\mu_2 - \beta_1)}\left(\mu_2 g^{(3)} + 2h^{(3)}(\mu_2 X_3^* + 1)\right) \\
b^{(3)} &= \frac{(\beta_1 - 1)c^{(3)}}{\mu_2(\mu_2 - \beta_1)}e^{X_3^*(1-\mu_2)} - g^{(3)}\left(X_3^* + \frac{2\mu_2 - \beta_1}{\mu_2(\mu_2 - \beta_1)}\right) \\
&\quad - h^{(3)}\left(X_3^{*2} + \frac{2(2\mu_2 - \beta_1)}{\mu_2(\mu_2 - \beta_1)}X_3^* + \frac{2}{\mu_2(\mu_2 - \beta_1)}\right) \\
c^{(3)} &= \frac{c^{(2)}}{1 + D} \\
d^{(3)} &= -\frac{(\Delta K - d^{(2)})}{1 + R} \\
g^{(3)} &= -\frac{k}{s\Delta K}\left(b^{(2)} - \frac{\sigma^2}{2s}g^{(2)}\right) \\
h^{(3)} &= -\frac{kg^{(2)}}{2s\Delta K}
\end{aligned}$$

$V_3^* = e^{X_3^*}$ is the solution of

$$(\beta_1 - 1)(\mu_2 - 1)c^{(3)}e^X + \mu_2\beta_1 d^{(3)} + \left((\mu_2 - \beta_1)(g^{(3)} + 2h^{(3)}X) + 2h^{(3)}\right) e^{\mu_2 X} \quad = \quad 0$$

The fourth approximation is given by

$$\begin{aligned} F^{(4)}(X) = &\quad (b^{(4)} + g^{(4)}X + h^{(4)}X^2 + i^{(4)}X^3)e^{\mu_2 X} + c^{(4)}e^X + d^{(4)} && X \geq X_4^* \\ f^{(4)}(X) = &\quad A^{(4)}e^{\beta_1 X} && X \leq X_4^* \end{aligned}$$

where

$$\begin{aligned}
A^{(4)} &= \frac{(\mu_2 - 1)c^{(4)}}{\beta_1(\mu_2 - \beta_1)} e^{X_4^*(1-\beta_1)} \\
&\quad + \frac{e^{X_2^*(\mu_2 - \beta_1)}}{\beta_1(\mu_2 - \beta_1)} \left(\mu_2 g^{(4)} + 2h^{(4)}(\mu_2 X_4^* + 1) + 3i^{(4)}(\mu_2 X_4^{*2} + 2X_4^*)\right) \\
b^{(4)} &= \frac{(\beta_1 - 1)c^{(4)}}{\mu_2(\mu_2 - \beta_1)} e^{X_4^*(1-\mu_2)} - g^{(4)} \left(X_4^* + \frac{2\mu_2 - \beta_1}{\mu_2(\mu_2 - \beta_1)}\right) \\
&\quad - h^{(4)} \left(X_4^{*2} + \frac{2(2\mu_2 - \beta_1)}{\mu_2(\mu_2 - \beta_1)} X_4^* + \frac{2}{\mu_2(\mu_2 - \beta_1)}\right) \\
&\quad - i^{(4)} \left(X_4^{*3} + \frac{3(2\mu_2 - \beta_1)}{\mu_2(\mu_2 - \beta_1)} X_4^{*2} + \frac{6}{\mu_2(\mu_2 - \beta_1)} X_4^*\right) \\
c^{(4)} &= \frac{c^{(3)}}{1 + D} \\
d^{(4)} &= -\frac{(\Delta K - d^{(3)})}{1 + R} \\
g^{(4)} &= -\frac{k}{s\Delta K} b^{(3)} - \frac{\sigma^2}{s} h^{(4)} \\
h^{(4)} &= -\frac{k}{2s\Delta K} g^{(3)} - \frac{3\sigma^2}{2s} i^{(4)} \\
i^{(4)} &= -\frac{k}{3s\Delta K} h^{(3)}
\end{aligned}$$

and $V_4^* = e^{X_4^*}$ is the solution of

$$\left((\mu_2 - \beta_1)(g^{(4)} + 2h^{(4)}X + 3i^{(4)}X^2) + 2h^{(4)} + 6i^{(4)}X\right) e^{\mu_2 X} + \ (\beta_1 - 1)(\mu_2 - 1)c^{(4)}e^X + \mu_2\beta_1 d^{(4)} = 0$$

The fifth approximation is given by

$$\begin{aligned} F^{(5)}(X) = &\quad (b^{(5)} + g^{(5)}X + h^{(5)}X^2 + i^{(5)}X^3 + j^{(5)}X^4)e^{\mu_2 X} + c^{(5)}e^X + d^{(5)} && X \geq X_5^* \\ f^{(5)}(X) = &\quad A^{(5)}e^{\beta_1 X} && X \leq X_5^* \end{aligned}$$

where

$$
\begin{aligned}
A^{(5)} &= \frac{(\mu_2-1)c^{(5)}}{\beta_1(\mu_2-\beta_1)}e^{X_4^*(1-\beta_1)} \\
&\quad + \frac{e^{X_2^*(\mu_2-\beta_1)}}{\beta_1(\mu_2-\beta_1)}\left(\mu_2 g^{(4)} + 2h^{(4)}(\mu_2 X_4^* + 1) + 3i^{(4)}(\mu_2 X_4^{*\,2} + 2X_4^*)\right)
\end{aligned}
$$

$$
\begin{aligned}
b^{(5)} &= \frac{(\beta_1-1)c^{(5)}}{\mu_2(\mu_2-\beta_1)}e^{X_4^*(1-\mu_2)} - g^{(4)}\left(X_4^* + \frac{2\mu_2-\beta_1}{\mu_2(\mu_2-\beta_1)}\right) \\
&\quad - h^{(4)}\left(X_4^{*\,2} + \frac{2(2\mu_2-\beta_1)}{\mu_2(\mu_2-\beta_1)}X_4^* + \frac{2}{\mu_2(\mu_2-\beta_1)}\right) \\
&\quad - i^{(4)}\left(X_4^{*3} + \frac{3(2\mu_2-\beta_1)}{\mu_2(\mu_2-\beta_1)}X_4^{*\,2} + \frac{6}{\mu_2(\mu_2-\beta_1)}X_4^*\right) \\
c^{(5)} &= \frac{c^{(4)}}{1+D} \\
d^{(5)} &= -\frac{(\Delta K - d^{(4)})}{1+R} \\
g^{(5)} &= -\frac{k}{s\Delta K}b^{(4)} - \frac{\sigma^2}{s}h^{(5)} \\
h^{(5)} &= -\frac{k}{2s\Delta K}g^{(4)} - \frac{3\sigma^2}{2s}i^{(5)} \\
i^{(5)} &= -\frac{k}{3s\Delta K}h^{(4)} - \frac{\sigma^2}{2s}j^{(5)} \\
j^{(5)} &= -\frac{k}{4s\Delta K}i^{(4)}
\end{aligned}
$$

and $V_5^* = e^{X_5^*}$ is the solution of

$$
\begin{aligned}
&\left((\mu_2-\beta_1)(g^{(5)} + 2h^{(5)}X + 3i^{(5)}X^2 + 4j^{(5)}X^3) + 2h^{(5)} + 6i^{(5)}X + 12j^{(5)}X^2\right)e^{\mu_2 X} \\
&\qquad + (\beta_1-1)(\mu_2-1)c^{(5)}e^X + \mu_2\beta_1 d^{(5)} = 0
\end{aligned}
$$

The Richardson extrapolations with two, three, four and five components are

$$
\begin{aligned}
V_{12}^* &= 2V_1^* - V_2^*, \\
V_{123}^* &= \frac{1}{2}V_1^* - 4V_2^* + \frac{9}{2}V_3^* \\
V_{1234}^* &= -\frac{1}{6}V_1^* + 4V_2^* - \frac{27}{2}V_3^* + \frac{32}{3}V_4^* \\
V_{12345}^* &= \frac{1}{24}V_1^* - \frac{8}{3}V_2^* + \frac{81}{4}V_3^* - \frac{128}{3}V_4^* + \frac{625}{24}V_5^*
\end{aligned}
$$

with similar expressions for the value function approximations.